

Syntactic - Semantic Axiomatic Theories in Mathematics

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Dedicated to Marie-Louise Nykamp

Abstract

A more careful consideration of the recently introduced "Grossone Theory" of Yaroslav Sergeev, [1], leads to a considerable enlargement of what can constitute possible legitimate mathematical theories by the introduction here of what we may call the *Syntactic - Semantic Axiomatic Theories in Mathematics*. The usual theories of mathematics, ever since the ancient times of Euclid, are in fact axiomatic, [1,2], which means that they are *syntactic* logical consequences of certain assumed axioms. In these usual mathematical theories *semantics* can only play an *indirect* role which is restricted to the inspiration and motivation that may lead to the formulation of axioms, definitions, and of the proofs of theorems. In a significant contradistinction to that, and as manifestly inspired and motivated by the mentioned Grossone Theory, here a *direct* involvement of *semantics* in the construction of axiomatic mathematical theories is presented, an involvement which gives semantics the possibility to act explicitly, effectively, and altogether directly upon the usual syntactic process of constructing the logical consequences of axioms. Two immediate objections to what appears to be an unprecedented and massive expansion of what may

now become legitimate mathematical theories given by the *syntactic - semantic axiomatic theories* introduced here can be the following : the mentioned direct role of semantics may, willingly or not, introduce in mathematical theories one, or both of the "eternal taboo-s" of *inconsistency* and *self-reference*. Fortunately however, such concerns can be alleviated due to recent developments in both inconsistent and self-referential mathematics, [1,2]. Grateful recognition is acknowledged here for long and most useful ongoing related discussions with Yaroslav Sergeev.

“There have been four sorts of ages in the world’s history. There have been ages when everybody thought they knew everything, ages when nobody thought they knew anything, ages when clever people thought they knew much and stupid people thought they knew little, and ages when stupid people thought they knew much and clever people thought they knew little. The first sort of age is one of stability, the second of slow decay, the third of progress, and the fourth of disaster.

Bertrand Russel, "On modern uncertainty" (20 July 1932) in *Mortals and Others*, p. 103-104.

“History is written with the feet ...”

Ex-Chairman Mao, of the Long March fame ...

“Of all things, good sense is the most fairly distributed : everyone thinks he is so well supplied with it that even those who are the hardest to satisfy in every other respect never desire more

of it than they already have.” :-) :-) :-)

R Descartes, Discourse de la Méthode

“Creativity often consists of finding hidden assumptions. And removing those assumptions can open up a new set of possibilities ...”

Henry R Sturman

“Science is not done scientifically, since it is mostly done by non-scientists ...”

Anonymous

“Science is nowadays not done scientifically, since it is mostly done by ... scientists ...”

Anonymous

“Physics is too important to be left only to physicists ...”

Anonymous

A “mathematical problem” ?

For quite sometime by now, American mathematicians have decided to hide their date of birth and not to mention it in their own academic CV. Why are they so blatantly against transparency in such an academically related matter ? Can one, therefore, trust American mathematicians, or for that matter, any other professional who behaves like that ?

Amusingly, Hollywood actors and actresses have their birth date easily available on Wikipedia. On the other hand, Hollywood movies have also for long by now been hiding the date of their production ...

A bemused non-American mathematician

1. Axiomatic Mathematical Theories as mere Models

Ever since Euclid axiomatized Geometry more than two millennia ago, there has been a widespread and strong tacit tendency, and not only among mathematicians, to *identify* the respective axiomatic theory with Geometry as such. Therefore the shock about two centuries earlier when non-Euclidean geometries have been discovered. A similar phenomenon happened more than a century back with the Peano Axioms of the natural numbers which were supposed to express all the relevant properties of such numbers. And thus the related shock of the Gödel Incompleteness Theorem in the early 1930s cannot by now be seen as a surprise.

The moral, of course, is that, on one hand, we may have a concept like "geometry" or "number", for instance, while on the other hand, we can have one or another axiomatic mathematical theory which aims to describe it. And the *gap* between these two sides may be hard to bridge, let alone eliminate. The reasons for that may be quite a few,

indeed. Here we mention some of the better known among them.

First, let us have a brief look at what is in fact an *axiomatic mathematical theory*, or more generally, an *axiomatic system*.

