Inconsistent countable set in second order ZFC and nonexistence of the strongly inaccessible cardinals. Non consistency results in topology.

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Abstract: In this article we derived an important example of the inconsistent countable set in second order ZFC (ZFC$_2$) with the full second-order semantic. Main results are:

(i) $\neg Con(ZFC_2)$, (ii) let $k$ be an inaccessible cardinal and $H_k$ is a set of all sets having hereditary size less than $k$, then $\neg Con(ZFC + (V = H_k))$.

Keywords: Gödel encoding, Completion of ZFC$_2$, Russell's paradox, $\omega$-model, Henkin semantics, full second-order semantic, strongly inaccessible cardinal

1. Introduction.

Let us remind that accordingly to naive set theory, any definable collection is a set. Let $R$ be the set of all sets that are not members of themselves. If $R$ qualifies as a member of itself, it would contradict its own definition as a set containing all sets that are not members of themselves. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox. In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and the Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory ZFC. "But how do we know that ZFC is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism)"— E. Nelson wrote in his not published paper [1]. However, it is deemed unlikely that even ZFC$_2$ which is a very stronger than ZFC harbors an unsuspected contradiction; it is widely believed that if ZFC$_2$ were inconsistent, that fact would have been uncovered by now. This much is certain — ZFC$_2$ is immune to the classic
paradoxes of naive set theory: Russell's paradox, the Burali-Forti paradox, and Cantor's paradox.

**Remark 1.1.** Note that in this paper we view the second order set theory $\mathit{ZFC}_2$ under the Henkin semantics [2],[3] and under the full second-order semantics [4],[5]. Thus we interpret the wff's of $\mathit{ZFC}_2$ language with the full second-order semantics as required in [4],[5].

**Designation 1.1.** We will denote by $\mathit{ZFC}_2^{\text{Hs}}$ set theory $\mathit{ZFC}_2$ with the Henkin semantics and we will denote by $\mathit{ZFC}_2^{\text{fss}}$ set theory $\mathit{ZFC}_2$ with the full second-order semantics.

**Remark 1.2.** There is no completeness theorem for second-order logic with the full second-order semantics. Nor do the axioms of $\mathit{ZFC}_2^{\text{fss}}$ imply a reflection principle which ensures that if a sentence $\mathcal{Z}$ of second-order set theory is true, then it is true in some (standard or nonstandard) model $M^{\mathit{ZFC}_2^{\text{fss}}}$ of $\mathit{ZFC}_2^{\text{fss}}$ [2]. Let $\mathcal{Z}$ be the conjunction of all the axioms of $\mathit{ZFC}_2^{\text{fss}}$. We assume now that:

$$\mathcal{Z}$$

is true, i.e. $\text{Con}(\mathit{ZFC}_2^{\text{fss}})$. It is known that the existence of a model for $\mathcal{Z}$ requires the existence of strongly inaccessible cardinals, i.e. under $\mathit{ZFC}$ it can be shown that $\kappa$ is a strongly inaccessible if and only if $(H_\kappa, \in)$ is a model of $\mathit{ZFC}_2^{\text{fss}}$. Thus $\neg\text{Con}(\mathit{ZFC}_2^{\text{fss}} + \exists M^{\mathit{ZFC}_2^{\text{fss}}}) \Rightarrow \neg\text{Con}(\mathit{ZFC} + (V = H_\kappa))$. In this paper we prove that $\mathit{ZFC}_2^{\text{fss}}$ is inconsistent. We will start from a simple naive consideration. Let $\mathfrak{I}$ be the countable collection of all sets $X$ such that $\mathit{ZFC}_2^{\text{fss}} \vdash \exists ! X \Psi(X)$, where $\Psi(X)$ is any 1-place open wff i.e.,

$$\forall Y \{ Y \in \mathfrak{I} \leftrightarrow \exists \Psi(\cdot) \exists ! X [\Psi(X) \land Y = X] \}.$$  \hspace{1cm} (1.1)

Let $X \not\in_{\mathit{ZFC}_2^{\text{fss}}} Y$ be a predicate such that $X \not\in_{\mathit{ZFC}_2^{\text{fss}}} Y \leftrightarrow \mathit{ZFC}_2^{\text{fss}} \vdash X \not\in Y$. Let $\mathfrak{R}$ be the countable collection of all sets such that

$$\forall X \left[ X \in \mathfrak{R} \leftrightarrow X \not\in_{\mathit{ZFC}_2^{\text{fss}}} X \right].$$ \hspace{1cm} (1.2)

From (1.2) one obtain

$$\mathfrak{R} \in \mathfrak{R} \leftrightarrow \mathfrak{R} \not\in_{\mathit{ZFC}_2^{\text{fss}}} \mathfrak{R}. $$ \hspace{1cm} (1.3)

But obviously this is a contradiction. However contradiction (1.3) it is not a contradiction inside $\mathit{ZFC}_2^{\text{fss}}$ for the reason that predicate $X \not\in_{\mathit{ZFC}_2^{\text{fss}}} Y$ not is a predicate of $\mathit{ZFC}_2^{\text{fss}}$ and therefore countable collections $\mathfrak{I}$ and $\mathfrak{R}$ not is a sets of $\mathit{ZFC}_2^{\text{fss}}$. Nevertheless by using Gödel encoding the above stated contradiction can be shipped in special consistent completion of $\mathit{ZFC}_2^{\text{fss}}$.

**Remark 1.3.** We note that in order to deduce $\neg\text{Con}(\mathit{ZFC}_2^{\text{Hs}})$ from $\text{Con}(\mathit{ZFC}_2^{\text{Hs}})$ by using Gödel encoding, one needs something more than the consistency of $\mathit{ZFC}_2^{\text{Hs}}$, e.g., that $\mathit{ZFC}_2^{\text{Hs}}$ has an omega-model $M_\omega^{\mathit{ZFC}_2^{\text{Hs}}}$ or an standard model $M^{\mathit{ZFC}_2^{\text{Hs}}}$ i.e., a model in which the integers are the standard integers [6]. To put it another way, why should we believe a statement just because there’s a $\mathit{ZFC}_2^{\text{Hs}}$-proof of it? It’s clear that if $\mathit{ZFC}_2^{\text{Hs}}$...
is inconsistent, then we won't believe $ZFC^H_2$-proofs. What's slightly more subtle is that the mere consistency of $ZFC_2$ isn't quite enough to get us to believe arithmetical theorems of $ZFC^H_2$: we must also believe that these arithmetical theorems are asserting something about the standard naturals. It is "conceivable" that $ZFC^H_2$ might be consistent but that the only nonstandard models $M^H_{Nst}$ it has are those in which the integers are nonstandard, in which case we might not "believe" an arithmetical statement such as "$ZFC^H_2$ is inconsistent" even if there is a $ZFC^H_2$-proof of it.

**Remark 1.4.** However assumption $\exists M^Z_{st}$ is not necessary. Note that in any nonstandard model $M^Z_{Nst}$ of the second-order arithmetic $Z^H_2$, the terms $\bar{0}, S\bar{0} = \bar{1}, S\bar{1} = \bar{2}, \ldots$ comprise the initial segment isomorphic to $M^Z_{st} \subset M^Z_{Nst}$. This initial segment is called the standard cut of the $M^Z_{Nst}$. The order type of any nonstandard model of $M^Z_{Nst}$ is equal to $\mathbb{N} + A \times \mathbb{Z}$ for some linear order $A$ [6],[7]. Thus one can to choose Gödel encoding inside $M^Z_{st}$.

**Remark 1.5.** However there is no any problem as mentioned above in second order set theory $ZFC_2$ with the full second-order semantics because corresponding second order arithmetic $Z^fss_2$ is categorical.

**Remark 1.6.** Note if we view second-order arithmetic $Z_2$ as a theory in first-order predicate calculus. Thus a model $M^Z_2$ of the language of second-order arithmetic $Z_2$ consists of a set $M$ (which forms the range of individual variables) together with a constant 0 (an element of $M$), a function $S$ from $M$ to $M$, two binary operations $+$ and $\times$ on $M$, a binary relation $<$ on $M$, and a collection $D$ of subsets of $M$, which is the range of the set variables. When $D$ is the full powerset of $M$, the model $M^Z_2$ is called a full model. The use of full second-order semantics is equivalent to limiting the models of second-order arithmetic to the full models. In fact, the axioms of second-order arithmetic have only one full model. This follows from the fact that the axioms of Peano arithmetic with the second-order induction axiom have only one model under second-order semantics, i.e. $Z_2$, with the full semantics, is categorical by Dedekind’s argument, so has only one model up to isomorphism. When $M$ is the usual set of natural numbers with its usual operations, $M^{\omega}_2$ is called an $\omega$-model. In this case we may identify the model with $D$, its collection of sets of naturals, because this set is enough to completely determine an $\omega$-model. The unique full omega-model $M^{\omega}_{st}$, which is the usual set of natural numbers with its usual structure and all its subsets, is called the intended or standard model of second-order arithmetic.

Main results are: $\neg \text{Con}(ZFC^H_2 + \exists (\omega\text{-model of } ZFC^H_2)), \neg \text{Con}(ZFC^{fss}_2)$.

2. Derivation inconsistent countable set in $ZFC^H_2 + \exists M^{ZFC^H_2}$. 

**Remark 2.1.** In this section we use second-order arithmetic $Z^H_2$ with first-order semantics. Notice that any standard model $M^Z_{st}$ of second-order arithmetic $Z^H_2$ consists of a set $\mathbb{N}$ of usual natural numbers (which forms the range of individual variables) together with a constant $0$ (an element of $\mathbb{N}$), a function $S$ from $\mathbb{N}$ to $\mathbb{N}$, two binary operations $+$ and $\cdot$ on $\mathbb{N}$, a binary relation $<$ on $\mathbb{N}$, and a collection $D \subseteq 2^\mathbb{N}$ of subsets of $\mathbb{N}$, which is the range of the set variables. Omitting $D$ produces a model of first order Peano arithmetic.

When $D = 2^\mathbb{N}$ is the full powerset of $\mathbb{N}$, the model $M^Z_{st}$ is called a full model. The use of full second-order semantics is equivalent to limiting the models of second-order arithmetic to the full models. In fact, the axioms of second-order arithmetic $Z^fss$ have only one full model. This follows from the fact that the axioms of Peano arithmetic with the second-order induction axiom have only one model under second-order semantics, see section 3.

Let $Th$ be some fixed, but unspecified, consistent formal theory. For later convenience, we assume that the encoding is done in some fixed formal second order theory $S$ and that $Th$ contains $S$. We assume through this paper that formal second order theory $S$ has an $\mathfrak{I}$-model $M^{S}_{\mathfrak{I}}$. The sense in which $S$ is contained in $Th$ is better exemplified than explained: if $S$ is a formal system of a second order arithmetic $Z^H_2$ and $Th$ is, say, $ZFC^H_2$, then $Th$ contains $S$ in the sense that there is a well-known embedding, or interpretation, of $S$ in $Th$. Since encoding is to take place in $M^{S}_{\mathfrak{I}}$, it will have to have a large supply of constants and closed terms to be used as codes. (e.g. in formal arithmetic, one has $\bar{0}, \bar{1}, \ldots$) $S$ will also have certain function symbols to be described shortly. To each formula, $\Phi$, of the language of $Th$ is assigned a closed term, $[\Phi]^c$, called the code of $\Phi$. We note that if $\Phi(x)$ is a formula with free variable $x$, then $[\Phi(x)]^c$ is a closed term encoding the formula $\Phi(x)$ with $x$ viewed as a syntactic object and not as a parameter. Corresponding to the logical connectives and quantifiers are function symbols, $\text{neg}(\cdot)$, $\text{imp}(\cdot)$, etc., such that, for all formulae $\Phi, \Psi : S \vdash \text{neg}([\Phi]^c) = [\neg \Phi]^c$, $S \vdash \text{imp}([\Phi]^c, [\Psi]^c) = [\Phi \rightarrow \Psi]^c$ etc. Of particular importance is the substitution operator, represented by the function symbol $\text{sub}(\cdot, \cdot)$. For formulae $\Phi(x)$, terms $t$ with codes $[t]^c$:

\[
S \vdash \text{sub}([\Phi(x)]^c, [t]^c) = [\Phi(t)]^c. \tag{2.1}
\]

It well known [8] that one can also encode derivations and have a binary relation $\text{Prov}_{Th}(x,y)$ (read "$x$ proves $y$" or "$x$ is a proof of $y$") such that for closed $t_1, t_2 : S \vdash \text{Prov}_{Th}(t_1, t_2)$ iff $t_1$ is the code of a derivation in $Th$ of the formula with code $t_2$. It follows that

\[
Th \vdash \Phi \text{ iff } S \vdash \text{Prov}_{Th}(t, [\Phi]^c) \tag{2.2}
\]

for some closed term $t$. Thus one can define

\[
Pr_{Th}(y) \leftrightarrow \exists x \text{Prov}_{Th}(x,y), \tag{2.3}
\]
and therefore one obtain a predicate asserting provability. We note that is not always the case that [8]:

\[ \text{Th} \vdash \phi \text{ if } S \vdash \text{Pr}_{\text{Th}}([\phi]^c), \] (2.4)

unless \( S \) is fairly sound, e.g. this is a case when \( S \) and \( \text{Th} \) replaced by \( S_\omega = S \uparrow M^\omega_{\text{Th}} \)

and \( \text{Th}_\omega = \text{Th} \uparrow M^\omega_{\text{Th}} \) correspondingly (see Designation 2.1).

**Remark 2.2.** Noticee that is always the case that:

\[ \text{Th}_\omega \vdash \phi \iff S_\omega \vdash \text{Pr}_{\text{Th}_\omega}([\phi_\omega]^c), \] (2.5)
i.e. that is the case when predicate \( \text{Pr}_{\text{Th}_\omega}(y), y \in M^\omega_{\text{Th}} : \)

\[ \text{Pr}_{\text{Th}_\omega}(y) \leftrightarrow \exists x (x \in M^\omega_{\text{Th}}) \text{prov}_{\text{Th}_\omega}(x, y) \] (2.6)

really asserts provability.

It well known [8] that the above encoding can be carried out in such a way that the following important conditions \( D_1, D_2 \) and \( D_3 \) are meet for all sentences [8]:

\[ D_1. \text{Th} \vdash \phi \text{ implies } S \vdash \text{Pr}_{\text{Th}}([\phi]^c), \]

\[ D_2. S \vdash \text{Pr}_{\text{Th}}([\phi]^c) \rightarrow \text{Pr}_{\text{Th}}([\text{Pr}_{\text{Th}}([\phi]^c)]^c), \] (2.7)

\[ D_3. S \vdash \text{Pr}_{\text{Th}}([\phi]^c) \land \text{Pr}_{\text{Th}}([\phi \rightarrow \psi]^c) \rightarrow \text{Pr}_{\text{Th}}([\psi]^c). \]

Conditions \( D_1, D_2 \) and \( D_3 \) are called the Derivability Conditions.

**Remark 2.3.** From (2.5)-(2.6) follows that

\[ D_4. \text{Th}_\omega \vdash \phi \iff S_\omega \vdash \text{Pr}_{\text{Th}_\omega}([\phi_\omega]^c), \]

\[ D_5. S_\omega \vdash \text{Pr}_{\text{Th}_\omega}([\phi_\omega]^c) \leftrightarrow \text{Pr}_{\text{Th}_\omega}([\text{Pr}_{\text{Th}_\omega}([\phi_\omega]^c)]^c), \] (2.8)

\[ D_6. S_\omega \vdash \text{Pr}_{\text{Th}_\omega}([\phi_\omega]^c) \land \text{Pr}_{\text{Th}_\omega}([\phi_\omega \rightarrow \psi_\omega]^c) \rightarrow \text{Pr}_{\text{Th}_\omega}([\psi_\omega]^c). \]

Conditions \( D_4, D_5 \) and \( D_6 \) are called the Strong Derivability Conditions.

**Definition 2.1.** Let \( \phi \) be well formed formula (wff) of \( \text{Th} \). Ten wff \( \phi \) is called \( \text{Th} \)-sentence iff it has no free variables.

**Designation 2.1.** (i) Assume that a theory \( \text{Th} \) has an \( \omega \)-model \( M^\omega_{\text{Th}} \) and \( \phi \) is an \( \text{Th} \)-sentence, then:

\( \Phi_{M^\omega_{\text{Th}}} \triangleq \phi \uparrow M^\omega_{\text{Th}} \) (we will write \( \Phi_{\omega \text{Th}} \) instead \( \Phi_{M^\omega_{\text{Th}}} \)) is a \( \text{Th} \)-sentence \( \phi \) with all quantifiers relativized to \( \omega \)-model \( M^\omega_{\text{Th}} \) [13],[23] and

\( \text{Th}_\omega \triangleq \text{Th} \uparrow M^\omega_{\text{Th}} \) is a theory \( \text{Th} \) relativized to model \( M^\omega_{\text{Th}} \), i.e., any \( \text{Th}_\omega \)-sentence has the form \( \Phi_{\omega \text{Th}} \) for some \( \text{Th} \)-sentence \( \phi \).

(ii) Assume that a theory \( \text{Th} \) has an non-standard model \( M^\text{Th}_{\text{Nst}} \) and \( \phi \) is an \( \text{Th} \)-sentence, then:

\( \Phi_{M^\text{Th}_{\text{Nst}}} \triangleq \phi \uparrow M^\text{Th}_{\text{Nst}} \) (we will write \( \Phi_{\text{Nst}} \) instead \( \Phi_{M^\text{Th}_{\text{Nst}}} \)) is a \( \text{Th} \)-sentence with all quantifiers relativized to non-standard model \( M^\text{Th}_{\text{Nst}} \), and

\( \text{Th}_{\text{Nst}} \triangleq \text{Th} \uparrow M^\text{Th}_{\text{Nst}} \) is a theory \( \text{Th} \) relativized to model \( M^\text{Th}_{\text{Nst}} \), i.e., any \( \text{Th}_{\text{Nst}} \)-sentence has
a form $\Phi_{Nst}$ for some $\mathbf{Th}$-sentence $\Phi$.

