Energy *is* Conserved in General Relativity

By Philip Gibbs

Abstract: Since the early days of relativity the question of conservation of energy in general relativity has been a controversial subject. There have been many assertions that energy is not exactly conserved except in special cases, or that the full conservation law as given by Noether's theorem reduces to a trivial identity. Here I refute each objection to show that the energy conservation law is exact, fully general and useful.

Introduction

Conservation of Energy is regarded as one of the most fundamental and best known laws of physics. It is a kind of meta-law that all systems of dynamics in physics are supposed to fulfil to be considered valid. Yet some physicists and cosmologists claim that the law of energy conservation and related laws for momentum actually break down in Einstein's theory of gravity described by general relativity. Some say that these conservation laws are approximate, or only valid in special cases, or that they can only be written in non-covariant terms making them unphysical. Others simply claim that they are satisfied only by equations that reduce to a uselessly trivial identity.

In fact none of those things are true. Energy conservation is an exact law in general relativity. It is fully general for all circumstances and it is certainly non-trivial. Furthermore, it is important for students of relativity to understand the conservation laws properly because they are important to further research at the forefront of physics.

In earlier papers I have described a covariant formulation of energy conservation law that is perfectly general. Some physicists continue to offer pedagogical arguments for why energy conservation in general relativity does not work. All of these are fallacies yet they seem convincing to many people including some physics experts. The purpose of this article is to refute all those objections in the simplest terms possible, citing more detailed analysis where necessary. For convenience and clarity I paraphrase each objection that I have encountered in discussions (often many times) giving it a number for reference and following with my answer in italics. I have ordered them to include the most common arguments first, but also in a way that makes a progressive dialog

The fallacies

(1) <u>Cosmic Microwave Background</u>: The cosmic microwave background consists of individual photons that cross space without interacting with matter. As the universe expands the photons are red-shifted. The energy of a photon is given by Planck's constant times its frequency, so as the frequency decreases over time the energy also decreases. Since the number of photons themselves remains constant this means that energy is being lost and hence energy cannot be conserved in general relativity.

To resolve this you must include the energy of the gravitational field in the calculation to get a conserved total energy. The photons are affected by the gravitational field via the expansion dynamics of the universe but there is also an opposite reaction and the gravitational field is affected by the energy and momentum of the photons so that the rate of expansion is slightly affected. As the photons lose energy the gravitational field gains energy at the same rate and energy is conserved.

The simplest model to consider is the Friedman-Robertson-Walker cosmology with flat space and a uniform matter and radiation density. The equation describing the total energy E *in an expanding volume of space* $V = a(t)^3$ *is*

$$E = Mc^2 + \frac{P}{a} - \frac{3a}{8\pi G}\left(\frac{da}{dt}\right)^2 = 0$$

Where M *is the fixed mass of matter in the volume , and* $\frac{P}{a}$ *is the energy in the cosmic background radiation with* P *constant so that the total energy is decreasing as* $a(t)$ *increase. The gravitational energy in the same volume is negative and given by the third term of the equation. If there were no radiation this term would be a constant with the expansion rate decreasing as the volume increases. When radiation is present this is modified so that the gravitational energy changes keeping the total energy constant as the energy in the radiation decreases.*

(2) <u>Dark Energy</u>: The status of energy conservation in an expanding universe is much worse when you include dark energy. Assuming that dark energy is accurately described by a cosmological constant (as observation confirms) we find that the amount of dark energy in a *fixed* volume of space is constant. This means that as the universe expands new energy is being created out of nowhere. Dark energy currently accounts for about three quarters of the total energy in the universe and the amount increases ever more rapidly as the universe expands. Clearly this is a strong violation of energy conservation

The solution to the problem when dark energy is included in the cosmological model is similar to the explanation for the cosmic radiation, except this time the amount of energy in an expanding volume is increasing. This is compensated for by a decreasing gravitational energy in the volume and since the gravitational energy is negative it can continue decreasing without limit as the dark energy increases.

