Weak fixed point property and schauder conjecture

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ABSTRACT. In this paper we introduce weak fixed point property and schauder conjecture. Firstly we investigate when various Banach algebras associated to a locally compact group C have the weak fixed point property for left reversible semigroups . Then we discuss the problem known as schauder conjecture.

1. INTRODUCTION

It is of interest to know that schauder conjecture is still open in some sepcial cases.now we are concerned with the following problem imposed by schauder: Does every compact convex set in a linear metric space have the fixed point property?

This problem was established by cauty(2001). Such a proof allows us to show that the convex compacta in metric linear spaces possess the simplidial approximation property introduced by kalton and roberts.

It is well known that the very important jobs completed by Nguyen To Nhu (Viet Nam) about the weakly admissible compact convex sets associated with schauder conjecture. From his former results,we discover a new property possed by the schauder fixed point.

Now, turn to the definition of weakly admissible compact convex sets.

Definition 1.1: a compact convex set X in a linear metric space is weakly admissible if and only if for every $\varepsilon > 0$ there exist compact convex subsets $X_1, X_2, \ldots X_n$ of X with X =conv($X_1 \cup \ldots \cup X_n$) and continuous maps f_i from X_i into finite dimensional subsets $E_i, i = 1, \ldots, n$ of X such that $\sum_1^n \| f_i(x_i) - x_i \| < \varepsilon$ for every $x_i \in X_i$,and $i = 1, \ldots, n$.

Theorem 1.1 :Any weakly admissible compact convex set has the fixed point propery?

Then we introduce the weak fixed point property.We recall the results proved by anthony To-ming lau and peter F. For example,that if G is a separable locally compact group with a compact neighborhood of the identity invariant under inner automorphisms, then the fourier-stieltjes algebra of G has the weak fixed point property for left reversible semigroups if G is compact.

Definition 1.2: Let E be a Banach space and K be a nonempty bounded closed convex subset of E. We say that K has the fixed point property if every nonexpansive mapping $T : K \to K(\|Tx - Ty\| \le \|x - y\|)$ has a fixed point.

Definition1.3 :We say that E has the weak fixed point property if every weakly compact Convex subset of E has the fixed point property.

Definition 1.4: Let S be a semitopological semigroup, S is a semigroup with a hausdorff topology such that for each $a \in S$,the mapping $S \to aS$ and $S \to Sa$ From S into S are continuous. S is called left reversible if $aS \cap bS \neq \varnothing$ for any $a, b \in S$.

firstly，we start with the weak fixed point property:

2. weak fixed point and error estimate

Theorem 2.1：

let G be a compact group，and let $\{D_a : a \in \Lambda\}$ be a decreasing net of bounded subsets of $B(G)$,and ϕ_m be a weak convergent sequence with weak limit ϕ .then：

$$\limsup \limsup \{\|\phi_M - \psi\| : \psi \in D_a\} = \limsup \{\|\phi - \psi\| : \psi \in D_a\}$$

$$+\limsup \|\phi_m - \phi\|$$

In this theorem, we classify the finite subsets $\sigma_1 \subset \sigma_2 \subset ...$ into three conditions, such that:

1. $\sum_{i\in\sigma_1} \|\psi_n(i)\| > (p-\varepsilon)/2$

2. $\sum_{i\in\sigma_2/\sigma_1} \|\psi_n(i)\| > (p-\varepsilon)/2$

3. $\sum_{i\in\sigma_3/\sigma_2} \|\psi_n(i)\| > (p-\varepsilon)/2$

Theorem 2.2：

Suppose that there exists a map $f:R^2 \to R^2$:If $x_1,x_2 \in R^2$ and $d(x_1,x_2) \in Q_+$,we have $d(f(x_1),f(x_2)) = d(x_1,x_2)$ .Then to every $x_1,x_2 \in R^2$,we obtain that $d(f(x_1),f(x_2)) = d(x_1,x_2)$, In this section, we apply theorem 2.2,to estimate the error of the weak fixed point type integral：