(iii) Assume that a theory $\mathbf{Th}$ has an $M^{\mathbf{Th}}$ and $\Phi$ is an $M^{\mathbf{Th}}$-sentence, then:

$\Phi_{M^{\mathbf{Th}}}$ is a $\mathbf{Th}$-sentence with all quantifiers relativized to model $M^{\mathbf{Th}}$, and $\mathbf{Th}_{M}$ is a theory $\mathbf{Th}$ relativized to model $M^{\mathbf{Th}}_{M}$, i.e., any $\mathbf{Th}_{M}$-sentence has a form $\Phi_{M}$ for some $\mathbf{Th}$-sentence $\Phi$.

**Designation 2.2.** (i) Assume that a theory $\mathbf{Th}$ has an $\omega$-model $M^{\mathbf{Th}}_{\omega}$ and there exist $\mathbf{Th}$-sentence denoted by $\text{Con}(\mathbf{Th}; M^{\mathbf{Th}}_{\omega})$ asserting that $\mathbf{Th}$ has a model $M^{\mathbf{Th}}_{\omega}$;

(ii) Assume that a theory $\mathbf{Th}$ has an non-standard model $M^{\mathbf{Th}}_{Nst}$ and there exist $\mathbf{Th}$-sentence denoted by $\text{Con}(\mathbf{Th}; M^{\mathbf{Th}}_{Nst})$ asserting that $\mathbf{Th}$ has a non-standard model $M^{\mathbf{Th}}_{Nst}$;

(iii) Assume that a theory $\mathbf{Th}$ has an model $M^{\mathbf{Th}}$ and there exist $\mathbf{Th}$-sentence denoted by $\text{Con}(\mathbf{Th}; M^{\mathbf{Th}})$ asserting that $\mathbf{Th}$ has a model $M^{\mathbf{Th}}$.

**Remark 2.4.** It is well known that there exist an $ZFC$-sentence $\text{Con}(ZFC; M^{ZFC})$ [21],[22].

Obviously there exist an $ZFC_{2}^{th}$-sentence $\text{Con}(ZFC_{2}^{th}; M^{ZFC_{2}})$ and there exist an $Z_{2}^{th}$-sentence $\text{Con}(Z_{2}^{th}; M^{Z_{2}})$.

**Designation 2.3.** Let $\text{Con}(\mathbf{Th})$ be the formula:

$$
\begin{aligned}
\text{Con}(\mathbf{Th}) & \triangleq \\
\forall t_1(t_1 \in M_{\omega}^{\mathbf{Th}}) \forall t'_1(t'_1 \in M_{\omega}^{\mathbf{Th}}) \forall t_2(t_2 \in M_{\omega}^{\mathbf{Th}}) \forall t'_2(t'_2 \in M_{\omega}^{\mathbf{Th}}) \\
& \quad \neg [\text{Prov}(t_1, [\Phi]^{c}) \land \text{Prov}(t_2, \neg ([\Phi]^{c}))], \\
& \text{or} \\
\text{Con}(\mathbf{Th}) & \triangleq \\
\forall t_1(t_1 \in M_{\omega}^{\mathbf{Th}}) \forall t_2(t_2 \in M_{\omega}^{\mathbf{Th}}) \\
& \quad \neg [\text{Prov}(t_1, [\Phi]^{c}) \land \text{Prov}(t_2, \neg ([\Phi]^{c}))]
\end{aligned}
$$

and where $t_1, t'_1, t_2, t'_2$ is a closed term.

**Lemma 2.1.** (I) Assume that: (i) $\text{Con}(\mathbf{Th}; M^{\mathbf{Th}})$, (ii) $M^{\mathbf{Th}} \models \text{Con}(\mathbf{Th})$ and (iii) $\mathbf{Th} \models \text{Pr}_{\mathbf{Th}}([\Phi]^{c})$, where $\Phi$ is a closed formula. Then $\mathbf{Th} \not\models \text{Pr}_{\mathbf{Th}}(\neg [\Phi]^{c})$.

(II) Assume that: (i) $\text{Con}(\mathbf{Th}; M_{\omega}^{\mathbf{Th}})$, (ii) $M_{\omega}^{\mathbf{Th}} \models \text{Con}(\mathbf{Th})$ and (iii) $\mathbf{Th}_{\omega} \models \text{Pr}_{\mathbf{Th}_{\omega}}([\Phi_{\omega}]^{c})$, where $\Phi_{\omega}$ is a closed formula. Then $\mathbf{Th}_{\omega} \not\models \text{Pr}_{\mathbf{Th}_{\omega}}(\neg [\Phi_{\omega}]^{c})$.

**Proof.** (I) Let $\text{Con}_{\mathbf{Th}}(\Phi)$ be the formula :
\[
Con_{\text{Th}}(\Phi) \triangleq \\
\forall t_1(t_1 \in M_{\text{Th}}^1)\forall t_2(t_2 \in M_{\text{Th}}^2) \neg [\text{Prov}_{\text{Th}}(t_1, [\Phi]^c) \land \text{Prov}_{\text{Th}}(t_2, \text{neg}([\Phi]^c))],
\]
\[
\forall t_1(t_1 \in M_{\text{Th}}^1)\forall t_2(t_2 \in M_{\text{Th}}^2) \neg [\text{Prov}_{\text{Th}}(t_1, [\Phi]^c) \land \text{Prov}_{\text{Th}}(t_2, \text{neg}([\Phi]^c))],
\]
\[
\leftrightarrow \neg \exists t_1(t_1 \in M_{\text{Th}}^1) \exists t_2(t_2 \in M_{\text{Th}}^2) [\text{Prov}_{\text{Th}}(t_1, [\Phi]^c) \land \text{Prov}_{\text{Th}}(t_2, \text{neg}([\Phi]^c))].
\]

where \( t_1, t_2 \) is a closed term. From (i)-(ii) follows that theory \( \text{Th} + Con(\text{Th}) \) is consistent. We note that

We assume now that:

Example 2.

Proof. Similarly as Lemma 2.1 above.

Example 2.1. (i) Let \( \text{Th} = \text{PA} \) be Peano arithmetic and \( \Phi \iff 0 = 1 \). Then obviously by Löbs theorem \( \text{PA} \vdash \text{Pr}_{\text{PA}}(0 \neq 1) \), and therefore \( \text{PA} \nvDash \text{Pr}_{\text{PA}}(0 = 1) \).

(ii) Let \( \text{PA}^* = \text{PA} + \text{Con}(\text{PA}) \) and \( \Phi \iff 0 = 1 \). Then obviously by Löbs theorem

\[
\text{PA}^* \vdash \text{Pr}_{\text{PA}^*}(0 \neq 1),
\]

and therefore

\[
\text{PA}^* \nvDash \text{Pr}_{\text{PA}^*}(0 = 1).
\]

However

\[
\text{PA}^* \vdash [\text{Pr}_{\text{PA}}(0 \neq 1)] \land [\text{Pr}_{\text{PA}}(0 = 1)].
\]

Remark 2.5. Notice that there is no standard model of \( \text{PA}^* \).

Assumption 2.1. Let \( \text{Th} \) be an second order theory with the Henkin semantics. We assume now that:

(i) the language of \( \text{Th} \) consists of:

- numerals \( \overline{0}, \overline{1}, \ldots \),
- countable set of the numerical variables: \( \{v_0, v_1, \ldots\} \),
- countable set \( \mathcal{F} \) of the set variables: \( \mathcal{F} = \{x, y, z, X, Y, Z, R, \ldots\} \),
- countable set of the \( n \)-ary function symbols: \( f_0^n, f_1^n, \ldots \),
- countable set of the \( n \)-ary relation symbols: \( R_0^n, R_1^n, \ldots \).
connectives: $\neg$, $\rightarrow$
quantifier: $\forall$

(i) $\text{Th}$ contains $ZFC_2$,
(ii) $\text{Th}$ has an $\omega$-model $M^{\text{Th}}_\omega$ or
(iv) $\text{Th}$ has a nonstandard model $M^{\text{Th}}_{\text{Nat}}$.

**Definition 2.1.** An $\text{Th}$-wff $\Phi$ (well-formed formula $\Phi$) is closed - i.e. $\Phi$ is a sentence - if it has no free variables; a wff is open if it has free variables. We’ll use the slang ‘$k$-place open wff ’ to mean a wff with $k$ distinct free variables.

**Definition 2.2.** We will say that, $\text{Th}^\#_\omega$ is a nice theory or a nice extension of the $\text{Th}$ if:
(i) $\text{Th}^\#_\omega$ contains $\text{Th}$;
(ii) Let $\Phi$ be any closed formula of $\text{Th}$, then $\text{Th} \vdash \text{Pr}_\text{Th}([\Phi]^c)$ implies $\text{Th}^\#_\omega \vdash \Phi$;
(iii) Let $\Phi_\omega$ be any closed formula of $\text{Th}^\#_\omega$, then $M^{\text{Th}}_\omega \models \Phi_\omega$ implies $M^{\text{Th}}_\omega \models \Phi_\omega$, i.e. $\text{Con}(\text{Th} + \Phi_\omega; M^{\text{Th}}_\omega)$ implies $M^{\text{Th}}_\omega \models \Phi_\omega$.

**Remark 2.6.** Notice that formulas $\text{Con}(\text{Th} + \Phi_\omega; M^{\text{Th}}_\omega)$ and $\text{Con}(\text{Th}^\#_\omega + \Phi_\omega; M^{\text{Th}}_\omega)$ is expressible in $\text{Th}^\#_\omega$.

**Definition 2.3.** Fix an classical propositional logic $L$. Recall that a set $\Delta$ of wff’s is said to be $L$ -consistent, or consistent for short, if $\Delta \not\vdash \bot$ and there are other equivalent formulations of consistency: (1) $\Delta$ is consistent, (2) $\text{Ded}(\Delta) := \{ A \mid \Delta \vdash A \}$ is not the set of all wff’s, (3) there is a formula such that $\Delta \not\vdash A$. (4) there are no formula $A$ such that $\Delta \vdash A$ and $\Delta \vdash \neg A$.

We will say that, $\text{Th}^\#_\omega$ is a maximally nice theory or a maximally nice extension of the $\text{Th}$ if

$\text{Th}^\#_\omega$ is consistent and for any consistent nice extension $\text{Th}^\#_\omega'$ of the $\text{Th}$:

$\text{Ded}(\text{Th}^\#_\omega) \subseteq \text{Ded}(\text{Th}^\#_\omega')$ implies $\text{Ded}(\text{Th}^\#_\omega) = \text{Ded}(\text{Th}^\#_\omega')$.

**Remark 2.7.** We note that a theory $\text{Th}^\#_\omega$ depend on model $M^{\text{Th}}_\omega$ or $M^{\text{Th}}_{\text{Nat}}$, i.e. $\text{Th}^\#_\omega = \text{Th}^\#_\omega[M^{\text{Th}}_\omega]$ or $\text{Th}^\#_\omega = \text{Th}^\#_\omega[M^{\text{Th}}_{\text{Nat}}]$ correspondingly. We will consider now the case

$\text{Th}^\#_\omega \triangleq \text{Th}^\#_\omega[M^{\text{Th}}_\omega]$ without loss of generality.

**Remark 2.8.** Notice that in order to prove the statement: $\neg \text{Con}(ZFC^{H_\delta}_2; M^{\text{Th}}_\omega)$, Proposition 2.1 is not necessary, see Proposition 2.18.

**Proposition 2.1.** (Generalized Lobs Theorem) (I) Assume that (i) $\text{Con}(\text{Th})$ (see 2.9) and
(ii) $\text{Th}$ has an $\omega$-model $M^{\text{Th}}_\omega$. Then theory $\text{Th}$ can be extended to a maximally consistent nice theory $\text{Th}^\#_\omega \triangleq \text{Th}^\#_\omega[M^{\text{Th}}_\omega]$.

(II) Assume that (i) $\text{Con}(\text{Th})$ and (ii) $\text{Th}$ has an $\omega$-model $M^{\text{Th}}_\omega$. Then theory
\( \text{Th}_\omega \) can be extended to a maximally consistent nice theory \( \text{Th}_\omega^\# \equiv \text{Th}_\omega^\# [M^\text{Th}_\omega] \).

**Proof.** (i) Let \(\Phi_1 \ldots \Phi_i \ldots\) be an enumeration of all closed wff's of the theory \(\text{Th}\) (this can be achieved if the set of propositional variables can be enumerated). Define a chain \(\mathcal{G} = \{\text{Th}_i^\# | i \in \mathbb{N}\}\), \(\text{Th}_i^\# = \text{Th}\) of consistent theories inductively as follows: assume that theory \(\text{Th}_i^\#\) is defined.

(i) Suppose that the statement (2.13) is satisfied
\[
[\text{Th}_i^\# \not\vdash \text{Pr}_{\text{Th}_i^\#}([\Phi_i]^c)] \land [\text{Th}_i^\# \not\vdash \Phi_i] \quad \text{and} \quad M^\text{Th}_\omega \models \Phi_i. \tag{2.13}
\]
Then we define a theory \(\text{Th}_{i+1}^\#\) as follows \(\text{Th}_{i+1}^\# \equiv \text{Th}_i^\# \cup \{\Phi_i\}\). We will rewrite the condition (2.13) using predicate \(\text{Pr}_{\text{Th}_{i+1}^\#}(\cdot)\) symbolically as follows:
\[
\begin{align*}
\text{Th}_{i+1}^\# & \equiv \text{Pr}_{\text{Th}_{i+1}^\#}([\Phi_i]^c), \\
\text{Pr}_{\text{Th}_{i+1}^\#}([\Phi_i]^c) & \Leftrightarrow \text{Pr}_{\text{Th}_i^\#}([\Phi_i]^c) \land [M^\text{Th}_\omega \models \Phi_i], \\
M^\text{Th}_\omega & \models \Phi_i \iff \text{Con}(\text{Th}_i^\# + \Phi_i; M^\text{Th}_\omega), \\
i.e. & \\
\text{Pr}_{\text{Th}_{i+1}^\#}([\Phi_i]^c) & \Leftrightarrow \text{Pr}_{\text{Th}_i^\#}([\Phi_i]^c), \\
\text{Pr}_{\text{Th}_{i+1}^\#}([\Phi_i]^c) & \Rightarrow \Phi_i, \\
\text{Pr}_{\text{Th}_{i+1}^\#}([\Phi_i]^c) & \Rightarrow \Phi_i.
\end{align*}
\]

(ii) Suppose that the statement (2.15) is satisfied
\[
[\text{Th}_i^\# \not\vdash \text{Pr}_{\text{Th}_i^\#}([\neg \Phi_i]^c)] \land [\text{Th}_i^\# \not\vdash \neg \Phi_i] \quad \text{and} \quad M^\text{Th}_\omega \models \neg \Phi_i. \tag{2.15}
\]
Then we define a theory \(\text{Th}_{i+1}^\#\) as follows \(\text{Th}_{i+1}^\# \equiv \text{Th}_i^\# \cup \{\Phi_i\}\). We will rewrite the condition (2.15) using predicate \(\text{Pr}_{\text{Th}_{i+1}^\#}(\cdot)\), symbolically as follows:
\[
\begin{align*}
\text{Th}_{i+1}^\# & \equiv \text{Pr}_{\text{Th}_{i+1}^\#}([\neg \Phi_i]^c), \\
\text{Pr}_{\text{Th}_{i+1}^\#}([\neg \Phi_i]^c) & \Leftrightarrow \text{Pr}_{\text{Th}_i^\#}([\neg \Phi_i]^c) \land [M^\text{Th}_\omega \models \neg \Phi_i], \\
M^\text{Th}_\omega & \models \neg \Phi_i \iff \text{Con}(\text{Th}_i^\# + \neg \Phi_i; M^\text{Th}_\omega), \\
i.e. & \\
\text{Pr}_{\text{Th}_{i+1}^\#}([\neg \Phi_i]^c) & \Leftrightarrow \text{Pr}_{\text{Th}_i^\#}([\neg \Phi_i]^c), \\
\text{Pr}_{\text{Th}_{i+1}^\#}([\neg \Phi_i]^c) & \Rightarrow \neg \Phi_i, \\
\text{Pr}_{\text{Th}_{i+1}^\#}([\neg \Phi_i]^c) & \Rightarrow \neg \Phi_i.
\end{align*}
\]
(iii) Suppose that the statement (2.17) is satisfied

\[
\text{Th}_i^\# \vdash \text{Pr}_{\text{Th}_i^\#}(\lbrack \Phi_i \rbrack^\#) \text{ and } \lbrack \text{Th}_i^\# \not\models \Phi_i \rbrack \land \lbrack M_{\omega^0}^\# \models \Phi_i \rbrack. \tag{2.17}
\]

Then we define a theory \(\text{Th}_{i+1}^\#\) as follows \(\text{Th}_{i+1}^\# \cong \text{Th}_i^\# \cup \lbrack \Phi_i \rbrack\). Using Lemma 2.1 and predicate \(\text{Pr}_{\text{Th}_{i+1}^\#}(\cdot)\), we will rewrite the condition (2.17) symbolically as follows:

\[
\text{Th}_{i+1}^\# \vdash \text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#),
\]

\[
\text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#) \iff \text{Pr}_{\text{Th}_i^\#}(\lbrack \Phi_i \rbrack^\#) \land [M_{\omega^0}^\# \models \Phi_i],
\]

\[
M_{\omega^0}^\# \models \Phi_i \iff \text{Con}(\text{Th}_i^\# + \Phi_i; M_{\omega^0}^\#),
\]

i.e.