One starts such a theory with a setup of a *formal deductive system*. Namely, let A be an *alphabet* which can be given by any nonvoid finite or infinite set. Then a procedure is given according to which one can in a finite number of steps effectively construct - by using the symbols in A - a set \mathcal{F} of *well formed formulas*, or in short, *wff*-s. Next, one chooses a nonvoid set \mathcal{R} of *logical deduction rules* which operate as follows

$$(1.1) \quad \mathcal{F} \supseteq P \xrightarrow{\mathcal{R}} Q \subseteq \mathcal{F}$$

that is, from any set P of wff-s which are the *premises*, it leads to a corresponding set Q of wff-s which are all the *consequences* of P . It will be convenient to assume that, for every set of well formed formulas $P \subseteq \mathcal{F}$, we have

$$(1.2) \quad P \subseteq \mathcal{R}(P) = \mathcal{R}(\mathcal{R}(P))$$

in other words, the premises P are supposed to be among the consequences $\mathcal{R}(P)$, and in addition, these consequences $\mathcal{R}(P)$ contain all the possible consequences of P , in other words, the iteration of \mathcal{R} does not produce further consequences of P .

Clearly, condition (1.2) does not lead to a loss of generality regarding \mathcal{R} in (1.1). Indeed, if the relation

$$\forall P \subseteq \mathcal{F} : P \subseteq \mathcal{R}(P)$$

is not satisfied, then this relation will obviously be satisfied by the modification of \mathcal{R} given by $\mathcal{R}'(P) = P \cup \mathcal{R}(P)$. Also, if the relation

$$\forall P \subseteq \mathcal{F} : \mathcal{R}(\mathcal{R}(P)) = \mathcal{R}(P)$$

is not satisfied, then this relation will obviously be satisfied by the modification of \mathcal{R} given by

$$\mathcal{R}''(P) = \mathcal{R}(P) \cup \mathcal{R}(\mathcal{R}(P)) \cup \mathcal{R}(\mathcal{R}(\mathcal{R}(P))) \cup \dots$$

And now come the *axioms* which can be any nonvoid subset $\mathcal{A} \subseteq \mathcal{F}$ of *wff*-s.

Once the above is established, the respective *axiomatic theory* follows easily as being the *smallest* subset $\mathcal{T} \subseteq \mathcal{F}$ with the properties

$$(1.3) \quad \mathcal{A} \subseteq \mathcal{T}$$

$$(1.4) \quad \mathcal{T} \supseteq P \xrightarrow{\mathcal{R}} Q \subseteq \mathcal{T}$$

in which case the *wff*-s in \mathcal{T} are called the *theorems* of the axiomatic system \mathcal{A} . In view of (1.3), clearly, all axioms in \mathcal{A} are also theorems.

Now an essential fact is that the set \mathcal{T} of theorems depends not only on the axioms in \mathcal{A} , but also on the logical deduction rules \mathcal{R} , and prior to that, on the set \mathcal{F} of well formed formulas. Consequently, it is appropriate to write

$$(1.5) \quad \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A}) \quad \text{or more simply} \quad \mathcal{T}_{\mathcal{R}}(\mathcal{A})$$

for the set \mathcal{T} of theorems.

Here are some of the relevant questions which can arise regarding such axiomatic systems :

- are the axioms in \mathcal{A} *independent* ?
- are the axioms in \mathcal{A} *consistent* ?
- are the axioms in \mathcal{A} *complete* ?

Independence means that for no axiom $\alpha \in \mathcal{A}$, do we have $\mathcal{T}_{\mathcal{R}}(\mathcal{A}) = \mathcal{T}_{\mathcal{R}}(\mathcal{B})$, where $\mathcal{B} = \mathcal{A} \setminus \{\alpha\}$. In other words, the axioms in \mathcal{A} are minimal in order to obtain the theorems in $\mathcal{T}_{\mathcal{R}}(\mathcal{A})$. This condition can be formulated equivalently, but more simply and sharply, by saying that for no axiom $\alpha \in \mathcal{A}$, do we have $\alpha \in \mathcal{T}_{\mathcal{R}}(\mathcal{B})$, where $\mathcal{B} = \mathcal{A} \setminus \{\alpha\}$.