By theorem2.1,we select $\varepsilon > 0$ be arbitrary then there exist an $m_1 \in M$ and N_1 such that for all $n \ge N_1$

$$\|\phi_{m1}\| > q - \varepsilon/4 \tag{1}$$

$$\|\phi_{m1} - \psi_n\| < r + \varepsilon/4 \tag{2}$$

$$\|\psi_n\| > r - q + p - \varepsilon/4 \tag{3}$$

We divide the inequality above by p,

$$\|\phi_{m1}\| / p > q/p - \varepsilon/4p \tag{4}$$

$$\|\psi_n\| / p > r/p - q/p + 1 - \varepsilon/4p \tag{5}$$

If， $q > p$ then，it follows that it suffices to get the results below:

$$\left\|\varphi_{m1}\right\|/p > q/p - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \varepsilon/4p \tag{6}$$

Because , $\sqrt{2}$ is an irrational number, so

$$\left|2q^2 - p^2\right| > 1 \tag{7}$$

Next,since (7) and substitute it to (4)(5)(6),we can write out:

$$q/p - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \varepsilon/4p > \frac{1}{4p^2} + \frac{\sqrt{2}}{2} - \varepsilon/4p \tag{8}$$

$$\left\|\varphi_{m1}\right\| > \frac{1}{4p} + \frac{\sqrt{2}}{2}p - \varepsilon/4 \tag{9}$$

Similarly,we have:

$$r - q + p > \left\|\psi_n\right\| > r - q + p - \varepsilon/4 \tag{10}$$

Which lead to:

$$\left\|\psi_n\right\|/p < \frac{r}{p} - \frac{q}{p} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1 \tag{11}$$

$$< \frac{r}{p} - \frac{1}{4p^2} + \frac{\sqrt{2}}{2} + 1$$

$$\left\|\psi_n\right\| < r - \frac{1}{4p} + \frac{\sqrt{2}}{2}p + p \tag{12}$$

$$\left\|\phi_{m1}\right\| - \left\|\psi_n\right\| > \left(\frac{1}{4p} + \frac{\sqrt{2}}{2}p - \varepsilon/4\right) - \left(r - \frac{1}{4p} + \frac{\sqrt{2}}{2}p + p\right) \tag{13}$$

$$> \frac{1}{2p} - r - p - \frac{\varepsilon}{4}$$

From theorem 2.2,we can assume that $\varepsilon < d(x,y)$, and select $r \in Q$, $d(x,y) - \varepsilon < r < d(x,y)$

Also associated with theorem 2.1,we set $r' = \varepsilon/4$, then, we find that :

$$r+r^{'}>r+\varepsilon/4 \tag{14}$$

$$>\left\|\phi_{m1}-\psi_n\right\|$$

$$>\left\|\phi_{m1}\right\|-\left\|\psi_n\right\|$$

$$>\frac{1}{2p}-r-p-\frac{\varepsilon}{4}$$

Immediate calculation show that:

$$r>\frac{1}{2p}-r-p-\frac{\varepsilon}{4}-r^{'} \tag{15}$$

$$>\frac{1}{2p}-r-p-\varepsilon/2$$

$$2r>\frac{1}{2p}-p-\varepsilon/2 \tag{16}$$

$$r>\frac{1}{4p}-\frac{p}{2}-\frac{\varepsilon}{4} \tag{17}$$

$$d(x,y)>r>\frac{1-2p^2}{4p}-\frac{\varepsilon}{4}=\frac{1-2p^2}{4p}-r^{'} \tag{18}$$

When, $p\in(0,\frac{\sqrt{2}}{2})$

From these inequalities we obtain, $r+r^{'}>\frac{1-2p^2}{4p}$ In addition, if $r+r^{'}>d(x,y)$ exists ,then $d(x,y)\in(\frac{1-2p^2}{4p}-r^{'},\frac{1-2p^2}{4p})$ Now,we pay attention to the size of $r^{'}$ and compare it with $\varepsilon/4$.

Combinate it with theorem 2.2,we obtained the following inequality under the condition of (1) with theorem 2.1, we have:

If $r' > \varepsilon/4$,we get:

$$r+r' > \frac{1}{2p} - r - p - \frac{\varepsilon}{4} > \frac{1}{2P} - r - p - r' \qquad (19)$$

Else if $r' < \varepsilon/4$

$$r - 2r' > \frac{1}{2p} - r - p - \varepsilon > \frac{1}{2p} - r - p - r \qquad (20)$$

In which，$r' = d(y,z)$

Similarly we repeat the methods above, apply them to the condition 2 and 3 as below:

2. $\sum_{i\in\sigma_2/\sigma_1} \|\psi_n(i)\| > (p-\varepsilon)/2$

3. $\sum_{i\in\sigma_3/\sigma_2} \|\psi_n(i)\| > (p-\varepsilon)/2$

where we can get the similar results.