\[
\text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#) \iff \text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#) \land \text{Con}(\text{Th}_i^\# + \Phi_i; M_{\omega^0}^\#),
\]

\[
\text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#) \iff \text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#),
\]

\[
\text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#) \iff \Phi_i.
\]

\[
\text{Remark 2.9. Notice that predicate } \text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#) \text{ is expressible in } \text{Th}_i^\# \text{ because } \text{Th}_i^\# \text{ is a finite extension of the recursive theory } \text{Th} \text{ and } \text{Con}(\text{Th}_i^\# + \Phi_i; M_{\omega^0}^\#) \in \text{Th}_i^\#.
\]

(iv) Suppose that a statement (2.19) is satisfied

\[
\text{Th}_i^\# \vdash \text{Pr}_{\text{Th}_i^\#}(\lbrack \neg \Phi_i \rbrack^\#) \text{ and } \lbrack \text{Th}_i^\# \not\models \neg \Phi_i \rbrack \land [M_{\omega^0}^\# \models \neg \Phi_i]. \tag{2.19}
\]

Then we define theory \(\text{Th}_{i+1}^\#\) as follows: \(\text{Th}_{i+1}^\# \cong \text{Th}_i^\# \cup \lbrack \neg \Phi_i \rbrack\). Using Lemma 2.2 and predicate \(\text{Pr}_{\text{Th}_{i+1}^\#}(\cdot)\), we will rewrite the condition (2.15) symbolically as follows:

\[
\text{Th}_{i+1}^\# \vdash \text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \neg \Phi_i \rbrack^\#),
\]

\[
\text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \neg \Phi_i \rbrack^\#) \iff \text{Pr}_{\text{Th}_i^\#}(\lbrack \neg \Phi_i \rbrack^\#) \land [M_{\omega^0}^\# \models \neg \Phi_i],
\]

\[
M_{\omega^0}^\# \models \neg \Phi_i \iff \text{Con}(\text{Th}_i^\# + \neg \Phi_i; M_{\omega^0}^\#),
\]

i.e.

\[
\text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \neg \Phi_i \rbrack^\#) \iff \text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \neg \Phi_i \rbrack^\#) \land \text{Con}(\text{Th}_i^\# + \neg \Phi_i; M_{\omega^0}^\#),
\]

\[
\text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#) \iff \text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#),
\]

\[
\text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \Phi_i \rbrack^\#) \iff \Phi_i.
\]

\[
\text{Remark 2.10. Notice that predicate } \text{Pr}_{\text{Th}_{i+1}^\#}(\lbrack \neg \Phi_i \rbrack^\#) \text{ is expressible in } \text{Th}_i^\# \text{ because } \text{Th}_i^\# \text{ is a finite extension of the recursive theory } \text{Th} \text{ and } \text{Con}(\text{Th}_i^\# + \neg \Phi_i; M_{\omega^0}^\#) \in \text{Th}_i^\#.
\]

(v) Suppose that the statement (2.21) is satisfied
If the statement (2.24) is satisfied, then we define a theory $\text{Th}^\#_{i+1}$ as follows: $\text{Th}^\#_{i+1} = \text{Th}^\#_i$. We define now a theory $\text{Th}^\#_\omega$ as follows:

$$\text{Th}^\#_\omega = \bigcup_{i \in \mathbb{N}} \text{Th}^\#_i.$$  

First, notice that each $\text{Th}^\#_i$ is consistent. This is done by induction on $i$ and by Lemmas 2.1-2.2. By assumption, the case is true when $i = 1$. Now, suppose $\text{Th}^\#_i$ is consistent. Then its deductive closure $\text{Ded}(\text{Th}^\#_i)$ is also consistent. If the statement (2.14) is satisfied, i.e. $\text{Th}^\#_{i+1} \vdash \text{Pr}_{\text{Th}^\#_{i+1}}([\Phi_i]^c)$ and $\text{Th}^\#_{i+1} \vdash \Phi_i$, then clearly $\text{Th}^\#_{i+1} = \text{Th}^\#_i \cup \{\Phi_i\}$ is consistent since it is a subset of closure $\text{Ded}(\text{Th}^\#_{i+1})$. If a statement (2.16) is satisfied, i.e. $\text{Th}^\#_{i+1} \vdash \text{Pr}_{\text{Th}^\#_{i+1}}([\neg \Phi_i]^c)$ and $\text{Th}^\#_{i+1} \vdash \neg \Phi_i$, then clearly $\text{Th}^\#_{i+1} = \text{Th}^\#_i \cup \{\neg \Phi_i\}$ is consistent since it is a subset of closure $\text{Ded}(\text{Th}^\#_{i+1})$. If the statement (2.18) is satisfied, i.e. $\text{Th}^\#_i \vdash \text{Pr}_{\text{Th}^\#_i}([\Phi_i]^c)$ and $[\text{Th}^\#_i \not\vdash \Phi_i] \land [M^\text{Th}_0 \vdash \Phi_i]$ then clearly $\text{Th}^\#_i = \text{Th}^\#_i \cup \{\Phi_i\}$ is consistent by Lemma 2.1 and by one of the standard properties of consistency: $\Delta \cup \{A\}$ is consistent iff $\Delta \not\vdash \neg A$. If the statement (2.20) is satisfied, i.e. $\text{Th}^\#_i \vdash \text{Pr}_{\text{Th}^\#_i}([\neg \Phi_i]^c)$ and $[\text{Th}^\#_i \not\vdash \neg \Phi_i] \land [M^\text{Th}_0 \vdash \neg \Phi_i]$ then clearly $\text{Th}^\#_{i+1} = \text{Th}^\#_i \cup \{\neg \Phi_i\}$ is consistent by Lemma 2.2 and by one of the standard properties of consistency: $\Delta \cup \{\neg A\}$ is consistent iff $\Delta \not\vdash A$. Next, notice $\text{Ded}(\text{Th}^\#_\omega)$ is maximally consistent nice extension of the $\text{Ded}(\text{Th})$. $\text{Ded}(\text{Th}^\#_\omega)$ is consistent because, by the standard Lemma 2.3 below, it is the union of a chain of consistent sets. To see that $\text{Ded}(\text{Th}^\#_\omega)$ is maximal, pick any wff $\Phi$. Then $\Phi$ is some $\Phi_i$ in the enumerated list of all wff’s. Therefore for any $\Phi$ such that $\text{Th}^\#_i \vdash \text{Pr}_{\text{Th}^\#_i}([\Phi]^c)$ or $\text{Th}^\#_i \vdash \text{Pr}_{\text{Th}^\#_i}([\neg \Phi]^c)$, either $\Phi \in \text{Th}^\#_\omega$ or $\neg \Phi \in \text{Th}^\#_\omega$. Since $\text{Ded}(\text{Th}^\#_{i+1}) \subseteq \text{Ded}(\text{Th}^\#_\omega)$, we have $\Phi \in \text{Ded}(\text{Th}^\#_\omega)$ or $\neg \Phi \in \text{Ded}(\text{Th}^\#_\omega)$, which implies that $\text{Ded}(\text{Th}^\#_\omega)$ is maximally consistent nice extension of
the Ded(Th).

Proof. (II) Let $\Phi_{\omega,1} \ldots \Phi_{\omega,i} \ldots$ be an enumeration of all closed wff’s of the theory Th$_\omega$ (this can be achieved if the set of propositional variables can be enumerated). Define a chain $\mathcal{O} = \{\text{Th}\#_{\omega,i} | i \in \mathbb{N}\},$ Th$_{\omega,1} = \text{Th}_\omega$ of consistent theories inductively as follows: assume that theory Th$_{\omega,i}$ is defined.

(i) Suppose that a statement (2.26) is satisfied

$\text{Th}_{\omega,i} \vdash \text{Pr}_{\text{Th}_{\omega,i}}([\Phi_{\omega,i}]^c)$ and $M^\text{Th}_\omega \models \Phi_i.$

Then we define a theory Th$_{\omega,i+1}$ as follows:

$\text{Th}_{\omega,i+1} \equiv \text{Th}_{\omega,i} \cup \{\Phi_{\omega,i}\}.$

We will rewrite now the conditions (2.26) and (2.27) symbolically as follows

$\text{Th}_{\omega,i+1} \vdash \text{Pr}_{\text{Th}_{\omega,i+1}}([\Phi_{\omega,i}]^c) \iff \text{Th}_{\omega,i+1} \vdash \Phi_{\omega,i},$

$\text{Pr}_{\text{Th}_{\omega,i+1}}([\Phi_i]^c) \equiv \text{Pr}_{\text{Th}_{\omega,i+1}}([\Phi_i]^c) \land \Phi_{\omega,i}.$

(ii) Suppose that a statement (2.29) is satisfied

$\text{Th}_{\omega,i} \vdash \text{Pr}_{\text{Th}_{\omega,i}}([\neg \Phi_{\omega,i}]^c)$ and $M^\text{Th}_\omega \models \neg \Phi_i.$

Then we define theory Th$_{\omega,i+1}$ as follows:

$\text{Th}_{\omega,i+1} \equiv \text{Th}_{\omega,i} \cup \{\neg \Phi_{\omega,i}\}.$

We will rewrite the conditions (2.25) and (2.26) symbolically as follows

$\text{Th}_{\omega,i+1} \vdash \text{Pr}_{\text{Th}_{\omega,i+1}}([\neg \Phi_{\omega,i}]^c) \iff \text{Th}_{\omega,i+1} \vdash \neg \Phi_{\omega,i},$

$\text{Pr}_{\text{Th}_{\omega,i+1}}([\neg \Phi_i]^c) \equiv \text{Pr}_{\text{Th}_{\omega,i+1}}([\neg \Phi_i]^c).$

(iii) Suppose that the following statement (2.28) is satisfied

$\text{Th}_{\omega,i} \vdash \text{Pr}_{\text{Th}_{\omega,i}}([\Phi_{\omega,i}]^c),$ and therefore by Derivability Conditions (2.8)

$\text{Th}_{\omega,i} \vdash \Phi_{\omega,i}.$

We will rewrite now the conditions (2.28) and (2.29) symbolically as follows

$\text{Pr}_{\text{Th}_{\omega,i}}([\Phi_{\omega,i}]^c) \iff \text{Th}_{\omega,i} \vdash \text{Pr}_{\text{Th}_{\omega,i}}([\Phi_{\omega,i}]^c),$  

(2.30)

Then we define a theory Th$_{\omega,i+1}$ as follows:

$\text{Th}_{\omega,i+1} \equiv \text{Th}_{\omega,i}.$

(iv) Suppose that the following statement (2.31) is satisfied

$\text{Th}_{\omega,i} \vdash \text{Pr}_{\text{Th}_{\omega,i}}([\neg \Phi_{\omega,i}]^c),$

and therefore by Derivability Conditions (2.8)
We will rewrite now the conditions (2.31) and (2.32) symbolically as follows

\[
\text{Th}_{\omega,i} \vdash \neg \Phi_{\omega,i}. 
\]  

(2.32)

Then we define a theory \( \text{Th}_{\omega,i+1} \) as follows: \( \text{Th}_{\omega,i+1} \triangleq \text{Th}_{\omega,i} \). We define now a theory \( \text{Th}_{\omega}^{\#} \) as follows:

\[
\text{Th}_{\omega}^{\#} \triangleq \bigcup_{i \in \mathbb{N}} \text{Th}_{\omega,i}. 
\]  

(2.34)

First, notice that each \( \text{Th}_{\omega,i} \) is consistent. This is done by induction on \( i \). Now, suppose \( \text{Th}_{\omega,i} \) is consistent. Then its deductive closure \( \text{Ded}(\text{Th}_{\omega,i}) \) is also consistent. If statement (2.22) is satisfied, i.e. \( \text{Th}_{\omega,i} \not\vdash \text{Pr}_{\text{Th}_{\omega,i}}([\Phi_{\omega,i}]^c) \) and \( M_{\omega}^{\text{Th}} \models \Phi_i \) then clearly \( \text{Th}_{\omega,i+1} \triangleq \text{Th}_{\omega,i} \cup \{\Phi_{\omega,i}\} \) is consistent. If statement (2.25) is satisfied, i.e. \( \text{Th}_{\omega,i} \vdash \text{Pr}_{\text{Th}_{\omega,i}}([\neg \Phi_{\omega,i}]^c) \) and \( M_{\omega}^{\text{Th}} \models \neg \Phi_i \), then clearly \( \text{Th}_{\omega,i+1} \triangleq \text{Th}_{\omega,i} \cup \{\neg \Phi_{\omega,i}\} \) is consistent. If the statement (2.28) is satisfied, i.e. \( \text{Th}_{\omega,i} \vdash \text{Pr}_{\text{Th}_{\omega,i}}([\Phi_{\omega,i}]^c) \), then clearly \( \text{Th}_{\omega,i+1} \triangleq \text{Th}_{\omega,i} \) is also consistent. If the statement (2.31) is satisfied, i.e. \( \text{Th}_{\omega,i} \vdash \text{Pr}_{\text{Th}_{\omega,i}}([\neg \Phi_{\omega,i}]^c) \), then clearly \( \text{Th}_{\omega,i+1} \triangleq \text{Th}_{\omega,i} \) is also consistent. Next, notice \( \text{Ded}(\text{Th}_{\omega}^{\#}) \) is maximally consistent nice extension of the \( \text{Ded}(\text{Th}_{\omega}^{\#}) \). The set \( \text{Ded}(\text{Th}_{\omega}^{\#}) \) is consistent because, by the standard Lemma 2.3 belov, it is the union of a chain of consistent sets.

**Lemma 2.3.** The union of a chain \( \wp = \{ \Gamma_i | i \in \mathbb{N} \} \) of consistent sets \( \Gamma_i \), ordered by \( \subseteq \), is consistent.

**Definition 2.4.** (I) We define now predicate \( \text{Pr}_{\text{Th}_{\omega}^{\#}} ([\Phi]^c) \) and predicate \( \text{Pr}_{\text{Th}_{\omega}^{\#}} ([\neg \Phi]^c) \) asserting provability in \( \text{Th}_{\omega}^{\#} \) by the following formulae

\[
\begin{align*}
\text{Pr}_{\text{Th}_{\omega}^{\#}} ([\Phi]^c) & \iff \exists i (\Phi \in \text{Th}_{\omega}^{\#}) [\text{Pr}_{\text{Th}_{\omega}^{\#}} ([\Phi]^c) \vee [\text{Pr}_{\text{Th}_{\omega}^{\#}} ([\Phi]^c) ]] \vee \\
& \quad \lor [((\Phi \in \text{Th}_{\omega}^{\#}) \land \text{Con}(\text{Th}_{\omega}^{\#} + \Phi; M_{\omega}^{\text{Th}})). \\
\text{Pr}_{\text{Th}_{\omega}^{\#}} ([\neg \Phi]^c) & \iff \exists i (\Phi \in \text{Th}_{\omega}^{\#}) [\text{Pr}_{\text{Th}_{\omega}^{\#}} ([\neg \Phi]^c) \vee [\text{Pr}_{\text{Th}_{\omega}^{\#}} ([\neg \Phi]^c) ]] \vee \\
& \quad \lor [((\Phi \in \text{Th}_{\omega}^{\#}) \land \text{Con}(\text{Th}_{\omega}^{\#} + \neg \Phi; M_{\omega}^{\text{Th}})).
\end{align*}
\]  

(2.35)

(II) We define now predicate \( \text{Pr}_{\text{Th}_{\omega}^{\#}} ([\Phi_{\omega}]^c) \) and predicate \( \text{Pr}_{\text{Th}_{\omega}^{\#}} ([\neg \Phi_{\omega}]^c) \) asserting provability in \( \text{Th}_{\omega}^{\#} \) by following formulae
Pr_{Th}^\#(\Phi_o)^c) \iff \\
\{\exists i(\Phi_o \in Th_{\omega,i}^\#) \left[ Pr_{Th_{\omega,i}^\#}^\#(\Phi_o)^c \right] \lor \left[ Pr_{Th_{\omega,i}^\#}^\#(\neg \Phi_o)^c \right] \} \lor \\
\left[ (\Phi_o \in Th_{x,\omega}^\#) \land Con(Th_{x,\omega}^\# + \Phi_o; M_{\omega}^Th) \right], \\
Pr_{Th_{x,\omega}^\#}(\neg \Phi_o)^c) \iff \\
\{\exists i(\Phi_o \in Th_{\omega,j}^\#) \left[ Pr_{Th_{\omega,j}^\#}^\#(\neg \Phi_o)^c \right] \lor \left[ Pr_{Th_{\omega,j}^\#}^\#(\neg \Phi_o)^c \right] \} \lor \\
\left[ (\Phi_o \in Th_{x,\omega}^\#) \land Con(Th_{x,\omega}^\# + \neg \Phi_o; M_{\omega}^Th) \right]. \\
(2.36)

Remark 2.11. (I) Notice that both predicate $Pr_{Th_{x}^\#}(\Phi)^c$ and predicate $Pr_{Th_{x}^\#}(\neg \Phi)^c$ are expressible in $Th_x^\#$ because for any $i$, $Th_i^\#$ is a finite extension of the recursive theory $Th$ and $Con(Th_i^\# + \Phi; M^Th) \in Th_i$, $Con(Th_i^\# + \neg \Phi; M^Th) \in Th_i$.