As for consistency, it means that there is no theorem $\tau \in \mathcal{T}_{\mathcal{R}}(\mathcal{A})$, such that for its negation $\text{non } \tau$, we have $\text{non } \tau \in \mathcal{T}_{\mathcal{R}}(\mathcal{A})$.

Completeness, in one possible formulation, means that, given any additional axiom $\beta \in \mathcal{F} \setminus \mathcal{A}$ which is independent from \mathcal{A} , the axiom system $\mathcal{B} = \mathcal{A} \cup \{\beta\}$ is inconsistent.

It is obvious, therefore, that in setting up axiomatic systems, there is a lot of freedom in choosing the alphabet A , the well formed formulas \mathcal{F} , the deduction rules \mathcal{R} and the axioms \mathcal{A} , all of which influence the resulting theorems \mathcal{T} . However, such a freedom is not necessarily a complete blessing when it comes to express all the possible relevant properties of concepts such as "geometry", "numbers", and so on. Indeed, each particular such choice may not only miss on certain relevant properties, but may actually introduce some strange and unintended ones as well.

In this regard, in modern times, it was the philosophy of neo-positivism, or the so called third positivism, which in the early 20th century brought to attention the fact that the very structure of language can significantly influence thinking and the results of thought, and in particular, can lead to pseudo-problems. Not much later, in linguistics, a similar idea arose with the Sapir-Whorf Hypothesis about the relativity of language.

A corresponding recognition in Mathematics, as mentioned, started to emerge in the early 1800s, even if tentatively, with the non-Euclidean geometries, and was later confirmed by further modern developments of various axiomatic mathematical theories, a most important moment in this regard being Gödel's Incompleteness Theorem, in the early 1930s.

In conclusion, it is appropriate to realize - even if in the day to day activity of the so called "working mathematicians" it may still be disregarded - that an axiomatic mathematical theory most likely fails to express all of the properties of the domain of mathematics which it is supposed to model, and in fact, may even introduce inappropriate properties. And that can already happen with such basic mathemat-

ical concepts like "geometry" and "numbers".

It will be convenient in the sequel to consider the following. Given a theorem $\tau \in \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$, we denote by

$$(1.6) \quad \mathcal{T}_{\mathcal{F},\mathcal{R},\mathcal{A}}(\tau)$$

which is the set of all elements $\sigma \in \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$ with the property :

$$(1.7) \quad \begin{aligned} &\exists B \subseteq \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A}) : \\ &*) \sigma \in \mathcal{R}(B \cup \{\tau\}) \\ &**) \sigma \notin \mathcal{R}(B) \end{aligned}$$

and we call the set $\mathcal{T}_{\mathcal{F},\mathcal{R},\mathcal{A}}(\tau)$ the *implications* in the axiomatic system \mathcal{A} of τ .

Example : the Peano Axioms versus the Digital Computers

Let us consider the Peano Axioms, denoted by \mathcal{P} , related to the natural integer numbers \mathbb{N} , where we take the usual view, and not that in the "Grossone Theory". Let us further consider the additional axiom which, obviously, is not in \mathcal{P} , namely

$$(1.8) \quad \exists M \gg 1 : M + 1 \leq M$$

Clearly, every electronic digital computer - although hardly ever noticed consciously - functions according to the obviously *inconsistent* set of axioms, denoted by \mathcal{A} , and given by the Peano Axiom, plus axiom (1.8), thus briefly, $\mathcal{A} = \mathcal{P} \cup \{(1.8)\}$.

Now, among the consequences of (1.8) are the following theorems in $\mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$, namely

$$(1.9) \quad M + 1 \leq M, \quad M + 2 \leq M, \quad \dots$$

If we take $\tau, \sigma \in \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$ as the respective above theorems $M + 1 \leq M$ and $M + 2 \leq M$, then, see (1.6), (1.7)

$$(1.10) \quad \sigma \in \mathcal{T}_{\mathcal{F},\mathcal{R},\mathcal{A}}(\tau)$$

Indeed, let $B = \mathcal{A} = \mathcal{P} \cup \{(1.8)\} \subseteq \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$, then obviously $\sigma \in \mathcal{R}(B)$. On the other hand, it is equally obvious that $\sigma \notin \mathcal{R}(B \setminus \{\tau\})$, since $B \setminus \{\tau\} = \mathcal{P}$.