This can be seen by including an extra term in the equation for a homogeneous universe to add dark energy

$$E = Mc^2 + \frac{P}{a} + \frac{\Lambda}{8\pi G}a^3 - \frac{3a}{8\pi G}\left(\frac{da}{dt}\right)^2 = 0$$

Whereas the cosmic radiation has only a slight effect on the expansion rate that cannot be observed, the effect of dark energy is more dramatic causing the rate of expansion to increase rather than *decrease. This has been observed by measuring the apparent brightness of supernovae from the early universe to the present.*

(3) <u>Background Gravity</u>: By Noether's theorem [1] energy is only conserved for a system that has no time dependence. When gravity is present this is only true in special cases where the gravitational field has a time-like Killing vector. We can see this because energy conservation is described by the equation $T^{\mu\nu}{}_{;\mu} = 0$. I.e. the covariant divergence of the symmetric energy-momentum stress tensor $T^{\mu\nu}$ is zero. In special relativity the covariant divergence reduces to ordinary divergence and this equation can be integrated to give a conservation law but in general relativity you need the divergence of a vector field to be zero in order to get an exact conservation law. This means that energy conservation in general relativity is only approximate except in special cases when the gravitational field admits a time-like Killing vector K_μ defined by the equation

$$K_{\mu;\nu} + K_{\nu;\mu} = 0$$

The stress tensor can then be contracted over the Killing vector to construct a conserved energy current

$$J^\mu = K_\nu T^{\mu\nu} \Rightarrow J^\mu{}_{;\mu} = K_{\nu;\mu} T^{\mu\nu} + K_\nu T^{\mu\nu}{}_{;u} = 0$$

These equations describe the situation for matter moving in a predetermined background gravitational field. This is unphysical because it does not take into account the changes in the gravitational field in reaction to the matter. If you consider the whole system of equations including the dynamics of the gravitational field in response to the energy and momentum of the matter then the complete system is independent of any time co-ordinate. This is just part of general covariance. This means that Noether's theorem applies for the general case and a conservation law can always be derived. As well as contributions from the matter field the energy current will include terms corresponding to the energy of the gravitational field.

To implement Noether's theorem we need to consider an infinitesimal change of co-ordinates given by $\delta x^\mu = \epsilon \xi^\mu$ where ϵ is vanishingly small and ξ^μ is any time-like vector field called the transport vector. Using the Einstein-Hilbert action for general relativity which is invariant under any co-ordinate change we can plug the change into Noether's theorem to derive the corresponding conserved current. After a considerable amount of tensor manipulation the answer resolves to [2]

$$J^\mu(\xi^\nu) = \xi^\nu T^\mu{}_\nu - \frac{1}{8\pi G}\left(\xi^\nu G^\mu{}_\nu + \xi^\mu \Lambda\right) + K^{\mu\nu}{}_{;\nu}$$

$$K^{\mu\nu} = \frac{1}{16\pi G}\left(\xi^{\nu;\mu} - \xi^{\mu;\nu}\right)$$

It is easy to confirm that the divergence of this current is zero because if the Einstein Field Equations are applied

$$G^{\mu\nu} + g^{\mu\nu}\Lambda = 8\pi G\, T^{\mu\nu}$$

Then the current reduces to

$$J^{\mu} = K^{\mu\nu}{}_{;\nu}$$

The divergence is then zero because $K^{\mu\nu}$ is an anti-symmetric tensor. This gives a valid energy conservation law for the fully general case in general relativity. When ξ^{μ} is a space-like vector rather than a time-like vector this equation gives a conserved current for momentum rather than energy.

(4) <u>Non-covariance</u>: One of the main problem's with energy conservation laws in general relativity arises from the fact that the Lagrangian includes second derivatives of the metric rather than just the metric and its first derivatives as required by Noether's theorem. To get round this problem physicists have modified the action principle to remove these troublesome second derivatives without affecting the equations of motion but this cannot be done in a covariant way. From this they derive a pseudo-tensor for the energy-momentum current which is co-ordinate dependent. This makes the resulting conservation law unphysical. Furthermore there are several different ways to derive the pseudo-tensor each giving different results.