(II) Notice that both predicate $Pr_{Th_{x,\omega}^\#}(\Phi)^c$ and predicate $Pr_{Th_{x,\omega}^\#}(\neg \Phi)^c$ are expressible in $Th_{x,\omega}^\#$ because for any $i$, $Th_{\omega,j}^\#$ is a finite extension of the recursive theory $Th_{\omega,j}$ and \\
$Con(Th_{\omega,j}^\# + \Phi; M^Th) \in Th_{\omega,j}, Con(Th_{\omega,j}^\# + \neg \Phi; M^Th) \in Th_{\omega,j}^\#$.

Definition 2.5. Let $\Psi = \Psi(x)$ be one-place open $Th$-wff such that the following condition:

$$Th \triangleq Th_1^# \vdash \exists! x.\Psi(x.\Psi)$$

(2.37) is satisfied.

Remark 2.12. We rewrite now the condition (2.37) using only language of the theory $Th_1^#$:

$$\{Th_1^# \vdash \exists! x.\Psi(x.\Psi)\} \iff Pr_{Th_1^#}(\exists! x.\Psi(x.\Psi))^c \land \\
\land \left\{Pr_{Th_1^#}(\exists! x.\Psi(x.\Psi))^c \Rightarrow \exists! x.\Psi(x.\Psi)\right\}. \\
(2.38)

Definition 2.6. We will say that, a set $y$ is a $Th_{x}^\#$-set if there exist one-place open wff $\Psi(x)$ such that $y = x.\Psi$. We write $y[\Theta^\#]$ iff $y$ is a $Th_{x}^\#$-set.

Remark 2.13. Note that

$$y[\Theta^\#] \iff \exists \Psi \left\{ (y = x.\Psi) \land Pr_{Th_1^#}(\exists! x.\Psi(x.\Psi))^c \right\} \\
\land \left\{Pr_{Th_1^#}(\exists! x.\Psi(x.\Psi))^c \Rightarrow \exists! x.\Psi(x.\Psi)\right\}. \\
(2.39)

Definition 2.7. Let $\mathfrak{I}_1$ be a collection such that:

$$\forall x \in \mathfrak{I}_1 \iff x \text{ is a } Th_{x}^\#\text{-set}. \\
(2.40)$$
**Proposition 2.2.** Collection \( S_1 \) is a \( \text{Th}_1^\# \)-set.

**Proof.** Let us consider an one-place open wff \( \Psi(x) \) such that conditions (2.37) is satisfied, i.e. \( \text{Th}_1^\# \vdash \exists x \Phi([\Psi(x)]) \). We note that there exists countable collection \( F_\Psi \) of the one-place open wff's \( F_\Psi = \{ \Psi_n(x) \}_{n \in \mathbb{N}} \) such that: (i) \( \Psi(x) \in F_\Psi \) and (ii)

\[
\text{Th} \triangleq \text{Th}_1^\# \vdash \exists! x \Phi([\Psi(x)]) \land \forall n \in \mathbb{N} \left[ \Psi(x) \leftrightarrow \Psi_n(x) \right].
\]

or in the equivalent form

\[
\text{Th} \triangleq \text{Th}_1^\# \vdash \left[ \Pr_{\text{Th}_1^\#} ([\exists! x \Phi([\Psi(x)])]^c) \right] \land \left[ \Pr_{\text{Th}_1^\#} ([\forall n \in \mathbb{N} \left[ \Psi(x) \leftrightarrow \Psi_n(x) \right]^c) \right] \land \Pr_{\text{Th}_1^\#} ([\forall n \in \mathbb{N} \left[ \Psi(x) \leftrightarrow \Psi_n(x) \right]^c) \Rightarrow \forall n \in \mathbb{N} \left[ \Psi(x) \leftrightarrow \Psi_n(x) \right]
\]

or in the following equivalent form

\[
\text{Th}_1^\# \vdash \exists x_1 [[\Psi_1(x_1)] \land \forall n \in \mathbb{N} \left[ \Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1) \right]]
\]

or

\[
\text{Th}_1^\# \vdash \left[ \Pr_{\text{Th}_1^\#} ([\exists x_1 \Psi(x_1)]^c) \right] \land \left[ \Pr_{\text{Th}_1^\#} ([\forall n \in \mathbb{N} \left[ \Psi(x_1) \leftrightarrow \Psi_{n,1}(x_1) \right]^c) \right] \land \Pr_{\text{Th}_1^\#} ([\forall n \in \mathbb{N} \left[ \Psi(x_1) \leftrightarrow \Psi_{n,1}(x_1) \right]^c) \Rightarrow \forall n \in \mathbb{N} \left[ \Psi(x_1) \leftrightarrow \Psi_{n,1}(x_1) \right],
\]

where we have set \( \Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1) \) and \( x_1 = x_1 \). We note that any collection \( F_{\Psi_k} = \{ \Psi_{n,k}(x) \}_{n \in \mathbb{N}}, k = 1, 2, \ldots \) such mentioned above, defines an unique set \( x_{\Psi_k}, \text{i.e. } F_{\Psi_{k_1}} \cap F_{\Psi_{k_2}} = \varnothing \text{ iff } x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}} \). We note that collections \( F_{\Psi_k}, k = 1, 2, \ldots \) is not a part of the \( ZFC_2 \), i.e. collection \( F_{\Psi_k} \) there is no set in sense of \( ZFC_2 \). However that is no problem, because by using Gödel numbering one can to replace any collection \( F_{\Psi_k}, k = 1, 2, \ldots \) by collection \( \Theta_k = g(F_{\Psi_k}) \) of the corresponding Gödel numbers such that

\[
\Theta_k = g(F_{\Psi_k}) = \{ g(\Psi_{n,k}(x_{k})) \}_{n \in \mathbb{N}}, k = 1, 2, \ldots.
\]

It is easy to prove that any collection \( \Theta_k = g(F_{\Psi_k}), k = 1, 2, \ldots \) is a \( \text{Th}_1^\# \)-set. This is done by Gödel encoding [8],[10] (2.43), by the statement (2.41) and by axiom schema of separation [9]. Let \( g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \ldots \) be a Gödel number of the wff \( \Psi_{n,k}(x_k) \). Therefore \( g(F_k) = \{ g_{n,k} \}_{n \in \mathbb{N}}, k = 1, 2, \ldots \) and

\[
\forall k \forall k \exists [\{ g_{n,k} \}_{n \in \mathbb{N}} \cap \{ g_{n,k_2} \}_{n \in \mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}].
\]

Let \( \{ \{ g_{n,k} \}_{n \in \mathbb{N}} \}_{k \in \mathbb{N}} \) be a family of the all sets \( \{ g_{n,k} \}_{n \in \mathbb{N}} \). By axiom of choice [9] one
obtains unique set $\mathcal{I}_1' = \{ g_k \}_{k \in \mathbb{N}}$ such that $\forall k [g_k \in \{ g_n \}_{n \in \mathbb{N}}]$. Finally one obtains a set $\mathcal{I}_1$ from the set $\mathcal{I}_1'$ by axiom schema of replacement [9].

**Proposition 2.3.** Any collection $\Theta_k = g(\mathcal{F}_k), k = 1, 2, \ldots$ is a $\text{Th}_i^\#$-set.

**Proof.** We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \text{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi(g_{n,k}, v_k)$

\[
\Pi(g_{n,k}, v_k) \leftrightarrow \text{Pr}_{\text{Th}_i^\#}([\exists x_k^c \Psi_{1,k}(x_1)])^c \wedge [\forall n(n \in \mathbb{N})\left[\text{Pr}_{\text{Th}_i^\#}([\Psi_{1,k}(x_k)])^c \leftrightarrow \text{Pr}_{\text{Th}_i^\#}(\text{Fr}(g_{n,k}, v_k))\right]].
\]  

(2.45)

We define now a set $\Theta_k$ such that

\[
\Theta_k = \Theta_k' \cup \{ g_k \}, \quad \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta_k \leftrightarrow \Pi(g_{n,k}, v_k)]
\]  

(2.46)

Obviously definitions (2.41) and (2.46) is equivalent.

**Definition 2.7.** We define now the following $\text{Th}_i^\#$-set $\mathcal{R}_1 \subseteq \mathcal{I}_1$:

\[
\forall x[x \in \mathcal{R}_1 \leftrightarrow (x \in \mathcal{I}_1) \wedge [\text{Pr}_{\text{Th}_i^\#}([\exists x \in x]^c)] \wedge [\text{Pr}_{\text{Th}_i^\#}([x \notin x]^c) \therefore x \notin x]].
\]  

(2.47)

**Proposition 2.4.** (i) $\text{Th}_i^\# \vdash \exists \mathcal{R}_1$, (ii) $\mathcal{R}_1$ is a countable $\text{Th}_i^\#$-set.

**Proof.** (i) Statement $\text{Th}_i^\# \vdash \exists \mathcal{R}_1$ follows immediately from the statement $\exists \mathcal{I}_1$ and axiom schema of separation [4], (ii) follows immediately from countability of a set $\mathcal{I}_1$.

**Proposition 2.5.** A set $\mathcal{R}_1$ is inconsistent.

**Proof.** From formula (2.47) we obtain

\[
\text{Th}_i^\# \vdash \mathcal{R}_1 \in \mathcal{R}_1 \leftrightarrow [\mathcal{R}_1 \notin \mathcal{R}_1]^c \wedge [\text{Pr}_{\text{Th}_i^\#}([\mathcal{R}_1 \notin \mathcal{R}_1]^c) \therefore \mathcal{R}_1 \notin \mathcal{R}_1].
\]  

(2.48)

From (2.48) we obtain

\[
\text{Th}_i^\# \vdash \mathcal{R}_1 \in \mathcal{R}_1 \leftrightarrow \mathcal{R}_1 \notin \mathcal{R}_1
\]  

(2.49)

and therefore

\[
\text{Th}_i^\# \vdash (\mathcal{R}_1 \in \mathcal{R}_1) \wedge (\mathcal{R}_1 \notin \mathcal{R}_1).
\]  

(2.50)

But this is a contradiction.

**Definition 2.8.** Let $\Psi = \Psi(x)$ be one-place open $\text{Th}$-wff such that the following condition:

\[
\text{Th}_i^\# \vdash \exists x_\Psi[\Psi(x_\Psi)]
\]  

(2.51)

is satisfied.

**Remark 2.14.** We rewrite now the condition (2.51) using only language of the theory $\text{Th}_i^\#$:
\[ \{ \text{Th}_{i}^{f} \vdash \exists ! x_{\Psi}[\Psi(x_{\Psi})] \} \iff \text{Pr}_{\text{Th}_{i}^{f}}(\exists ! x_{\Psi}[\Psi(x_{\Psi})])^{c} \land \]
\[ \land \{ \text{Pr}_{\text{Th}_{i}^{f}}((\exists ! x_{\Psi}[\Psi(x_{\Psi})])^{c}) \implies \exists ! x_{\Psi}[\Psi(x_{\Psi})] \}. \]  

(2.52)

**Definition 2.9.** We will say that, a set \( y \) is a \( \text{Th}_{i}^{f} \)-set if there exist one-place open wff \( \Psi(x) \)

such that \( y = x_{\Psi} \). We write \( y[\text{Th}_{i}^{f}] \) iff \( y \) is a \( \text{Th}_{i}^{f} \)-set.

**Remark 2.15.** Note that

\[ y[\text{Th}_{i}^{f}] \iff \exists \Psi \{ (y = x_{\Psi}) \land \text{Pr}_{\text{Th}_{i}^{f}}((\exists ! x_{\Psi}[\Psi(x_{\Psi})])^{c}) \}
\]

\[ \{ \text{Pr}_{\text{Th}_{i}^{f}}((\exists ! x_{\Psi}[\Psi(x_{\Psi})])^{c}) \implies \exists ! x_{\Psi}[\Psi(x_{\Psi})] \}. \]  

(2.53)

**Definition 2.10.** Let \( \mathcal{I}_{i} \) be a collection such that:

\[ \forall x \{ x \in \mathcal{I}_{i} \iff x \text{ is a } \text{Th}_{i}^{f} \text{-set} \}. \]  

(2.54)

**Proposition 2.6.** Collection \( \mathcal{I}_{i} \) is a \( \text{Th}_{i}^{f} \)-set.

**Proof.** Let us consider an one-place open wff \( \Psi(x) \) such that conditions (2.51) is satisfied, i.e. \( \text{Th}_{i}^{f} \vdash \exists ! x_{\Psi}[\Psi(x_{\Psi})] \). We note that there exists countable collection \( \mathcal{F}_{\Psi} \) of the one-place open wff's \( \mathcal{F}_{\Psi} = \{ \psi(n) \}_{n \in \mathbb{N}} \) such that: (i) \( \Psi(x) \in \mathcal{F}_{\Psi} \) and (ii)

\[ \text{Th}_{i}^{f} \vdash \exists ! x_{\Psi}[\Psi(x_{\Psi})] \land \{ \forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_{n}(x_{\Psi})] \} \]

or in the equivalent form

\[ \text{Th}_{i}^{f} \vdash \text{Pr}_{\text{Th}_{i}^{f}}((\exists ! x_{\Psi}[\Psi(x_{\Psi})])^{c}) \land \]

\[ \{ \text{Pr}_{\text{Th}_{i}^{f}}((\exists ! x_{\Psi}[\Psi(x_{\Psi})])^{c}) \implies \exists ! x_{\Psi}[\Psi(x_{\Psi})] \} \land \]

\[ \text{Pr}_{\text{Th}_{i}^{f}}((\forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_{n}(x_{\Psi})])^{c}) \land \]

\[ \text{Pr}_{\text{Th}_{i}^{f}}((\forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_{n}(x_{\Psi})]^{c}) \implies \forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_{n}(x_{\Psi})] \} \]

(2.55)

or in the following equivalent form

\[ \text{Th}_{i}^{f} \vdash \exists ! x_{1}[\Psi_{1}(x_{1})] \land \{ \forall n(n \in \mathbb{N})[\Psi_{1}(x_{1}) \leftrightarrow \Psi_{n,1}(x_{1})] \} \]

or

\[ \text{Th}_{i}^{f} \vdash \]

\[ \text{Pr}_{\text{Th}_{i}^{f}}((\exists ! x_{1}[\Psi(x_{1})])^{c}) \land \]

\[ \{ \text{Pr}_{\text{Th}_{i}^{f}}((\exists ! x_{1}[\Psi(x_{1})])^{c}) \implies \exists ! x_{1}[\Psi(x_{1})] \} \land \]

\[ \text{Pr}_{\text{Th}_{i}^{f}}((\forall n(n \in \mathbb{N})[\Psi(x_{1}) \leftrightarrow \Psi_{n}(x_{1})])^{c}) \land \]

\[ \text{Pr}_{\text{Th}_{i}^{f}}((\forall n(n \in \mathbb{N})[\Psi(x_{1}) \leftrightarrow \Psi_{n}(x_{1})]^{c}) \implies \forall n(n \in \mathbb{N})[\Psi(x_{1}) \leftrightarrow \Psi_{n}(x_{1})] \}. \]  

(2.56)

where we have set \( \Psi(x) = \Psi_{1}(x_{1}), \Psi_{n}(x_{1}) = \Psi_{n,1}(x_{1}) \) and \( x_{\Psi} = x_{1} \). We note that any collection \( \mathcal{F}_{\Psi_{k}} = \{ \psi_{n,k}(x) \}_{n \in \mathbb{N}}, k = 1, 2, \ldots \) such mentioned above, defines an unique set \( x_{\Psi_{k}} \), i.e. \( \mathcal{F}_{\Psi_{k}} \cap \mathcal{F}_{\Psi_{k+1}} = \emptyset \) iff \( x_{\Psi_{k+1}} \neq x_{\Psi_{k}} \). We note that collections \( \mathcal{F}_{\Psi_{k}}, k = 1, 2, \ldots \) is not a
part of the $ZFC_2$, i.e. collection $\mathcal{F}_{\Psi_i}$ there is no set in sense of $ZFC_2$. However that is no problem, because by using G"odel numbering one can to replace any collection $\mathcal{F}_{\Psi_i}, k = 1, 2, \ldots$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_i})$ of the corresponding G"odel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_i}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \ldots. \quad (2.57)$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_i}), k = 1, 2, \ldots$ is a $\text{Th}^\#_i$-set. This is done by G"odel encoding [8],[10] (2.57), by the statement (2.51) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \ldots$ be a G"odel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}},$ where we have set $\mathcal{F}_k = \mathcal{F}_{\Psi_i}, k = 1, 2, \ldots$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \iff x_{k_1} \neq x_{k_2}]. \quad (2.58)$$

Let $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$ be a family of all the sets $\{g_{n,k}\}_{n \in \mathbb{N}}$. By axiom of choice [9] one obtains unique set $\mathcal{I}_i = \{g_k\}_{k \in \mathbb{N}}$ such that $\forall k [g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$. Finally one obtains a set $\mathcal{I}_i$ from the set $\mathcal{I}_i$ by axiom schema of replacement [9].

**Proposition 2.8.** Any collection $\Theta_k = g(\mathcal{F}_{\Psi_i}), k = 1, 2, \ldots$ is a $\text{Th}^\#_i$-set.

**Proof.** We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \text{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi_i(g_{n,k}, v_k)$

$$\Pi_i(g_{n,k}, v_k) \leftrightarrow \text{Pr}_{\text{Th}^\#_i}([\exists! x_k[\Psi_{1,k}(x_k)]]^c) \land$$

$$\land \exists! x_k(v_k = [x_k]^c)[\forall n(n \in \mathbb{N})[\text{Pr}_{\text{Th}^\#_i}([\Psi_{1,k}(x_k)]^c) \leftrightarrow \text{Pr}_{\text{Th}^\#_i}(\text{Fr}(g_{n,k}, v_k))]]. \quad (2.59)$$

We define now a set $\Theta_k$ such that

$$\Theta_k = \Theta_k' \cup \{g_k\},$$

$$\forall n(n \in \mathbb{N})[g_{n,k} \in \Theta_k' \leftrightarrow \Pi_i(g_{n,k}, v_k)]. \quad (2.60)$$

Obviously definitions (2.55) and (2.60) is equivalent.