2. A Brief Review of the Theory of : Grossone = ①

Recently, a remarkable avenue - called the "Grossone Theory" - has been proposed and developed in [1,2] for an effective computation with infinitesimal and infinitely large numbers, a computation which - as a first in the literature - is implementable on usual digital computers.

The presentation in [1,2] is based on three general *postulates* which are supposed to set up the framework for a usual mathematical type *axiom* that has three components.

Here it is important to point out the unprecedented novelty in modern Mathematics of this approach, in which *semantical* type postulates determine the scope of action of usual *syntactic* type axioms. And it is precisely due to this feature of the "Grossone Theory" introduced in [1,2] that its more rigorous foundational status has not yet been obtained, [3].

This is precisely why the present paper suggests a novel approach given by the *syntactic-semantic axiomatic* method which is introduced in the sequel. Such an approach may contribute to the rigorous foundation of the "Grossone Theory". And considerably beyond that, it may open the way for a far larger class of valid mathematical theories than the usual axiomatic ones employed ever since Euclid.

And now, we briefly recall the presentation of the "Grossone Theory" in [1,2], which starts with the following three *postulates* :

P1. "We postulate the existence of infinite and infinitesimal objects but accept that human beings and machines are able to execute only

a finite number of operations.”

P2. ”We shall not tell what are the mathematical objects we deal with. Instead, we shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.”

As a consequence of P2, an essential distinction is made in [1,2] between ”numbers” which are supposed to be the mathematical objects about which one only talks *indirectly*, namely, through corresponding ”numerals”. And it is pointed out that, usually, this distinction fails to be made in Mathematics where, instead, ”numerals” are identified with ”numbers”, although the various related axiomatic mathematical theories can only talk about ”numerals”, and not ”numbers” as well. Indeed, as it is clear ever since the Gödel Incompleteness Theorem, the Peano Axioms, for instance, do not give the natural integer numbers, but only a representation of them by ”numerals”, whose infinite set is denoted by \mathbb{N} .

P3. ”We adopt the principle : ’The part is less than the whole’, and apply it to all numbers, be they finite, infinite, or infinitesimal, as well as to all sets and processes, be they finite or infinite.”

The development of the consequences of these three postulates in [1,2] starts with section 3, where the “numeral” $\textcircled{1}$ - called “grossone” - is introduced as ”the infinite unit of measure” which is declared to be, or rather, to represent ”the number of elements of the set of natural numbers, or rather ”numerals”, \mathbb{N} ”.

The following step is the introduction of the so called ”Infinite Unit Axiom”, or in short, IUA, which consists of three parts, namely :

Infinity : Any finite natural number n is less than the grossone $\textcircled{1}$, that is, $n < \textcircled{1}$.

Identity : The following relations hold :

$$0.\textcircled{1} = \textcircled{1}.0 = 0, \quad \textcircled{1} - \textcircled{1} = 0, \quad \textcircled{1}/\textcircled{1} = 1$$

$$\mathbb{1}^0 = 1, \quad 1^{\mathbb{1}} = 1, \quad 0^{\mathbb{1}} = 0$$

Divisibility : For any finite natural number n , the infinite sets $\mathbb{N}_{k,n} = \{k, k + n, k + 2n, k + 3n, \dots\}$, $1 \leq k \leq n$ have the same number of elements given by the numeral $\mathbb{1}/n$.

Obviously, the numeral $\mathbb{1} / n$ above is considered to be infinite, and therefore, its inverse $n / \mathbb{1}$ is considered to be infinitesimal. Furthermore, $\mathbb{1} / n$ is, in fact, considered to be an infinite integer.

3. How the Syntactic - Semantic Axiomatic Theories in Mathematics are Set Up

As mentioned, an essential feature - and peculiarity - of the "Grossone Theory" is the thorough *interplay* in its axiomatic development between the three Postulates 1,2 and 3, and on the other hand, the three components of the "Infinite Unit Axiom". And manifestly, the respective Postulates have a clearly *distinct* nature from the mentioned axiom, least of all due to the rather general, informal formulations of the former. Indeed, the postulates are formulated in what me justifiably seen as a kind of "philosophical language", while on the contrary, the three components of the "Infinite Unit Axiom" are expressed in simple mathematical terms.