There are three equally valid ways to refute this particular objection which I shall label (a), (b) and (c)

(a) It is in fact possible to generalise Noether's theorem to work for higher derivatives. This is what I did when I derived the energy current given above [2]. *This does have a cost. The energy current then includes second derivatives of the metric as well as first. This is unusual in an expression for energy, but it is not a problem conceptually. In this formulation the energy current takes a unique natural form.*

(b) Although pseudo-tensor formulations of energy conservation are rather ugly, there is no sense in which they are unphysical just because they are not expressed in covariant form. They are co-ordinate dependent but co-ordinates can be constructed in real space and time by any arbitrary convention and then the pseudo-tensors work perfectly well. The have been used with complete success in physical problems such as calculating the energy carried by gravitational waves. These results are in perfect agreement with experimental observations where energy is radiated by binary pulsars.

Several different formulations of pseudo-tensors for energy-momentum have been formulated, e.g. by Einstein [3], *Dirac* [4], *Landau and Lifshitz* [5] *and Weinberg* [6]. *There is no conflict between them but some are better than others for specific situations.*

(c) A third way to tackle the problem of second derivatives is to use some of the newer formulations of general relativity in terms of non-metric variables where second derivatives of field variables do not appear in the action.

(5) <u>Non-locality</u>: An important limitation of energy conservation laws in general relativity is that they do not work globally except in specific cases. For example, you can work out the energy in a closed universe but you just get the answer zero. Another special case that can be accounted for is an isolated system where space-time is asymptotically flat as you go towards special infinity (the cosmological constant must therefore be zero) In this case you can calculate the ADM mass/energy of the system and show that it changes by an amount

consistent with the energy radiated away by gravitational radiation. In all other cases it makes no sense to talk about total energy.

Obviously it does not make sense to talk about total energy in an infinite cosmology where total energy is unlimited, but that is not the way global conservation laws are treated. Locally we can express a conservation law in the form of a current J^{μ} whose divergence vanishes $J^{\mu}{}_{;\mu} = 0$. To see how the conservation law works globally we need to convert this to an integral form. It is not necessary to integrate over the whole universe to get a total constant energy. We can integrate over some finite volume V and consider the flux of energy across the bounding surface S. The flux must be equal to the rate of change of energy inside the volume if energy is conserved.

To derive the integral form we must first convert the covariant divergence to an ordinary divergence using this identity

$$J^{\mu}{}_{;\mu}\sqrt{-g} = \left(J^{\mu}\sqrt{-g}\right)_{,\mu}$$

Where g is the determinant of the metric tensor.

The simplest way to see how the integral form works is to use ordinary co-ordinates (t,x,y,z) Then we can write

$$J^{\mu}{}_{;\mu} = 0 \ \Rightarrow \frac{d}{dt}\left(J^{0}\sqrt{-g}\right) + \nabla\cdot\left(\boldsymbol{J}\sqrt{-g}\right) = 0$$

Integrating over the volume and using the divergence theorem

$$E = \iiint_{V} J^{0}\sqrt{-g}\, dV$$

$$\Rightarrow \frac{dE}{dt} = -\iiint_{V} \nabla\cdot\left(\boldsymbol{J}\sqrt{-g}\right) dV = -\oiint_{S} \left(\boldsymbol{J}\sqrt{-g}\right)\cdot d\boldsymbol{S} = -\Phi$$

So the rate of change of energy E in the volume V is the negative of the flux Φ of the current through the bounding surface S. This expresses the global conservation law in integral form.