**Definition 2.11.** We define now the following $\text{Th}^\#_i$-set $\mathcal{R}_i \subseteq \mathcal{I}_i$ :

$$\forall x[x \in \mathcal{R}_i \iff (x \in \mathcal{I}_i) \land \text{Pr}_{\text{Th}^\#_i}([x \not\in x]^c) \land \text{Pr}_{\text{Th}^\#_i}([x \not\in x]^c) \Rightarrow x \not\in x]. \quad (2.61)$$

**Proposition 2.9.** (i) $\text{Th}^\#_i \vdash \exists \mathcal{R}_i$, (ii) $\mathcal{R}_i$ is a countable $\text{Th}^\#_i$-set, $i \in \mathbb{N}$.

**Proof.** (i) Statement $\text{Th}^\#_i \vdash \exists \mathcal{R}_i$ follows immediately by using statement $\exists \mathcal{I}_i$ and axiom schema of separation [4], (ii) follows immediately from countability of a set $\mathcal{I}_i$.

**Proposition 2.10.** Any set $\mathcal{R}_i, i \in \mathbb{N}$ is inconsistent.

**Proof.** From formula (2.61) we obtain

$$\text{Th}^\#_i \vdash \mathcal{R}_i \in \mathcal{R}_i \iff \text{Pr}_{\text{Th}^\#_i}([\mathcal{R}_i \not\in \mathcal{R}_i]^c) \land \text{Pr}_{\text{Th}^\#_i}([\mathcal{R}_i \not\in \mathcal{R}_i]^c) \Rightarrow \mathcal{R}_i \not\in \mathcal{R}_i]. \quad (2.62)$$

From (2.62) we obtain

$$\text{Th}^\#_i \vdash \mathcal{R}_i \in \mathcal{R}_i \iff \mathcal{R}_i \not\in \mathcal{R}_i \quad (2.63)$$

and therefore
\( \text{Th}_i^\# \vdash (\mathcal{R}_i \in \mathcal{R}_i) \land (\mathcal{R}_i \not\in \mathcal{R}_i). \)  \hspace{1cm} (2.64)

But this is a contradiction.

**Definition 2.12.** An \( \text{Th}_x^\# \)-wff \( \Phi_x \) that is: (i) \( \text{Th} \)-wff \( \Phi \) or (ii) well-formed formula \( \Phi_x \) which contains predicate \( \text{Pr}_{\text{Th}_x^\#}([\Phi]^c) \) given by formula (2.35). An \( \text{Th}_x^\# \)-wff \( \Phi_x \) (well-formed formula \( \Phi_x \)) is closed - i.e. \( \Phi \) is a sentence - if it has no free variables; a wff is open if it has free variables.

**Definition 2.13.** Let \( \Psi = \Psi(x) \) be one-place open \( \text{Th}_x^\# \)-wff such that the following condition:

\[
\text{Th}_x^\# \vdash \exists x \Psi([x]) \tag{2.65}
\]

is satisfied.

**Remark 2.16.** We rewrite now the condition (2.65) using only language of the theory \( \text{Th}_x^\# \):

\[
\{ \text{Th}_x^\# \vdash \exists x \Psi([x]) \} \iff \text{Pr}_{\text{Th}_x^\#}([\exists x \Psi([x])]^c) \land \\
\text{Pr}_{\text{Th}_x^\#}([\exists x \Psi([x])]^c) \Rightarrow \exists x \Psi([x]). \tag{2.66}
\]

**Definition 2.14.** We will say that, a set \( y \) is a \( \text{Th}_x^\# \)-set if there exist one-place open wff \( \Psi(x) \) such that \( y = x \). We write \( y[\text{Th}_x^\#] \) iff \( y \) is a \( \text{Th}_x^\# \)-set.

**Definition 2.15.** Let \( \mathcal{X}_x \) be a collection such that: \( \forall x \left[ x \in \mathcal{X}_x \iff x \text{ is a } \text{Th}_x^\# \text{-set} \right] \).

**Proposition 2.11.** Collection \( \mathcal{X}_x \) is a \( \text{Th}_x^\# \)-set.

**Proof.** Let us consider an one-place open wff \( \Psi(x) \) such that condition (2.65) is satisfied, i.e. \( \text{Th}_x^\# \vdash \exists x \Psi([x]) \). We note that there exists countable collection \( \mathcal{F}_x \) of the one-place open wff’s \( \mathcal{F}_x = \{ \Psi_n(x) \}_{n \in \mathbb{N}} \) such that: (i) \( \Psi(x) \in \mathcal{F}_x \) and (ii)

\[
\text{Th}_x^\# \vdash \exists x \Psi([\Psi(x)]) \land \{ \forall n (n \in \mathbb{N})[\Psi(x) \leftrightarrow \Psi_n(x)] \} \]

or in the equivalent form

\[
\text{Th}_x^\# \vdash \text{Pr}_{\text{Th}_x^\#}([\exists x \Psi([x])]^c) \land \\
\text{Pr}_{\text{Th}_x^\#}([\exists x \Psi([x])]^c) \Rightarrow \exists x \Psi([x]) \land \\
[\text{Pr}_{\text{Th}_x^\#}([\forall n (n \in \mathbb{N})[\Psi(x) \leftrightarrow \Psi_n(x)]])^c] \land \\
\text{Pr}_{\text{Th}_x^\#}([\forall n (n \in \mathbb{N})[\Psi(x) \leftrightarrow \Psi_n(x)]])^c) \Rightarrow \forall n (n \in \mathbb{N})[\Psi(x) \leftrightarrow \Psi_n(x)] \]  \hspace{1cm} (2.67)

or in the following equivalent form
Let \( \mathcal{F} \) be a family of the all sets \( n \in \mathbb{N} \). We define now a set \( \Theta_k = g(\mathcal{F}_{\psi_k}) \) of the corresponding Gödel numbers such that

\[
\forall k \in \mathbb{N}, k = 1, 2, \ldots
\]

It is easy to prove that any collection \( \Theta_k = g(\mathcal{F}_{\psi_k}), k = 1, 2, \ldots \) is a Th\( ^{\#} \)-set. This is done by Gödel encoding [8],[10] by the statement (2.66) and by axiom schema of separation [9]. Let \( g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \ldots \) be a Gödel number of the wff \( \Psi_{n,k}(x_k) \).

Therefore \( g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}} \), where we have set \( \mathcal{F}_k = \mathcal{F}_{\psi_k}, k = 1, 2, \ldots \) and

\[
\forall k \in \mathbb{N} \exists k \in \mathbb{N} \exists k \in \mathbb{N} \bigcap \{g_{n,k}\}_{n \in \mathbb{N}} = \varnothing \leftrightarrow x_k \neq x_k.
\]

Let \( \{g_{n,k}\}_{n \in \mathbb{N}} \) be a family of the all sets \( \{g_{n,k}\}_{n \in \mathbb{N}} \). By axiom of choice [9] one obtains unique set \( \mathcal{I} = \{g_k\}_{k \in \mathbb{N}} \) such that \( \forall k \in \mathbb{N} \in \{g_{n,k}\}_{n \in \mathbb{N}} \). Finally one obtains a set \( \mathcal{I}_\infty \) from the set \( \mathcal{I}_\infty \) by axiom schema of replacement [9]. Thus one can define Th\( ^{\#} \)-set

\[
\mathcal{R}_\infty \subseteq \mathcal{I}_\infty : \quad \forall x [x \in \mathcal{R}_\infty \iff (x \in \mathcal{I} \land \text{Pr}_{\text{Th}_\infty}([x \neq x]^c)]].
\]

**Proposition 2.12.** Any collection \( \Theta_k = g(\mathcal{F}_{\psi_k}), k = 1, 2, \ldots \) is a Th\( ^{\#} \)-set.

**Proof.** We define \( g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c \). Therefore \( g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \text{Fr}(g_{n,k}, v_k) \) (see [10]). Let us define now predicate \( \Pi(\mathcal{I}, g_{n,k}, v_k) \)

\[
\Pi(\mathcal{I}, g_{n,k}, v_k) \leftrightarrow \text{Pr}_{\text{Th}_\infty}([\exists x_k(\Psi_{1,k}(x_k)]^c)]^c) \land
\]

\[
\forall x_k(v_k = [x_k]^c) \forall n \in \mathbb{N} \{\text{Pr}_{\text{Th}_\infty}([\Psi_{1,k}(x_k)]^c) \leftrightarrow \text{Pr}_{\text{Th}_\infty}([\text{Fr}(g_{n,k}, v_k)])].
\]

We define now a set \( \Theta_k \) such that

\[
\Theta_k = \Theta_k \cup \{g_k\},
\]

\( \forall n \in \mathbb{N} \{g_{n,k} \in \Theta_k \leftrightarrow \Pi(\mathcal{I}, g_{n,k}, v_k) \}
\]

(2.73)

Obviously definitions (2.66) and (2.73) is equivalent by Proposition 2.1.
Proposition 2.13. (i) $\text{Th}_\# \vdash \exists R_x, (ii) R_x$ is a countable $\text{Th}_\#^*$-set.
Proof. (i) Statement $\text{Th}_\# \vdash \exists R_c$ follows immediately from the statement $\exists \exists$ and axiom schema of separation [9], (ii) follows immediately from countability of the set $\exists \infty$.

Proposition 2.14. Set $R_x$ is inconsistent.
Proof. From the formula (2.71) we obtain
$$\text{Th}_\# \vdash R_x \in R_x \leftrightarrow \Pr_{\text{Th}_\#}(\exists R_x \not\in R_x)^c).$$
From the formula (2.74) and Proposition 2.1 one obtain
$$\text{Th}_\# \vdash R_x \in R_x \leftrightarrow R_x \not\in R_x$$
and therefore
$$\text{Th}_\# \vdash (R_x \in R_x) \land (R_x \not\in R_x).$$
But this is a contradiction.

Definition 2.16. An $\text{Th}_{\#1}$-wff $\Phi_{\#1}$ that is: (i) $\text{Th}_{\#1}$-wff $\Phi_{\#1}$ or (ii) well-formed formula which contains predicate $\Pr_{\text{Th}_{\#1}}(\exists \Phi)^c$ given by formula (2.36). An $\text{Th}_{\#1}$-wff $\Phi_{\#1}$ (well-formed formula $\Phi_{\#1}$) is closed - i.e. $\Phi_{\#1}$ is a sentence - if it has no free variables; a wff is open if it has free variables.

Definition 2.17. Let $\Psi = \Psi(x)$ be one-place open $\text{Th}$-wff such that the following condition:
$$\text{Th}_\# \triangleq \text{Th}_\# \vdash \exists x \Psi(x)$$
is satisfied.
Remark 2.17. We rewrite now the condition (2.77) using only language of the theory $\text{Th}_{\#1}$:
$$\{\text{Th}_{\#1} \vdash \exists x \Psi(x)\} \iff \Pr_{\text{Th}_{\#1}}(\exists \exists x \Psi(x))^{c}.$$

Definition 2.18. We will say that, a set $y$ is a $\text{Th}_{\#1}$-set if there exist one-place open wff $\Psi(x)$ such that $y = x \Psi$. We write $y[\text{Th}_{\#1}]$ iff $y$ is a $\text{Th}_{\#1}$-set.

Remark 2.18. Note that
$$y[\text{Th}_{\#1}] \iff \exists \Psi \{ (y = x \Psi) \land \Pr_{\text{Th}_{\#1}}(\exists x \Psi(x)^c)$$
$$\Pr_{\text{Th}_{\#1}}(\exists x \Psi(x)^c) \supset \exists x \Psi(x)\}.$$
Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions (2.37) is satisfied, i.e. $Th^{\#}_{a,1} \vdash \exists!x\Psi[\Psi(x)]$. We note that there exists countable collection $F_{\Psi}$ of the one-place open wff's $F_{\Psi} = \{\Psi_n(x)\}_{n \in \mathbb{N}}$ such that: (i) $\Psi(x) \in F_{\Psi}$ and (ii)

$$Th^{\#}_{a,1} \vdash \exists!x[\Psi(x)] \land \forall n \in \mathbb{N}[\Psi(x) \leftrightarrow \Psi_n(x)]$$

or in the equivalent form

$$Th^{\#}_{a,1} \vdash Pr_{Th^{\#}_{a,1}}([\exists!x[\Psi(x)]]^c) \land

\left[ Pr_{Th^{\#}_{a,1}}([\forall n \in \mathbb{N}[\Psi(x) \leftrightarrow \Psi_n(x)]]^c) \right].$$

or in the following equivalent form

$$Th^{\#}_{a,1} \vdash \exists x_1[[\Psi_1(x_1)] \land \forall n \in \mathbb{N}[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]$$

or

$$Th^{\#}_{a,1} \vdash Pr_{Th^{\#}_{a,1}}([\exists!x_1[\Psi_1(x_1)]]^c) \land

\left[ Pr_{Th^{\#}_{a,1}}([\forall n \in \mathbb{N}[\Psi_1(x) \leftrightarrow \Psi_{n,1}(x_1)]]^c) \right],$$

where we have set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_{\Psi} = x_1$. We note that any collection $F_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}, k = 1, 2, ...$ such mentioned above, defines an unique set $x_{\Psi_k}$, i.e. $F_{\Psi_{k_1}} \cap F_{\Psi_{k_2}} = \emptyset$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$. We note that collections $F_{\Psi_k}, k = 1, 2, ...$ is not a part of the ZFC_2, i.e. collection $F_{\Psi_k}$ there is no set in sense of ZFC_2. However that is no problem, because by using Gödel numbering one can to replace any collection $F_{\Psi_k}, k = 1, 2, ...$ by collection $\Theta_k = g(F_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(F_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, ...$$

It is easy to prove that any collection $\Theta_k = g(F_{\Psi_k}), k = 1, 2, ...$ is a $Th^{\#}_{a,1}$-set. This is done by Gödel encoding [8],[10] (2.83), by the statement (2.81) and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,1}(x_k)), k = 1, 2, ...$ be a Gödel number of the wff $\Psi_{n,1}(x_k)$. Therefore

$$g(F_{\Psi_k}) = \{g_{n,k}\}_{n \in \mathbb{N}},$$

where we have set $F_k = F_{\Psi_k}, k = 1, 2, ...$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \leftrightarrow x_{k_1} \neq x_{k_2}].$$

Let $\{g_{n,k}\}_{n \in \mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n \in \mathbb{N}}$. By axiom of choice [9] one obtains unique set $\mathcal{A} = \{g_{k}\}_{k \in \mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}].$ Finally one obtains a set $\mathcal{A}_{a,1}$ from the set $\mathcal{A}$ by axiom schema of replacement [9].

**Proposition 2.16.** Any collection $\Theta_k = g(F_{\Psi_k}), k = 1, 2, ...$ is a $Th^{\#}_{a,1}$-set.

**Proof.** We define $g_{n,k} = g(\Psi_{n,1}(x_k)) = [\Psi_{n,1}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,1}(x_k)) \leftrightarrow Fr(g_{n,k},v_k)$ (see [10]). Let us define now predicate $\Pi(g_{n,k},v_k)$
\[ \Pi(g_{n,k}, v_k) \leftrightarrow \text{Pr}_{\text{Th}_{n,k}^{\#}}([\exists!x_k[\Psi_{1,k}(x_1)]]^c) \land \exists x_k (v_k = [x_k]^c)[\forall n(n \in \mathbb{N})[\text{Pr}_{\text{Th}_{n,k}^{\#}}([[[\Psi_{1,k}(x_1)]]^c) \leftrightarrow \text{Pr}_{\text{Th}_{n,k}^{\#}}(\text{Fr}(g_{n,k}, v_k))]]. \] (2.85)

We define now a set \( \Theta_k \) such that

\[
\Theta_k = \Theta_k' \cup \{g_k\}, \\
\forall n(n \in \mathbb{N})[g_{n,k} \in \Theta_k' \leftrightarrow \Pi(g_{n,k}, v_k)]
\] (2.86)

Obviously definitions (2.81) and (2.86) is equivalent.

**Definition 2.20.** We define now the following \( \text{Th}_{\omega,1}^{\#} \)-set \( R_{\omega,1} \subseteq \mathbb{I}_{\omega,1} : \)

\[
\forall x[x \in R_{\omega,1} \leftrightarrow (x \in \mathbb{I}_{\omega,1}) \land \text{Pr}_{\text{Th}_{\omega,1}^{\#}}([x \notin x]^c)].
\] (2.87)

**Proposition 2.17.** (i) \( \text{Th}_{\omega,1}^{\#} \vdash \exists R_{\omega,1} \), (ii) \( R_{\omega,1} \) is a countable \( \text{Th}_{\omega,1}^{\#} \)-set.

**Proof.** (i) Statement \( \text{Th}_{\omega,1}^{\#} \vdash \exists R_{\omega,1} \) follows immediately from the statement \( \exists \mathbb{I}_{\omega,1} \) and axiom schema of separation [4], (ii) follows immediately from countability of a set \( \mathbb{I}_{\omega,1} \).