To put it briefly, the postulates are expressed, and will be acting in the setting up of the "Grossone Theory" as *semantic* data. And in fact, they simply cannot act in any other manner, given their formulations which do not belong to Mathematics. In this way, it is only the three components of the "Infinite Unit Axiom" which can, and will act as *syntactic* data, that is, as is customary in usual axiomatic mathematical theories.

It is, therefore, precisely that distinction between the postulates and the axiom which bring up the issue of a rigorous foundational status of the "Grossone Theory". And this is, then, the issue which inspired and motivated the introduction in this paper of the *syntactic - seman-*

tic axiomatic theories in Mathematics.

For illustration, and as the source of the idea of the general syntactic - semantic axiomatic approach to mathematical theoreis introduced in this paper, let us recall a few examples of the way the semantically formulated postulates act in the "Grossone Theory", [1,2].

Needless to say, among the three above postulates, the more difficult to have its action described in any somewhat more satisfactory manner is P1. Let us cite from [2, p. 101] the following section :

"Let us consider, for example, the operation of constructing the successor element widely used in number and set theories. In traditional Mathematics, the question of whether this operation can be executed is not taken into consideration; it is supposed that it is always possible to execute the operation $k = n + 1$ starting from any integer n . Thus there is not any distinction between the existence of the number k and the possibility to execute the operation $n+1$ and to express its result, i.e., to have a numeral that can express k .

Postulates 1 and 2 emphasize this distinction and tell us that :

i) in order to execute the operation it is necessary to have a numeral system allowing one to express both numbers n and k ;

ii) for any numerical system there always exists a number k that cannot be expressed in it."

As for the action of P3, let us cite from [2, p.101] this section :

"Due to this declared applied statement, (our note : of Postulate 3), it becomes clear that the subject of this paper is outside Cantor's approach and, as a consequence,

outside of nonstandard Analysis of Robinson. Such concepts as bijection, numerable and continuum sets, cardinal and ordinal numbers cannot be used in this paper because they belong to a theory working with different assumptions. However, the approach used here does not contradict Cantor and Robinson. It can be viewed just as a stronger lens of a mathematical microscope that allows one to distinguish more objects and work with them.”

An immediate, clear and simple consequence of P3, [1,2], is the fact that $\textcircled{1}$ being a ”numeral”, also $\textcircled{1} - 1$ is a ”numeral”, and furthermore, $\textcircled{1} - 1 < \textcircled{1}$. This is, of course, unlike with the usual infinity ∞ , for which it is assumed that $\infty - 1 = \infty$. It is also different from the case of the Cantor’s cardinal numbers, where for every infinite cardinal c , we have $c - 1 = c$.

Before going further, let us note that modern Mathematics, and specifically the branch of Mathematical Logic called Model Theory, [7], is fully aware of the *content* of the above Postulate 2. In fact, this postulate is simply one of the *conclusions* of the modern developments in the study of the axiomatic method in Mathematics. Consequently, in itself, it does not act upon the various axiomatic mathematical theories. Certainly, the awareness, and even more so the relevance of this content was already dramatically underlined back in the early 1930s, and before the emergence of Model Theory, by Gödel’s Incompleteness Theorem regarding, among others, the Peano Axioms.

Finally, let us turn to the introduction of the general *syntactic - semantic axiomatic method* in mathematical theories which is the main object of this paper. For that purpose, we shall use the notations in section 1.

We start, therefore, with any nonvoid set A which serves as the *alphabet* of a given syntactic - semantic axiomatic mathematical theory under consideration. Further, according to a specified procedure, we construct a nonvoid set F of *well formed formulas*, each of which is obtained effectively by a finite number of steps from the alphabet A . Then, we choose a nonvoid set R of *logical deduction rules* which op-

erate accordint to (1.1). Last, we choose a novoid set $\mathcal{A} \subseteq \mathcal{F}$ of *axioms*.

And here, and unlike with the usual axiomatic method as presented in section 1, an *additional* stage is introduced. The essence of this additional stage is that due to the novelty of the presence of *semantic* type postulates or other considerations - which we shall jointly denote by \mathcal{S} - the resulting axiomatic theory is *no longer* given by all the theorems in $\mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$, see (1.5). Instead, depending on \mathcal{S} , we have specified a nonvoid subset of so called *\mathcal{S} -valid theorems*, namely

$$(4.1) \quad \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A}) \subseteq \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$$

which has the property that, for every $\tau \in \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A}) \setminus \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A})$, we have, see (1.6)

$$(4.2) \quad \mathcal{T}_{\mathcal{F},\mathcal{R},\mathcal{A}}(\tau) \cap \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A}) = \phi$$

We conclude with

Definition 4.1.