There is one further integration that can be performed for the energy itself when we apply the field equations to get

$$J^{\mu} = K^{\mu\nu}{}_{;\nu}$$

Where $K^{\mu\nu}$ is antisymmetric as above. Once again this covariant divergence can be changed to an ordinary divergence

$$K^{\mu\nu}{}_{;\nu}\sqrt{-g} = \left(K^{\mu\nu}\sqrt{-g}\right)_{,\nu}$$

(Note that this only works for antisymmetric tensors which is why the zero divergence of the symmetric energy-momentum stress tensor cannot be integrated)

Taking the time components only for the ν index leaves three components of a vector $\boldsymbol{K}$

$$J^0\sqrt{-g} = \nabla \cdot \left(\boldsymbol{K}\sqrt{-g}\right)$$

Which means the expression for energy can also be integrated to a surface term

$$E = \iiint_V J^0\sqrt{-g}\, dV = \iiint_V \nabla \cdot \left(\boldsymbol{K}\sqrt{-g}\right) dV = \oiint_S \left(\boldsymbol{K}\sqrt{-g}\right) \cdot d\boldsymbol{S}$$

This tells us that we can calculate the energy in a region from knowledge of the gravitational field on a boundary surface. One further consequence is that if we calculate total energy in a spatially closed universe so that there is no boundary, the answer must be zero

(6) <u>Triviality</u>: The problem when you try to include the gravitational energy in addition to the energy in the matter fields using Noether's theorem is that the energy current becomes zero. It boils down to a trivial, useless identity. You see this immediately from the fact that the energy-momentum stress tensor is given by varying the metric in the action for the matter fields. If you include the curvature tensor for the gravitational action you will just get the field equations from the least action principle which sum to zero.

The pseudotensor expressions for energy and momentum can be written as the sum of two terms the first of which is zero when the field equations hold and the second whose divergence vanishes as a trivial identity. This is also true for the covariant energy current expression given above and in this case the second term can even be made zero by choosing the transport vector to be given by a gradiant expression

$$\xi^\mu = g^{\mu\nu}\frac{\partial\phi}{\partial x^\nu} \Rightarrow K^{\mu\nu} = \frac{1}{16\pi G}\left(\xi^{\nu;\mu} - \xi^{\mu;\nu}\right) = \frac{1}{16\pi G}\left(\phi^{;\nu\mu} - \phi^{;\mu\nu}\right) = 0$$

Here the field ϕ can be any scalar field and in the case of energy it can be conveniently taken to coincide with the time variable $\phi(x^\mu) = t \Rightarrow \xi_\mu = (1,0,0,0)$ to show that the energy current is always zero. Even when the transport vector is not selected in this way we are free to redefine the energy current by adding any terms whose divergence is identically zero so this last term is superfluous. You might as well define the energy current in the gravitational field to be the negative of the energy in the matter field but that serves no useful purpose.

This apparently sophisticated chain of argument has fooled a lot of experts but it is nevertheless a fallacy just as much as the earlier claims. It contains a series of not too subtle errors.

Firstly the energy current is not derived from Noether's theorem using changes to the action when varying the metric tensor. For the matter fields this does give the energy-momentum stress tensor and it is a general property of gauge fields that this is directly related to the term in the energy current for the matter fields. However, you cannot extend this to the gravitational part of the action.

The correct application of Noether's theorem to calculate the expression for the conserved current involves variations over all the matter fields as well as the metric fields and is completely different in structure.

It is true that if we are free to choose the transport vector ξ^μ *in a way dependent on the gravitational metric then we can make the energy current zero as above. However, the correct thing to do is to fix the vector in contravariant form* $\xi^\mu = (1,0,0,0)$*, not in covariant form as above. This is important to ensure that space-like surfaces at constant time are not transported to surfaces that may not be space-like.*

It is true that the energy current is zero in some special cases. A good example of this is the FRW metric for a homogeneous cosmology. This does not mean that energy conservation reduces to a trivial identity. The energy equation for this case is sufficient to determine the evolution of space-time. As we have seen earlier the total energy is also zero in any closed universe but a good example where is does not work out to zero is the case of ADM energy in an asymptotically flat spacetime.