**Proposition 2.18.** A set \( R_{\omega,1} \) is inconsistent.

**Proof.** From formula (2.87) we obtain

\[ \text{Th}_{\omega,1}^{\#} \vdash R_{\omega,1} \in R_{\omega,1} \leftrightarrow \text{Pr}_{\text{Th}_{\omega,1}^{\#}}([R_{\omega,1} \notin R_{\omega,1}]^c). \] (2.88)

From (2.88) we obtain

\[ \text{Th}_{\omega,1}^{\#} \vdash R_{\omega,1} \in R_{\omega,1} \leftrightarrow R_{\omega,1} \notin R_{\omega,1} \] (2.89)

and therefore

\[ \text{Th}_{\omega,1}^{\#} \vdash (R_{\omega,1} \in R_{\omega,1}) \land (R_{\omega,1} \notin R_{\omega,1}). \] (2.90)

But this is a contradiction.

**Definition 2.21.** Let \( \Psi = \Psi(x) \) be one-place open \( \text{Th} \)-wff such that the following condition:

\[ \text{Th}_{\omega,1}^{\#} \vdash \exists!x_\Psi[\Psi(x_\Psi)] \] (2.91)

is satisfied.

**Remark 2.19.** We rewrite now the condition (2.91) using only language of the theory \( \text{Th}_{\omega,1}^{\#} : \)

\[ \{\text{Th}_{\omega,1}^{\#} \vdash \exists!x_\Psi[\Psi(x_\Psi)]\} \leftrightarrow \text{Pr}_{\text{Th}_{\omega,1}^{\#}}([\exists!x_\Psi[\Psi(x_\Psi)]]^c). \] (2.92)

**Definition 2.22.** We will say that, a set \( y \) is a \( \text{Th}_{\omega,1}^{\#} \)-set if there exist one-place open wff \( \Psi(x) \) such that \( y = x_\Psi \). We write \( y[\text{Th}_{\omega,1}^{\#}] \) iff \( y \) is a \( \text{Th}_{\omega,1}^{\#} \)-set.

**Remark 2.20.** Note that
\[ y[\text{Th}_{\omega,i}^\#] \Leftrightarrow \exists \Psi \left( (y = x_{\Psi}) \land \Pr_{\text{Th}_{\omega,i}^\#}([\exists ! x_{\Psi}[\Psi(x_{\Psi})]]^c) \right). \] (2.93)

**Definition 2.23.** Let \( \mathfrak{I}_{\omega,i} \) be a collection such that:
\[ \forall x \left[ x \in \mathfrak{I}_{\omega,i} \iff x \text{ is a } \text{Th}_{\omega,i}^\# \text{-set} \right]. \] (2.94)

**Proposition 2.19.** Collection \( \mathfrak{I}_{\omega,i} \) is a \( \text{Th}_{\omega,i}^\# \)-set.

**Proof.** Let us consider an one-place open wff \( \Psi(x) \) such that conditions (2.91) is satisfied, i.e. \( \text{Th}_{\omega,i}^\# \vdash \exists ! x_{\Psi}[\Psi(x_{\Psi})] \). We note that there exists countable collection \( \mathcal{F}_\Psi \) of the one-place open wff’s \( \mathcal{F}_\Psi = \{ \Psi_n(x) \}_{n \in \mathbb{N}} \) such that: (i) \( \Psi(x) \in \mathcal{F}_\Psi \) and (ii)
\[
\text{Th}_{\omega,i}^\# \vdash \exists ! x_{\Psi}[[\Psi(x_{\Psi})] \land \{ \forall n \in \mathbb{N}[\Psi(x_{\Psi}) \iff \Psi_n(x_{\Psi})] \}]
\]
or in the equivalent form
\[
\text{Th}_{\omega,i}^\# \vdash \Pr_{\text{Th}_{\omega,i}^\#}([\exists ! x_{\Psi}[\Psi(x_{\Psi})]]^c) \land
\left[ \Pr_{\text{Th}_{\omega,i}^\#}([\forall n \in \mathbb{N}[\Psi(x_{\Psi}) \iff \Psi_n(x_{\Psi})]]^c) \right],
\]
or in the following equivalent form
\[
\text{Th}_{\omega,i}^\# \vdash \exists ! x_1[[\Psi_1(x_1)] \land \{ \forall n \in \mathbb{N}[\Psi_1(x_1) \iff \Psi_n(x_1)] \}]
\]
or
\[
\text{Th}_{\omega,i}^\# \vdash \Pr_{\text{Th}_{\omega,i}^\#}([\exists ! x_1[\Psi_1(x_1)]]^c) \land
\left[ \Pr_{\text{Th}_{\omega,i}^\#}([\forall n \in \mathbb{N}[\Psi_1(x_1) \iff \Psi_n(x_1)]]]^c) \right].
\]
where we have set \( \Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1) \) and \( x_{\Psi} = x_1 \). We note that any collection \( \mathcal{F}_{\Psi_k} = \{ \Psi_{n,k}(x) \}_{n \in \mathbb{N}, k = 1, 2, \ldots} \) such mentioned above, defines an unique set \( x_{\Psi_k}, \) i.e. \( \mathcal{F}_{\Psi_1} \cap \mathcal{F}_{\Psi_2} = \emptyset \) if \( x_{\Psi_1} \neq x_{\Psi_2} \). We note that collections \( \mathcal{F}_{\Psi_k}, k = 1, 2, \ldots \) is not a part of the \( \text{ZFC}_2 \), i.e. collection \( \mathcal{F}_{\Psi_k} \) there is no set in sense of \( \text{ZFC}_2 \). However that is no problem, because by using Gödel numbering one can to replace any collection \( \mathcal{F}_{\Psi_k}, k = 1, 2, \ldots \) by collection \( \Theta_k = g(\mathcal{F}_{\Psi_k}) \) of the corresponding Gödel numbers such that
\[
\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{ g(\Psi_{n,k}(x_k)) \}_{y \in \mathbb{N}}, k = 1, 2, \ldots. \] (2.97)

It is easy to prove that any collection \( \Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \ldots \) is a \( \text{Th}_{\omega,i}^\# \)-set. This is done by Gödel encoding [8],[10] (2.97), by the statement (2.91) and by axiom schema of separation [9]. Let \( g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \ldots \) be a Gödel number of the wff \( \Psi_{n,k}(x_k) \). Therefore \( g(\mathcal{F}_k) = \{ g_{n,k} \}_{n \in \mathbb{N}} \) where we have set \( \mathcal{F}_k = \mathcal{F}_{\Psi_k}, k = 1, 2, \ldots \) and
\[
\forall k_1 \forall k_2 \{ [g_{n,k_1}]_{n \in \mathbb{N}} \cap \{ g_{n,k_2} \}_{n \in \mathbb{N}} = \emptyset \iff x_{k_1} \neq x_{k_2} \}.
\] (2.98)

Let \( \{ g_{n,k} \}_{n \in \mathbb{N}} \) be a family of the all sets \( \{ g_{n,k} \}_{n \in \mathbb{N}} \). By axiom of choice [9] one obtains unique set \( \mathfrak{I}_i = \{ g_k \}_{k \in \mathbb{N}} \) such that \( \forall k [g_k \in \{ g_{n,k} \}_{n \in \mathbb{N}}]. \) Finally one obtains a set
\( \exists_{a,i} \) from the set \( \exists_i' \) by axiom schema of replacement [9].

**Proposition 2.20.** Any collection \( \Theta_k = g(\mathcal{F}_k), k = 1, 2, \ldots \) is a \( \Theta_{a,i} \)-set.

**Proof.** We define \( g_{n,k} = g(\Psi_{n,k}(x)) = [\Psi_{n,k}(x)]^c, v_k = [x_k]^c \). Therefore \( g_{n,k} = g(\Psi_{n,k}(x)) \leftrightarrow \Pr(g_{n,k}, v_k) \) (see [10]). Let us define now predicate \( \Pi_{a,i}(g_{n,k}, v_k) \)

\[
\Pi_{a,i}(g_{n,k}, v_k) \leftrightarrow \Pr_{\Theta_{a,i}}(\exists x_k [\Psi_{1,1}(x_1)])^c \land \\
\exists x_k (v_k = [x_k]^c)[\forall n (n \in \mathbb{N})[\Pr_{\Theta_{a,i}}([\exists x_k [\Psi_{1,1}(x_k)])^c \leftrightarrow \Pr_{\Theta_{a,i}}(\Pr(g_{n,k}, v_k))]].
\]

(2.99)

We define now a set \( \Theta_k \) such that

\[
\Theta_k = \Theta_k' \cup \{ g_k \}, \quad \forall n (n \in \mathbb{N})[g_{n,k} \in \Theta_k' \leftrightarrow \Pi_{a,i}(g_{n,k}, v_k)].
\]

(2.100)

Obviously definitions (2.91) and (2.100) is equivalent.

**Definition 2.24.** We define now the following \( \Theta_{a,i} \)-set \( \mathcal{R}_{a,i} \subseteq \exists_{a,i} : \)

\[
\forall x [x \in \mathcal{R}_{a,i} \leftrightarrow (x \in \exists_{a,i}) \land \Pr_{\Theta_{a,i}}([x \not\in x]^c)].
\]

(2.101)

**Proposition 2.21.** (i) \( \Theta_{a,i} \vdash \exists \mathcal{R}_{a,i} \), (ii) \( \mathcal{R}_{a,i} \) is a countable \( \Theta_{a,i} \)-set, \( i \in \mathbb{N} \).

**Proof.** (i) Statement \( \Theta_{a,i} \vdash \exists \mathcal{R}_{a,i} \) follows immediately by using statement \( \exists \exists_{a,i} \) and axiom schema of separation [9]. (ii) follows immediately from countability of a set \( \exists_{a,i} \).

**Proposition 2.22.** Any set \( \mathcal{R}_{a,i}, i \in \mathbb{N} \) is inconsistent.

**Proof.** From formula (2.101) we obtain

\[
\Theta_{a,i} \vdash \mathcal{R}_{a,i} \in \mathcal{R}_{a,i} \leftrightarrow \Pr_{\Theta_{a,i}}([\mathcal{R}_{a,i} \not\in \mathcal{R}_{a,i}]^c).
\]

(2.102)

From (2.102) we obtain

\[
\Theta_{a,i} \vdash \mathcal{R}_{a,i} \in \mathcal{R}_{a,i} \leftrightarrow \mathcal{R}_{a,i} \not\in \mathcal{R}_{a,i}
\]

(2.103)

and therefore

\[
\Theta_{a,i} \vdash (\mathcal{R}_{a,i} \in \mathcal{R}_{a,i}) \land (\mathcal{R}_{a,i} \not\in \mathcal{R}_{a,i}.
\]

(2.104)

But this is a contradiction.

**Definition 2.25.** Let \( \Psi = \Psi(x) \) be one-place open \( \Theta_{x,\omega} \)-wff such that the following condition:

\[
\Theta_{x,\omega} \vdash \exists x_\Psi [\Psi(x_\Psi)]
\]

(2.105)

is satisfied.

**Remark 2.20.** We rewrite now the condition (2.65) using only language of the theory \( \Theta_{x,\omega} : \)

\[
\{ \Theta_{x,\omega} \vdash \exists x_\Psi [\Psi(x_\Psi)] \} \leftrightarrow \Pr_{\Theta_{x,\omega}}([\exists x_\Psi [\Psi(x_\Psi)]^c]
\]

(2.106)

**Definition 2.26.** We will say that, a set \( y \) is a \( \Theta_{x,\omega} \)-set if there exist one-place open wff
\[\Psi(x)\text{ such that } y = x_\Psi. \text{ We write } y[\text{Th}_{x_{\Psi}}] \text{ iff } y \text{ is a } \text{Th}_{x_{\Psi}}^\# \text{-set.}\]

**Definition 2.27.** Let \(\mathcal{J}_{x_{\Psi}}\) be a collection such that

\[\forall x \left( x \in \mathcal{J}_{x_{\Psi}} \iff x \text{ is a } \text{Th}_{x_{\Psi}}^\# \text{-set} \right).\]

**Proposition 2.23.** Collection \(\mathcal{J}_{x_{\Psi}}\) is a \(\text{Th}_{x_{\Psi}}^\#\)-set.

**Proof.** Let us consider an one-place open wff \(\Psi(x)\) such that condition (2.65) is satisfied, i.e. \(\text{Th}_{x_{\Psi}}^\# \vdash \exists! x_\Psi[\Psi(x_\Psi)]\). We note that there exists countable collection \(\mathcal{F}_\Psi\) of the one-place open wff's \(\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n \in \mathbb{N}}\) such that: (i) \(\Psi(x) \in \mathcal{F}_\Psi\) and (ii)

\[\text{Th}_{x_{\Psi}}^\# \vdash \exists! x_\Psi[[\Psi(x_\Psi)] \land (\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)])] \]

or in the equivalent form

\[\text{Th}_{x_{\Psi}}^\# \vdash \exists! x_\Psi[[\Psi(x_\Psi)] \land (\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)])] \cong (2.107)\]

or in the following equivalent form

\[\text{Th}_{x_{\Psi}}^\# \vdash \exists! x_\Psi[[\Psi_1(1.1)] \land (\forall n(n \in \mathbb{N})[\Psi_1(1.1) \leftrightarrow \Psi_n(1.1)])] \]

or

\[\text{Th}_{x_{\Psi}}^\# \vdash \exists! x_\Psi[[\Psi_1(1.1)] \land (\forall n(n \in \mathbb{N})[\Psi_1(1.1) \leftrightarrow \Psi_n(1.1)])] \cong (2.108)\]

where we set \(\Psi(x) = \Psi_1(1.1), \Psi_n(x_1) = \Psi_{n,1}(x_1)\) and \(x_\Psi = x_1\). We note that any collection \(\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}, k = 1, 2, \ldots\) such above defines an unique set \(x_{\Psi_k}\), i.e. \(\mathcal{F}_{\Psi_k} \bigcap \mathcal{F}_{\Psi_1} = \emptyset \text{ iff } x_{\Psi_k} \neq x_{\Psi_1}\). We note that collections \(\mathcal{F}_{\Psi_k}, k = 1, 2, \ldots\) is no part of the \(\text{ZFC}_2\), i.e. collection \(\mathcal{F}_{\Psi_k}\) there is no set in sense of \(\text{ZFC}_2\). However that is no problem, because by using Gödel numbering one can to replace any collection \(\mathcal{F}_{\Psi_k}, k = 1, 2, \ldots\) by collection \(\Theta_k = g(\mathcal{F}_{\Psi_k})\) of the corresponding Gödel numbers such that

\[\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \ldots\] \(\cong (2.109)\)

It is easy to prove that any collection \(\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \ldots\) is a \(\text{Th}_{x_{\Psi}}^\#\)-set. This is done by Gödel encoding [8],[10] by the statement (2.109) and by axiom schema of separation [9]. Let \(g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \ldots\) be a Gödel number of the wff \(\Psi_{n,k}(x_k)\). Therefore \(g(\mathcal{F}_{k}) = \{g_{n,k}\}_{n \in \mathbb{N}},\) where we have set \(\mathcal{F}_{k} = \mathcal{F}_{\Psi_k}, k = 1, 2, \ldots\) and

\[\forall k_1 \forall k_2 \left[\{g_{n,k_1}\}_{n \in \mathbb{N}} \bigcap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \iff x_{k_1} \neq x_{k_2}\right]. \cong (2.110)\]

Let \(\{g_{n,k}\}_{n \in \mathbb{N}}\) be a family of the all sets \(\{g_{n,k}\}_{n \in \mathbb{N}}\). By axiom of choice [9] one obtains unique set \(\mathcal{J}' = \{g_{k}\}_{k \in \mathbb{N}}\) such that \(\forall k[g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]\). Finally one obtains a set \(\mathcal{J}'_{x_{\Psi}}\) from the set \(\mathcal{J}'_{x_{\Psi}}\) by axiom schema of replacement [9]. Thus one can define

\(\text{Th}_{x_{\Psi}}^\#\)-set \(\mathcal{J}'_{x_{\Psi}} \subseteq \mathcal{J}_{x_{\Psi}}:\)
∀x[x ∈ ℜ_{x_{c_{0}}} ⇔ (x ∈ ℐ_x) ∧ Pr_{Th_{x_{c_{0}}}}^+([x ≠ x]^c)].

(2.111)

**Proposition 2.24.** Any collection $Θ_k = g(ℱ_{Ψ_k})$, $k = 1, 2, \ldots$ is a $Th_{x_{c_{0}}}^#$-set.

**Proof.** We define $g_{n,k} = g(Ψ_{n,k}(x_k)) = [Ψ_{n,k}(x_k)]^c$, $v_k = [x_k]^c$. Therefore $g_{n,k} = g(Ψ_{n,k}(x_k)) \iff Fr(g_{n,k}, ν_k)$ (see [10]). Let us define now predicate $Π_{x_{c_{0}}}(g_{n,k}, v_k)$

$$Π_{x_{c_{0}}}(g_{n,k}, v_k) \iff Pr_{Th_{x_{c_{0}}}^+}^+(∃!x_k[Ψ_{1,k}(x_k)]^c) \land$$

$$∃!x_k(v_k = [x_k]^c)[∀n(n ∈ ℤ)[Pr_{Th_{x_{c_{0}}}^+}^+(([Ψ_{1,k}(x_k)]^c) \iff Pr_{Th_{x_{c_{0}}}^+}^+(Fr(g_{n,k}, v_k))]]].

(2.112)

We define now a set $Θ_k$ such that

$$Θ_k = Θ_k \cup \{g_k\}, \forall n(n ∈ ℤ)[g_{n,k} ∈ Θ_k ⇔ Π_{x_{c_{0}}}(g_{n,k}, v_k)]

(2.113)

Obviously definitions (2.106) and (2.113) is equivalent by Proposition 2.1.