By an *axiomatic mathematical theory* one means a structure

$$(4.3) \quad (A, \mathcal{F}, \mathcal{R}, \mathcal{A}, \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A}))$$

as specified in (1.1) - (1.5).

By a *semantic - syntactic axiomatic mathematical theory* one means a structure

$$(4.4) \quad (A, \mathcal{F}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A}))$$

as specified in (4.1) , (4.2).

Remark 4.1.

Clearly, if in (4.1) we have

$$(4.5) \quad \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A}) = \phi$$

or alternatively, we have

$$(4.6) \quad \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A}) = \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$$

then, in both these cases, condition (4.2) is satisfied by default.

Obviously the case (4.6) can be seen as the usual case of axiomatic mathematical theories corresponding to (4.3).

The case (4.5) corresponds to the trivial situation when the syntactic restrictions imposed by \mathcal{S} are so extremely severe, as not to allow any theorem from $\mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$.

6. The case of the Grossone Theory

Strictly formally, one of the axioms of the Grossone Theory may recall (1.8), namely, the so called **Infinity Axiom**, see section 2

$$(6.1.) \quad n < \textcircled{1}$$

except that in (1.8) the number M is supposed to be a usual natural integer, although quite large, for instance, typically larger than 10^{100} , while in (6.1), the grossone $\textcircled{1}$ is definitely not supposed to be a finite "numeral".

But let us now see how the Grossone Theory is supposed to be put together as a *semantic - syntactic axiomatic mathematical theory*.

One possible way in this regard may be as follows.

We start with the usual Peano Axioms to which we add the three Infinite Unit Axioms, see section 2. Let us denote by \mathcal{A} the set of all these axioms.

Now, as a fundamental novelty of the Grossone Theory, we denote by

\mathcal{S} the three Postulates, see section 2, P1, P2 and P3, which are supposed to impose *semantically*, and *not* syntactically, the *restrictions* on the logical deductions which follow from the axioms \mathcal{A} , as would be the case in usual axiomatic mathematical theories.

Clearly, the mentioned three postulates which make up \mathcal{S} are themselves formulated *semantically*, and *not* syntactically, therefore, they simply *cannot* operate in any of the usual syntactic ways.

And how *does* then \mathcal{S} operate *semantically* ?

Well, in principle, this is very simple indeed. Namely, according to (4.1), see also (4.4), all one has to do is to specify the set

$$(6.2) \quad \mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A})$$

in the given setup of the Grossone Theory, and do so as a *strict subset* of $\mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{A})$.

Here however, the doors open up to the considerable complexities of possible semantical approaches, when compared with the syntactical ones. And to mention one single indication of such complexities, one that is particularly relevant in this case, it suffices to recall the problems faced by automatic translation from one language to another when it comes to deal with the semantics of languages, and not only with their syntax.

In short, so far, in the Grossone Theory it has been considered as *inadmissible* - in view of the *semantical* action of \mathcal{S} - to specify $\mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A})$ in (6.2) as a usual Cantorian set. Furthermore, the considerably more strong restriction is practiced, according to which, it is simply considered to be *meaningless* merely to conceive of $\mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A})$ as being a usual Cantorian set ...

In conclusion, the example of the Grossone Theory points already to a sharp *dichotomy* in the realms of syntactic-semantic axiomatic mathematical theories, namely :

- Syntactic - Semantic Axiomatic Mathematical Theories in which

the respective $\mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A})$, see (4.1), (4.4), (6.2), are considered to be usual Cantorian sets.

- Syntactic - Semantic Axiomatic Mathematical Theories in which - due to the semantic restrictions - it is meaningless to consider the respective $\mathcal{T}_{\mathcal{F},\mathcal{R}}(\mathcal{S}, \mathcal{A})$, see (4.1), (4.4), (6.2), to be usual Cantorian sets.

As mentioned, the Grossone Theory seems so far to belong to the second type of syntactic-semantic axiomatic mathematical theories.

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