The statement that energy conservation reduced to a useless identity goes back to the early days of relativity [7]. *The fact that the energy current splits into two terms as described above does not make it trivial. It has been shown that energy is carried by gravitational waves in a meaningful way. It is important to recognise that energy takes a form that is a sum over contributions from different types of field including matter fields, radiation fields, gravity and dark energy. When you use the field equations to simplify the expression you change it into a different form that depends only on the gravitational field. This can be integrated and is identically conserved but this does not make energy conservation trivial*

(7) <u>Locality</u>: If you attempt to calculate a local energy-momentum current including gravity using an inertial reference frame you will always find that it is zero. Even worse, if you consider flat empty Minkowski space and work out the current in accelerating or rotating reference frames you find that the energy current is non-zero! This clearly shows that no local formulation of gravitational energy can be valid.

These things are true using either the pseudo-tensor formulations or the covariant energy current formula given earlier. However, rather than being faults they are in fact salient features of gravitational energy and momentum.

In an inertial reference frame a local observer sees no gravitational effect assuming tidal forces are negligible. It is therefore normal and correct that the local expressions for energy and momentum currents in an inertial reference frame give the answer zero. Likewise, in an accelerated reference frame an observer would experience forces similar to those produced by gravity. This is a direct consequence of the equivalence principle that forms the foundation of general relativity. If we are happy to associate potential energy with a gravitational field on Earth then by the equivalence principle we should be equally happy to associate energy with the apparent gravitational field in a non-inertial reference frame, even if space-time is flat. The pseudo-tensor and covariant current formulations for energy momentum in general relativity are perfectly valid local expressions and they reflect the local nature of gravitational energy correctly.

(8) **<u>No 4-vector</u>: A good lesson we learnt from special relativity is that the energy of a particle is one component of a 4-vector where it is joined with the three vector components of momentum. In general relativity 4-vectors are local concepts. It is invalid to add 4-vectors at different locations in space so any attempt to form a global description of energy-momentum by integrating up over space is bound to fail. In the case of field theory the situation is worse because in special relativity the energy-momentum is embedded in the energy-momentum stress tensor which forms a current for the energy and momentum. In general relativity you cannot integrate this quantity over space.**

The only exception is the case of asymptotically flat space-time where in the limit as you go to infinity you find a flat space-time. In this case it turns out that you can assign a valid 4-vector for energy-momentum because it is valid at infinity. Unfortunately due to dark energy the real universe cannot be asymptotically flat and this case is only valid as an approximation for isolated systems on sufficiently small scales, such as stars and galaxies.

The simplest demonstration that this argument is faulty is the fact that we can develop an exact integral form for global energy conservation as shown above. Nevertheless we still need to explain directly why it works and why the 4-vector argument fails.

A 4-vector is a representation of the Lorentz group or Poincare group that describes the global symmetry in special relativity. In general relativity these groups are only valid in local reference frames. Globally the symmetry is described by diffeomorphism invariance and there is no representation of the full group that acts on 4-vectors. It should therefore be unsurprising that energy and momentum are described by a more complex object globally.

In the covariant current formulation the energy current is given by an expression with an explicit linear dependency on the transport vector field that generates diffeomorphisms. This can be regarded as forming an infinite dimensional vector space that is acted on by a representation of the diffeomorphism group for general relativity in the same way as the energy-momenetum 4-vector and energy-momentum-stress tensor are acted on by representations of the Poincare group in special relativity. But we can go further by integrating over a volume of a spatial hyper-surface to get global expressions for energy or momenta. These do not form a 4-vector except in the special case of asymptotically flat space-time where a global Poincare symmetry is valid at infinity. The reason we are able to do the integration despite the given argument to the contrary, is that the spatial hypersurface has a normal 4-vector which can be contracted locally with the local current 4-vector to form a scalar for energy density with an explicit dependency on the transport vector field. Unlike vectors and tensors these scalars can be integrated over space in general relativity.