**Proposition 2.25.** (i) $Th_{x_{c_{0}}}^# \vdash ∃ ℜ_{x_{c_{0}}}$, (ii) $ℜ_{x_{c_{0}}}$ is a countable $Th_{x_{c_{0}}}^#$-set.

**Proof.** (i) Statement $Th_{x_{c_{0}}}^# \vdash ∃ ℜ_{x_{c_{0}}}$ follows immediately from the statement $∃Θ$ and axiom of schema of separation [9], (ii) follows immediately from countability of the set $ℐ_θ$.

**Proposition 2.26.** Set $ℜ_{x_{c_{0}}}$ is inconsistent.

**Proof.** From the formula (2.71) we obtain

$$Th_{x_{c_{0}}}^# \vdash ℜ_{x_{c_{0}}} \in ℜ_{x_{c_{0}}} \iff Pr_{Th_{x_{c_{0}}}^+}^+(([ℜ_{x_{c_{0}}} \not\in ℜ_{x_{c_{0}}}]^c)).

(2.114)

From the formula (2.114) and Proposition 2.1 we obtain

$$Th_{x_{c_{0}}}^# \vdash ℜ_{x_{c_{0}}} \in ℜ_{x_{c_{0}}} \iff ℜ_{x_{c_{0}}} \not\in ℜ_{x_{c_{0}}}

(2.115)

and therefore

$$Th_{x_{c_{0}}}^# \vdash (ℜ_{x_{c_{0}}} \in ℜ_{x_{c_{0}}}) \land (ℜ_{x_{c_{0}}} \not\in ℜ_{x_{c_{0}}}).

(2.116)

But this is a contradiction.

**Proposition 2.26.** Assume that (i) $Con(Th)$ and (ii) $Th$ has a nonstandard model $M_{Th}^{Th}$ and $M_{Th}^{Th} \subset M_{Th}^{Th}$. Then theory $Th$ can be extended to a maximally consistent nice theory $Th_{x_{c_{0}}}^\# \equiv Th_{x_{c_{0}}}^\# [M_{Th}^{Th}]$.

**Proof.** Let $Φ_1, \ldots Φ_i, \ldots$ be an enumeration of all wff’s of the theory $Th$ (this can be achieved if the set of propositional variables can be enumerated). Define a chain $Θ = \{Θ_{Nst,l} | l ∈ ℤ\}$, $Θ_{Nst,l} = Th$ of consistent theories inductively as follows: assume that theory $Th_i$ is defined. (i) Suppose that a statement (2.117) is satisfied

$$Th_{Nst,l}^\# \vdash Pr_{Th_{Nst,l}^\#}^+(([Φ_i]^c) \land [Th_{Nst,l}^\# \not\vDash Φ_i]) \land [M_{Th}^{Th} \vDash Φ_i].

(2.117)

Then we define a theory $Th_{Nst,l+1}$ as follows $Th_{Nst,l+1} \equiv Th_{Nst,l} \cup \{Φ_i\}$. Using Lemma 2.1 we will rewrite the condition (2.117) symbolically as follows
(ii) Suppose that the statement (2.119) is satisfied

\[ \text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right), \]

and \([\text{Th}_{Nst,i}^\# \vdash \neg \Phi_i] \wedge [M_{Th}^\# \models \Phi_i]. \) \hfill (2.119)

Then we define theory \( \text{Th}_{i+1} \) as follows: \( \text{Th}_{i+1} \triangleq \text{Th}_{i} \cup \{\neg \Phi_i\} \). Using Lemma 2.2 we will rewrite the condition (2.119) symbolically as follows

\[ \left\{ \begin{array}{l}
\text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right), \\
\text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right) \iff \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right) \wedge [M_{Th}^\# \models \neg \Phi_i].
\end{array} \right. \] \hfill (2.120)

(iii) Suppose that a statement (2.21) is satisfied

\[ \text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\Phi_i]^c \right) \text{ and } \text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\Phi_i]^c \right) \implies \Phi_i. \] \hfill (2.21)

We will rewrite the condition (2.21) symbolically as follows

\[ \left\{ \begin{array}{l}
\text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\Phi_i]^c \right), \\
\text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\Phi_i]^c \right) \iff \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\Phi_i]^c \right) \wedge [\text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\Phi_i]^c \right) \implies \Phi_i].
\end{array} \right. \] \hfill (2.22)

Then we define a theory \( \text{Th}_{Nst,i+1}^\# \) as follows: \( \text{Th}_{Nst,i+1}^\# \triangleq \text{Th}_{Nst,i}^\#. \)

(iv) Suppose that the statement (2.23) is satisfied

\[ \text{Th}_{Nst,i+1}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right) \text{ and } \text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right) \implies \neg \Phi_i. \] \hfill (2.23)

We will rewrite the condition (2.23) symbolically as follows

\[ \left\{ \begin{array}{l}
\text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right), \\
\text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right) \iff \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right) \wedge \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right) \implies \neg \Phi_i.
\end{array} \right. \] \hfill (2.24)

Then we define a theory \( \text{Th}_{Nst,i+1}^\# \) as follows: \( \text{Th}_{Nst,i+1}^\# \triangleq \text{Th}_{Nst,i}^\#. \) We define now a theory \( \text{Th}_{x;Nst}^\# \) as follows:

\[ \text{Th}_{x;Nst}^\# \triangleq \bigcup_{i \in \mathbb{N}} \text{Th}_{Nst,i}^\#. \] \hfill (2.25)

First, notice that each \( \text{Th}_{Nst,i}^\# \) is consistent. This is done by induction on \( i \) and by Lemmas 2.1-2.2. By assumption, the case is true when \( i = 1 \). Now, suppose \( \text{Th}_{Nst,i}^\# \) is consistent. Then its deductive closure \( \text{Ded} \left( \text{Th}_{Nst,i}^\# \right) \triangleq \{ A | \text{Th}_{Nst,i}^\# \vdash A \} \) is also consistent. If a statement (2.21) is satisfied, i.e. \( \text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\Phi_i]^c \right) \text{ and } \text{Th}_{Nst,i}^\# \vdash \Phi_i \), then clearly \( \text{Th}_{Nst,i+1}^\# \triangleq \text{Th}_{Nst,i}^\# \cup \{ \Phi_i \} \) is consistent since it is a subset of closure \( \text{Ded} \left( \text{Th}_{Nst,i}^\# \right) \). If a statement (2.23) is satisfied, i.e. \( \text{Th}_{Nst,i}^\# \vdash \text{Pr}_{\text{Th}_{Nst,i}^\#}^\# \left( [\neg \Phi_i]^c \right) \text{ and } \text{Th}_{Nst,i}^\# \vdash \neg \Phi_i \), then
Th^\#_{Nst,i} \vdash \neg \Phi_i$, then clearly $Th^\#_{Nst,i+1} = Th^\#_{Nst,i} \cup \{ \neg \Phi_i \}$ is consistent since it is a subset of closure Ded$(Th^\#_{Nst,i})$. If a statement (2.117) is satisfied, i.e. $Th^\#_{Nst,i} \vdash Pr_{Th^\#_{Nst}}(\{\Phi_i\}^c)$ and $[Th^\#_{Nst,i} \not\vdash \Phi_i] \land [M^\#_{Th} \models \Phi_i]$ then clearly $Th^\#_{Nst,i+1} = Th^\#_{Nst,i} \cup \{ \Phi_i \}$ is consistent by Lemma 2.1 and by one of the standard properties of consistency: $\Delta \cup \{ A \}$ is consistent iff $\Delta \vdash \neg A$. If a statement (2.119) is satisfied, i.e. $Th^\#_{Nst,i} \vdash Pr_{Th^\#_{Nst}}(\neg \Phi_i)^c)$ and $[Th^\#_{Nst,i} \not\vdash \neg \Phi_i] \land [M^\#_{Th} \models \neg \Phi_i]$ then clearly $Th^\#_{Nst,i+1} = Th^\#_{Nst,i} \cup \{ \neg \Phi_i \}$ is consistent by Lemma 2.2 and by one of the standard properties of consistency: $\Delta \cup \{ \neg A \}$ is consistent iff $\Delta \vdash \neg A$. Next, notice Ded$(Th^\#_{x:Nst})$ is maximally consistent nice extension of the Ded$(Th)$. Ded$(Th^\#_{x:Nst})$ is consistent because, by the standard Lemma 2.3 above, it is the union of a chain of consistent sets. To see that Ded$(Th^\#_{x:Nst})$ is maximal, pick any wff $\Phi$. Then $\Phi$ is some $\Phi_i$ in the enumerated list of all wff’s. Therefore for any $\Phi$ such that $Th^\#_{Nst,i} \vdash Pr_{Th^\#_{Nst}}(\{\Phi\}^c)$ or $Th^\#_{Nst,i} \vdash Pr_{Th^\#_{Nst}}(\neg \Phi)^c)$, either $\Phi \in Th^\#_{x:Nst}$ or $\neg \Phi \in Th^\#_{x:Nst}$. Since Ded$(Th^\#_{Nst,i+1}) \subseteq Ded(Th^\#_{x:Nst})$, we have $\Phi \in Ded(Th^\#_{x:Nst})$ or $\neg \Phi \in Ded(Th^\#_{x:Nst})$, which implies that $Ded(Th^\#_{x:Nst})$ is maximally consistent nice extension of the Ded$(Th)$.

**Definition 2.28.** We define now predicate $Pr_{Th^\#}(\{\Phi_i\}^c)$ asserting provability in $Th^\#_{x:Nst}$:

\[
\begin{align*}
Pr_{Th^\#_{x:Nst}}(\{\Phi_i\}^c) & \iff \left[ Pr_{Th^\#_{x:Nst}}(\{\Phi_i\}^c) \right] \lor \left[ Pr^{*}_{Th^\#_{x:Nst}}(\{\Phi_i\}^c) \right], \\
Pr^{*}_{Th^\#_{x:Nst}}(\neg \Phi_i)^c) & \iff \left[ Pr^{*}_{Th^\#_{x:Nst}}(\neg \Phi_i)^c) \right] \lor \left[ Pr^{*}_{Th^\#_{x:Nst}}(\neg \Phi_i)^c) \right].
\end{align*}
\]

**Definition 2.29.** Let $\Psi = \Psi(x)$ be one-place open wff such that the conditions:

(*) $Th^\#_{x:Nst} \vdash \exists!x\Psi[\Psi(x)]$ or

(**) $Th^\#_{x:Nst} \vdash Pr_{Th^\#_{x:Nst}}(\exists!x\Psi[\Psi(x)])^c)$ and $M^\#_{Th} \models \exists!x\Psi[\Psi(x)]$ is satisfied.

Then we said that, a set $y$ is a $Th^\#$-set iff there is exist one-place open wff $\Psi(x)$ such that $y = x_\Psi$. We write $y[Th^\#_{x:Nst}]$ iff $y$ is a $Th^\#_{x:Nst}$-set.

**Remark 2.21.** Note that $[(* \lor (**)) \Rightarrow Th^\#_{x:Nst} \vdash \exists!x\Psi[\Psi(x)]$.

**Remark 2.22.** Note that $y[Th^\#_{x:Nst}] \iff \exists \Psi \left[ (y = x_\Psi) \land Pr^{*}_{Th^\#_{x:Nst}}(\exists!x\Psi[\Psi(x)])^c) \right]$

**Definition 2.30.** Let $\mathcal{I}^\#_{x:Nst}$ be a collection such that $\forall x \in \mathcal{I}^\#_{x:Nst} \leftrightarrow x$ is a $Th^\#$-set.

**Proposition 2.27.** Collection $\mathcal{I}^\#_{x:Nst}$ is a $Th^\#_{x:Nst}$-set.

**Proof.** Let us consider an one-place open wff $\Psi(x)$ such that conditions (*) or (**) is satisfied, i.e. $Th^\# \vdash \exists!x\Psi[\Psi(x)]$. We note that there exists countable collection $\mathcal{F}_\Psi$ of the one-place open wff’s $\mathcal{F}_\Psi = \{ \Psi_n(x) \}_{n \in \mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_\Psi$ and (ii)
\[
\text{Th}_{x:Nst}^{\#} \vdash \exists! x_{\Psi} \left[ (x_{\Psi}) \land \left\{ \forall n \left( n \in M_{\omega}^{ZFC} \right) \left( \Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi}) \right) \right\} \right]
\]

or

\[
\text{Th}_{x:Nst}^{\#} \vdash \exists! x_{\Psi} \left[ \text{Pr}_{\text{Th}_{x:Nst}^{\#}} \left( (x_{\Psi}) \land \left\{ \forall n \left( n \in M_{\omega}^{ZFC} \right) \text{Pr}_{\text{Th}_{x:Nst}^{\#}} \left( (x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi}) \right) \right\} \right) \right]
\]

and

\[
M_{\text{Th}_{x:Nst}^{\#}} \vdash \exists! x_{\Psi} \left[ \left( \forall n \left( n \in M_{\omega}^{ZFC} \right) \left( \Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi}) \right) \right) \right]
\]

or the equivalent form

\[
\text{Th}_{x:Nst}^{\#} \vdash \exists! x_1 \left[ (x_1(1)) \land \left\{ \forall n \left( n \in M_{\omega}^{ZFC} \right) \left( x_1(1) \leftrightarrow x_{n,1}(1) \right) \right\} \right]
\]

or

\[
\text{Th}_{x:Nst}^{\#} \vdash \exists! x_{\Psi} \left[ \text{Pr}_{\text{Th}_{x:Nst}^{\#}} \left( (x_{\Psi}(1)) \land \left\{ \forall n \left( n \in M_{\omega}^{ZFC} \right) \text{Pr}_{\text{Th}_{x:Nst}^{\#}} \left( (x_{\Psi}(1)) \leftrightarrow x_{n}(1) \right) \right\} \right) \right]
\]

and

\[
M_{\text{Th}_{x:Nst}^{\#}} \vdash \exists! x_{\Psi} \left[ \left( \forall n \left( n \in M_{\omega}^{ZFC} \right) \left( x_{\Psi}(1) \leftrightarrow x_{n}(1) \right) \right) \right]
\]

where we set \( \Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1) \) and \( x_{\Psi} = x_1 \). We note that any collection \( F_{\psi_k} = \{ \Psi_{n,k}(x) \}_{n \in \mathbb{N}}, k = 1,2,.. \) such above defines an unique set \( x_{\Psi_k}, \) i.e. \( F_{\psi_k} \cap F_{\psi_{k_2}} = \emptyset \) iff \( x_{\Psi_k} \neq x_{\Psi_{k_2}} \). We note that collections \( F_{\psi_k}, k = 1,2,.. \) is no part of the \( ZFC_{2}^{Hs} \), i.e. collection \( F_{\psi_k} \) there is no set in sense of \( ZFC_{2}^{Hs} \). However that is no problem, because by using Gödel numbering one can to replace any collection \( F_{\psi_k}, k = 1,2,.. \) by collection \( \Theta_k = g(F_{\psi_k}) \) of the corresponding Gödel numbers such that

\[
\Theta_k = g(F_{\psi_k}) = \{ g(\Psi_{n,k}(x_k)) \}_{n \in \mathbb{N}}, k = 1,2,.. \ .
\]

It is easy to prove that any collection \( \Theta_k = g(F_{\psi_k}), k = 1,2,.. \) is a \( \text{Th}_{x:Nst}^{\#} \)-set. This is done by Gödel encoding \([8],[10]\) \((2.129)\) and by axiom schema of separation \([9]\). Let \( g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,.. \) be a Gödel number of the wff \( \Psi_{n,k}(x_k) \). Therefore \( g(F_{k}) = \{ g_{n,k} \}_{n \in \mathbb{N}}, \) where we set \( F_{k} = F_{\psi_k}, k = 1,2,.. \) and

\[
\forall k_1 \forall k_2 \left( \{ g_{n,k_1} \}_{n \in \mathbb{N}} \cap \{ g_{n,k_2} \}_{n \in \mathbb{N}} = \emptyset \leftrightarrow x_{k_1} \neq x_{k_2} \right).
\]

Let \( \{ g_{n,k} \}_{n \in \mathbb{N}} \) be a family of the all sets \( \{ g_{n,k} \}_{n \in \mathbb{N}} \). By axiom of choice \([9]\) one obtain unique set \( \Xi_{x:Nst}^{\#} = \{ g_k \}_{k \in \mathbb{N}} \) such that \( \forall k \left[ g_k \in \{ g_{n,k} \}_{n \in \mathbb{N}} \right] \). Finally one obtain a set \( \Xi_{x:Nst}^{\#} \) from a set \( \Xi_{x:Nst}^{\#} \) by axiom schema of replacement \([9]\). Thus we can define a \( \text{Th}_{x:Nst}^{\#} \)-set

\[
\mathcal{R}_{x:Nst}^{\#} = \Xi_{x:Nst}^{\#}:
\]

\[
\forall x \left[ x \in \mathcal{R}_{x:Nst}^{\#} \leftrightarrow \left( x \in \Xi_{x:Nst}^{\#} \land \text{Pr}_{\text{Th}_{x:Nst}^{\#}} \left( (x \not\epsilon x)^c \right) \right) \land \right.
\]

\[
\left( \text{Pr}_{\text{Th}_{x:Nst}^{\#}} \left( \left( x \not\epsilon x \right)^c \Rightarrow x \not\epsilon x \right) \right).
\]

\]
**Proposition 2.28.** Any collection \( \Theta_k = g(F_{\psi_k}), k = 1, 2, \ldots \) is a \( \text{Th}^{|\alpha|_{\text{Nat}}} \)-set.

**Proof.** We define \( g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c \). Therefore \( g_{n,k} = g(\Psi_{n,k}(x_k)) \iff Fr(g_{n,k}, v_k) \) (see [10]). Let us define now predicate \( \Pi_\alpha(g_{n,k}, v_k) \)

\[
\Pi_\alpha(g_{n,k}, v_k) \iff \text{Pr}_{\text{Th}^{|\alpha|_{\text{Nat}}}}([\Psi_{1,k}(x_1)]) \land \exists!x_k(v_k = [x_k]^c)
\]

\[
\forall n(n \in M_{st}^{ZFC}) \left[ \text{Pr}_{\text{Th}^{|\alpha|_{\text{Nat}}}}([\Psi_{1,k}(x_k)]) \iff \text{Pr}_{\text{Th}^{|\alpha|_{\text{Nat}}}}(Fr(g_{n,k}, v_k)) \right]
\]

(2.132)

We define now a set \( \Theta_k \) such that

\[
\Theta_k = \Theta'_k \cup \{g_k\},
\]

\[
\forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k \iff \Pi_\alpha(g_{n,k}, v_k)]
\]

(2.133)

But obviously definitions (2.29) and (2.133) is equivalent by Proposition 2.26.

**Proposition 2.28.** (i) \( \text{Th}^{|\alpha|_{\text{Nat}}} \vdash \exists \text{Th}^{|\alpha|_{\text{Nat}}}_c \), (ii) \( \text{Th}^{|\alpha|_{\text{Nat}}} \) is a countable \( \text{Th}^{|\alpha|_{\text{Nat}}} \)-set.

**Proof.** (i) Statement \( \text{Th}^{|\alpha|_{\text{Nat}}} \vdash \exists \text{Th}^{|\alpha|_{\text{Nat}}}_c \) follows immediately from the statement \( \exists \text{Th}^{|\alpha|_{\text{Nat}}} \) and axiom schema of separation [9]. (ii) follows immediately from countability of the set \( \text{Th}^{|\alpha|_{\text{Nat}}} \).

**Proposition 2.29.** A set \( \text{Th}^{|\alpha|_{\text{Nat}}} \) is inconsistent.

**Proof.** From formula (2.131) we obtain

\[
\text{Th}^{|\alpha|_{\text{Nat}}} \vdash \exists \text{Th}^{|\alpha|_{\text{Nat}}} \iff \exists \text{Th}^{|\alpha|_{\text{Nat}}} \not\in \text{Th}^{|\alpha|_{\text{Nat}}}_c.
\]

(2.134)

From formula (2.41) and Proposition 2.6 one obtain

\[
\text{Th}^{|\alpha|_{\text{Nat}}} \vdash \forall \text{Th}^{|\alpha|_{\text{Nat}}} \iff \forall \text{Th}^{|\alpha|_{\text{Nat}}} \not\in \text{Th}^{|\alpha|_{\text{Nat}}}_c
\]

(2.135)

and therefore

\[
\text{Th}^{|\alpha|_{\text{Nat}}} \vdash (\text{Th}^{|\alpha|_{\text{Nat}}} \in \text{Th}^{|\alpha|_{\text{Nat}}}_c) \land (\text{Th}^{|\alpha|_{\text{Nat}}} \not\in \text{Th}^{|\alpha|_{\text{Nat}}}_c).
\]

(2.136)

But this is a contradiction.

**Remark 2.23.** A cardinal \( \kappa \) is inaccessible if and only if \( \kappa \) has the following reflection property: for all subsets \( U \subset V_\kappa \), there exists \( \alpha < \kappa \) such that \( (V_\alpha, \in, U \cap V_\alpha) \) is an elementary substructure of \( (V_\kappa, \in, U) \). (In fact, the set of such \( \alpha \) is closed unbounded in \( \kappa \).) Equivalently, \( \kappa \) is \( \Pi^0_n \)-indescribable for all \( n \geq 0 \).

**Remark 2.24.** Under ZFC it can be shown that \( \kappa \) is inaccessible if and only if \( (V_\kappa, \in) \) is a model of second order ZFC, [5].

**Remark 2.25.** By the reflection property, there exists \( \alpha < \kappa \) such that \( (V_\alpha, \in) \) is a standard model of (first order) ZFC. Hence, the existence of an inaccessible cardinal is a stronger hypothesis than the existence of the standard model of \( \text{ZFC}^{H_2} \).

3. Derivation inconsistent countable set in set theory \( \text{ZFC}_2 \)
with the full semantics.

Let $\text{Th} = \text{Th}^{fss}$ be an second order theory with the full second order semantics. We assume now that: (i) $\text{Th}$ contains $\text{ZFC}^\text{fss}_2$, (ii) $\text{Th}$ has no any model. We will write for short $\text{Th}$, instead $\text{Th}^{fss}$.

**Remark 3.1.** Notice that $M$ is a model of $\text{ZFC}^\text{fss}_2$ if and only if it is isomorphic to a model of
the form $V_\kappa, \in \cap (V_\kappa \times V_\kappa)$, for $\kappa$ a strongly inaccessible ordinal.

**Remark 3.2.** Notice that a standard model for the language of first-order set theory is an ordered pair $\langle D, I \rangle$. Its domain, $D$, is a nonempty set and its interpretation function, $I$, assigns a set of ordered pairs to the two-place predicate "$\in$". A sentence is true in $\langle D, I \rangle$ just in case it is satisfied by all assignments of first-order variables to members of $D$ and second-order variables to subsets of $D$; a sentence is satisfiable just in case it is true in some standard model; finally, a sentence is valid just in case it is true in all standard models.

**Remark 3.3.** Notice that:

(I) The assumption that $D$ and $I$ be sets is not without consequence. An immediate effect of this stipulation is that no standard model provides the language of set theory with its intended interpretation. In other words, there is no standard model $\langle D, I \rangle$ in which $D$ consists of all sets and $I$ assigns the standard element-set relation to "$\in$". For it is a theorem of $\text{ZFC}$ that there is no set of all sets and that there is no set of ordered-pairs $\langle x, y \rangle$ for $x$ an element of $y$.

(II) Thus, on the standard definition of model:

(1) it is not at all obvious that the validity of a sentence is a guarantee of its truth;

(2) similarly, it is far from evident that the truth of a sentence is a guarantee of its satisfiability in some standard model.

(3) If there is a connection between satisfiability, truth, and validity, it is not one that can be

"read off" standard model theory.

(III) Nevertheless this is not a problem in the first-order case since set theory provides us

with two reassuring results for the language of first-order set theory. One result is the first

order completeness theorem according to which first-order sentences are provable, if

true in all models. Granted the truth of the axioms of the first-order predicate calculus

and the truth preserving character of its rules of inference, we know that a sentence of the first-order language of set theory is true, if it is provable. Thus, since valid sentences are provable and provable sentences are true, we know that valid sentences
are true. The connection between truth and satisfiability immediately follows: if \( \phi \) is unsatisfiable, then \( \neg \phi \), its negation, is true in all models and hence valid. Therefore, \( \neg \phi \) is true and \( \phi \) is false.

**Definition 3.1.** The language of second order arithmetic \( Z_2 \) is a two-sorted language: there are two kinds of terms, numeric terms and set terms.

1. \( 0 \) is a numeric term,
2. There are infinitely many numeric variables, \( x_0, x_1, \ldots, x_n, \ldots \) each of which is a numeric term;
3. If \( s \) is a numeric term then \( Ss \) is a numeric term;
4. If \( s, t \) are numeric terms then \( +st \) and \( \cdot st \) are numeric terms (abbreviated \( s + t \) and \( s \cdot t \));
5. There are infinitely many set variables, \( X_0, X_1, \ldots, X_n, \ldots \) each of which is a set term;
6. If \( t \) is a numeric term and \( S \) then \( t \) is an atomic formula (abbreviated \( t \in S \));
7. If \( s \) and \( t \) are numeric terms then \( = st \) and \( < st \) are atomic formulas (abbreviated \( s = t \) and \( s < t \) correspondingly).

The formulas are built from the atomic formulas in the usual way.

As the examples in the definition suggest, we use upper case letters for set variables and lower case letters for numeric terms. (Note that the only set terms are the variables.) It will be more convenient to work with functions instead of sets, but within arithmetic, these are equivalent: one can use the pairing operation, and say that \( X \) represents a function if for each \( n \) there is exactly one \( m \) such that the pair \( (n, m) \) belongs to \( X \).

We have to consider what we intend the semantics of this language to be. One possibility is the semantics of full second order logic: a model consists of a set \( M \), representing the numeric objects, and interpretations of the various functions and relations (probably with the requirement that equality be the genuine equality relation), and a statement \( \forall X \exists \Phi (X) \) is satisfied by the model if for every possible subset of \( M \), the corresponding statement holds.

**Remark 3.1.** Full second order logic has no corresponding proof system. An easy way to see this is to observe that it has no compactness theorem. For example, the only model (up to isomorphism) of Peano arithmetic together with the second order induction axiom: \( \forall X(0 \in X \land \forall x(x \in X \Rightarrow Sx \in X) \Rightarrow \forall x(x \in X)) \) is the standard model \( \mathbb{N} \). This is easily seen: any model of Peano arithmetic has an initial segment isomorphic to \( \mathbb{N} \); applying the induction axiom to this set, we see that it must be the whole of the
Definition 3.2. Using formula (2.3) one can define predicate \( \Pr_{\text{Th}}(y) \) really asserting provability in \( \text{Th} = ZFC^\text{fs}_2 \)

\[
\Pr_{\text{Th}}(y) \leftrightarrow \exists x \left( x \in M^Z_{\omega^2} \right) \text{Prov}_{\text{Th}}(x, y),
\]

(3.1)

Theorem 3.1.[12]. (Löb’s Theorem for \( ZFC^\text{fs}_2 \)) Let \( \Phi \) be any closed formula with code \( y = [\Phi]^c \in M^Z_{\omega^2} \), then \( \text{Th} \vdash \Pr_{\text{Th}}([\Phi]^c) \) implies \( \text{Th} \vdash \Phi \) (see [12] Theorem 5.1).

Proof. Assume that

(1) \( \text{Th} \notvdash \neg \Phi \). Otherwise one obtains \( \text{Th} \vdash \Pr_{\text{Th}}([\neg \Phi]^c) \land \Pr_{\text{Th}}([\Phi]^c) \), but this is a contradiction.

(2) Assume now that (2.i) \( \text{Th} \vdash \Pr_{\text{Th}}([\Phi]^c) \) and (2.ii) \( \text{Th} \notvdash \Phi \).

From (1) and (2.ii) follows that

(3) \( \text{Th} \notvdash \neg \Phi \) and \( \text{Th} \notvdash \Phi \).

Let \( \text{Th}_{\neg \Phi} \) be a theory

(4) \( \text{Th}_{\neg \Phi} \triangleq \text{Th} \cup \{ \neg \Phi \} \). From (3) follows that

(5) \( \text{Con}(\text{Th}_{\neg \Phi}) \).

From (4) and (5) follows that

(6) \( \text{Th}_{\neg \Phi} \vdash \Pr_{\text{Th}_{\neg \Phi}}([\neg \Phi]^c) \).

From (4) and (#) follows that

(7) \( \text{Th}_{\neg \Phi} \vdash \Pr_{\text{Th}_{\neg \Phi}}([\Phi]^c) \).

From (6) and (7) follows that

(8) \( \text{Th}_{\neg \Phi} \vdash \Pr_{\text{Th}_{\neg \Phi}}([\Phi]^c) \land \Pr_{\text{Th}_{\neg \Phi}}([\neg \Phi]^c) \), but this is a contradiction.

Definition 3.3. Let \( \Psi = \Psi(x) \) be one-place open wff such that the conditions:

\[
\text{Th} \vdash \exists! x_{\Psi}[\Psi(x_{\Psi})]
\]

(3.2)

Then we said that, a set \( y \) is a \( \text{Th} \)-set iff there is exist one-place open wff \( \Psi(x) \) such that

\[
y = x_{\Psi}.
\]

We write \( y[\text{Th}] \) iff \( y \) is a \( \text{Th} \)-set.

Remark 3.2. Note that \( y[\text{Th}] \iff \exists \Psi[(y = x_{\Psi}) \land \Pr_{\text{Th}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^c)] \)

Definition 3.4. Let \( \mathcal{I} \) be a collection such that : \( \forall x [x \in \mathcal{I} \iff x \text{ is a } \text{Th}-\text{set}] \).

Proposition 3.1. Collection \( \mathcal{I} \) is a \( \text{Th} \)-set.

Definition 3.4. We define now a \( \text{Th} \)-set \( \mathcal{R}_c \subseteq \mathcal{I} \):

\[
\forall x [x \in \mathcal{R}_c \iff (x \in \mathcal{I} \land \Pr_{\text{Th}}([x \not\in x]^c))].
\]

(3.3)

Proposition 3.2. (i) \( \text{Th} \vdash \exists \mathcal{R}_c \), (ii) \( \mathcal{R}_c \) is a countable \( \text{Th} \)-set.

Proof. (i) Statement \( \text{Th} \vdash \exists \mathcal{R}_c \) follows immediately by using statement \( \exists \mathcal{I} \) and axiom schema of separation [4]. (ii) follows immediately from countability of a set \( \mathcal{I} \).
Proposition 3.3. A set $\mathcal{R}_c$ is inconsistent.

Proof. From formula (3.2) one obtains

$$Th \vdash \mathcal{R}_c \in \mathcal{R}_c \iff \Pr_{Th}([\mathcal{R}_c \notin \mathcal{R}_c]^c).$$

(3.4)

From formula (3.3) and definition 3.4 and by Theorem 3.1 one obtains

$$Th \vdash \mathcal{R}_c \in \mathcal{R}_c \iff \mathcal{R}_c \notin \mathcal{R}_c$$

(3.5)

and therefore

$$Th \vdash (\mathcal{R}_c \in \mathcal{R}_c) \land (\mathcal{R}_c \notin \mathcal{R}_c).$$

(3.6)

But this is a contradiction.

Therefore finally we obtain:

**Theorem 3.2.** [12]. \(\neg \text{Con}(ZFC_{2s}^{SS})\).

It well known that under ZFC it can be shown that $\kappa$ is inaccessible if and only if $(V_{\kappa}, \in)$ is a model of $ZFC_2$ [5],[11]. Thus finally we obtain.

**Theorem 3.3.** [12]. \(\neg \text{Con}(ZFC + (V = H_k))\).

4. Non consistency Results in Topology.

**Definition 4.1.** [19]. A Lindelöf space is indestructible if it remains Lindelöf after forcing with

any countably closed partial order.

**Theorem 4.1.** [20]. If it is consistent with ZFC that there is an inaccessible cardinal, then it

is consistent with ZFC that every Lindelöf $T_3$ indestructible space of weight $\leq \aleph_1$ has size

$\leq \aleph_1$.

**Corollary 4.1.** [20] The existence of an inaccessible cardinal and the statement:

$L[T_3, \leq \aleph_1, \leq \aleph_1] \triangleq \text{“every Lindelöf } T_3 \text{ indestructible space of weight } \leq \aleph_1 \text{ has size } \leq \aleph_1”$

are equiconsistent.

**Theorem 4.2.** [12]. \(\neg \text{Con}(ZFC + L[T_3, \leq \aleph_1, \leq \aleph_1])\).

Proof. Theorem 4.2 immediately follows from Theorem 3.3 and Corollary 4.1.

**Definition 4.2.** The $\aleph_1$-Borel Conjecture is the statement: $BC[\aleph_1] \triangleq \text{“a Lindelöf space is}$

indestructible if and only if all of its continuous images in $[0; 1]^{\omega_1}$ have cardinality

$\leq \aleph_1”$.

**Theorem 4.3.** [12]. If it is consistent with ZFC that there is an inaccessible cardinal, then it

is consistent with ZFC that the $\aleph_1$-Borel Conjecture holds.

**Corollary 4.2.** The $\aleph_1$-Borel Conjecture and the existence of an inaccessible cardinal are
Theorem 4.4.[12] \( \neg \text{Con}(ZFC + BC[\aleph_1]) \).
Proof. Theorem 4.4 immediately follows from Theorem 3.3 and Corollary 4.2.

**Theorem 4.5.**[20] If \( \omega_2 \) is not weakly compact in \( L \), then there is a Lindelöf \( T_3 \) indestructible space of pseudocharacter \( \leq \aleph_1 \) and size \( \aleph_2 \).

**Corollary 4.3.** The existence of a weakly compact cardinal and the statement:
\[ \exists \kappa \in \mathbb{L}[T_3, \leq \aleph_1, \aleph_2] \iff \text{there is no Lindelöf } T_3 \text{ indestructible space of pseudocharacter } \leq \aleph_1 \]
and size \( \aleph_2 \) are equiconsistent.

**Theorem 4.6.**[12] There is a Lindelöf \( T_3 \) indestructible space of pseudocharacter \( \leq \aleph_1 \) and size \( \aleph_2 \) in \( L \).
Proof. Theorem 4.6 immediately follows from Theorem 3.3 and Theorem 4.5.

**Theorem 4.7.**[12] \( \neg \text{Con}(ZFC + \exists \kappa \in \mathbb{L}[T_3, \leq \aleph_1, \aleph_2]) \).
Proof. Theorem 4.7 immediately follows from Theorem 3.3 and Corollary 4.3.

5. Conclusion.

In this paper we have proved that the second order \( ZFC \) with the full second-order semantic is inconsistent, i.e. \( \neg \text{Con}(ZFC_{\text{ess}}^2) \). Main result is: let \( k \) be an inaccessible cardinal and \( H_k \) is a set of all sets having hereditary size less then \( k \), then \( \neg \text{Con}(ZFC + (V = H_k)) \). This result also was obtained in [7],[12],[13] by using essentially another approach. For the first time this result has been declared to AMS in [14],[15]. An important applications in topology and homotopy theory are obtained in [16],[17],[18].

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