(9) **<u>Infinite Ambiguity</u>: A big issue with energy conservation in general relativity is that there are an infinite number of different answers. With the pseudo-tensor methods you get a different answer depending on what co-ordinates you select. On top of that there is a different pseudo-tensor formalism for every physicist who ever considered it and their answers are not all equivalent. Even in the covariant energy current formulation the answer depends on the infinite dimensional field of the transport vector. With such an infinitude of possibilities the energy conservation law loses all meaning.**

Although there are several different pseudo-tensor formlations they all seem to work equally well for energy conservations when applied to specific situations such as gravitational waves. Some methods have advantages over others, for example if the pseudo-tensor is symmetric it can be used to formulate conservation of angular momentum.

In the covariant formulism there is only one method but there is a dependency of the transport vector field. This means there could be potentially an infinite number of independent conservation laws but in practice that appears not to be the case. For the isolated system in asymptotically flat space-time there are ten independent conservation laws corresponding to energy, momentum, angular momentum and initial position of centre of mass. It is likely that there are only ten independent conservation laws in all cases (not including internal gauge charges such as electric charge).

In any case this argument does not seem to invalidate any methodology in practice.

(10) <u>**Black Holes**</u>**: The laws of general relativity must break down at the centre of a black hole where they predict a singularity with infinite density that defies the laws of physics. Furthermore it is not possible to know what is happening inside a black-hole so the total energy in a system that includes black-holes should be unknown.**

These statements are not true. To define energy over a region including a black hole we need a space-like hypersurface that passes through it. It is a feature of ordinary black-holes that the singularity inside is in the future light cone so the hypersurface can easily be chosen to avoid it. Indeed the total energy of a black hole is given by the energy equivalent of its mass which is easily determined from outside.

There remains q question over how to deal with more complex space-times which can have naked singularities or wormholes through to disconnected regions of spacetime. Of course this does not invalidate energy conservation laws for any situations that are known to really exist.

(11) <u>**The Hamiltonian**</u>**: When you work through the Hamiltonian formulation for general relativity you find that the Hamiltonian equation of motion is replaced by a Hamiltonian constraint that fixes its value to zero. The Hamiltonian is used to formulate the quantum version of the theory and represents the energy observable. The Hamiltonian constraint means that we must select only quantum states where the energy is zero. This shows that not only is the energy of a system always zero, but it must be fixed to be zero as a kinematic constraint, not a dynamical result.**

The first thing to notice here is that although energy in a closed universe or a region of a homogeneous universe is indeed zero, this is not the case for other model cosmologies such as an isolated system in an asymptotically flat space-time. If the Hamiltonian constraint says total energy is zero then we must conclude either that there is something wrong with the formulation of the Hamiltonian constraint or that it has been wrongly interpreted.

As for quantum gravity, no complete theory has been found so it would be dangerous to draw too many conclusions from the incomplete versions so far known. Given that energy is not always zero in classical systems it would be a failure of the correspondence principle if it were zero in quantum systems.

(12) **<u>Freedom</u>: Although you might be able to formulate some kind of energy conservation law in general relativity it is complicated and not really a useful concept. In new areas of physics we are free to extend laws from earlier physical systems in any way we find most practical. The easiest way to extend energy conservation to general relativity is just to accept that it does not hold. This may seem surprising but it is the most helpful way to proceed when teaching students of the subject.**

This is a sentiment with which I absolutely disagree. Energy conservation in general relativity may be more complicated to understand than in special relativity but it is still an important subject with non-trivial results. It has practical applications to the theory of gravitational wave generation and propagation for example. On a conceptual level it is also very important and learning the law of energy conservation in general relativity is an exercise that any serious student of the subject should undertake to improve their understanding.

Energy is closely related to the Hamiltonian used in the quantum theory so it is likely to be significant in a proper relativistic formulation of quantum gravity. The holographic nature of energy that allows it to be determined for a system from a boundary surrounding the system may be closely connected to the holographic principle which is also considered an important element of quantum gravity. Much more research could be done to understand energy conservation by exploring in more detail its transformation rules and how it works for formulations of gravity using alternative connection variables instead of a metric etc. The opportunity to explore these areas is being lost because of a lack of understanding among scholars of the subject.

In my view it is a great pity that very few modern general relativity text books treat the conservation laws fully. A window of understanding on the universe has been blacked out.

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