Now we will ask a technical problem : If we use $\sqrt{k}$ to substitute $\sqrt{2}$，what results we will get ?

For example ：

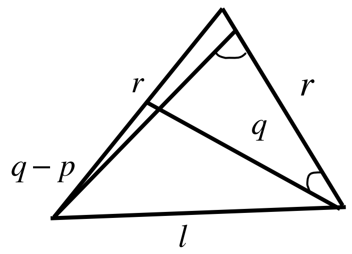
We consider the intervals

$$0 < \frac{\sqrt{k}}{k} < \frac{\sqrt{k-1}}{k-1} < \ldots\ldots < \frac{\sqrt{3}}{3} < \frac{\sqrt{2}}{2} < 1 \qquad (21)$$

Set p be included in these intervals,from (19) and (20) above，we can ensure what interval p is in. then we can estimate the value of r .

Next，we should estimate the angle：

Figure 1 ：



This figure describe the relation of the norms from (1) to (3)

By observation of this figure ,we have

$$\cos\theta = \frac{r^2 + (r-q+p)^2 - q^2}{2r(r-q+p)} \tag{22}$$

In which， $\theta_1 = 180^\circ - \theta$

Also consider sin and cos ,we have the relations:

$$\frac{\sin\theta}{l} = \frac{\sin(\pi/2 - \theta/2)}{r}$$

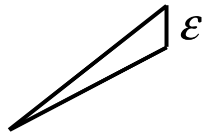
$$\cos\theta_1 = \frac{2r^2 - l^2}{2r^2} \tag{23}$$

By immediate calculation,we have ： $q = \dfrac{4r^3 + 4r^2 p + 3rp^2}{6rp + 4r^2}$

then， we can omit q again. In this way， we can use polar coordinate to estimate the integrals.

By using the method in this example ,we can estimate the volumn of the figure below:

Figure 2:



Use formula ： $\int_{0}^{\frac{\varepsilon}{4}} + \int_{\frac{\varepsilon}{4}}^{\varepsilon}$ ， in which

$$q = \frac{4r^2 + 4r \quad p + 3p^2}{6p + 4r} \tag{24}$$

Meanwhile ， we use inequality in number theory. Because, $\sqrt{2}$ is an irrational number，so $\left|2q^2 - p^2\right| > 1$

and $\left|q/p - \frac{\sqrt{2}}{2}\right| > \frac{1}{4p^2}$

3. projection operator and the neumann algebra

Definition 3.1: a banach space X is said to have the weak fixed point property for left reversible semigroups is whenever S is a left reversible semigroup and K is a nonempty weakly compact convex subset of X for which the action of S on K is separately continuously and nonexpansive, then S has a common fixed point.

Theorem 3.1：

Assume that $\{D_a : a \in \Lambda\}$ is a decreasing net of bounded subsets of M_* and $(\varepsilon_M)_{M\geq 1}$ is a bounded and disjointly supported sequence in M_*(in the sense that there is a sequence of mutually disjoint sequence $(e_m)_{m\geq 1}$ in M_{proj} such that for every $m \geq 1$ $\varepsilon_m = e_m \varepsilon_m e_m$) ,then

$$\lim\sup\{\|\psi\| : \psi \in D_a\} + \lim\|\varepsilon_m\| = \lim\lim\sup\{\|\varepsilon_m - \psi\| : \psi \in D_a\}$$

Theorem 3.2：

Assume that X is a banach space， and $A,B\in BL(X)$. Then $\sigma(AB)$ and $\sigma(BA)$ differ {0} at most.

In this section,we discuss the projection operators in neumann algebra which are related with weak fixed point property.

From theorem 3.1,we can assume that the operators π_1,π_k,ψ_n as follow:

$$\psi_n = A \tag{25}$$

$$\pi_1 = B_1$$

$$\pi_K - \pi_{K-1} = B_K - B_{K-1}$$

Where $$\pi_K = \sum_1^{m_k} e_j$$

Also note that : $k\notin\sigma(AB), k\neq 0$

According to theorem 3.2,we can also assume that:

$$F = AB - kI \tag{26}$$

$$G = BA - kI$$

Then ， from theorem 3.2 ,we have

$$FA = AG \tag{27}$$

$$GB = BF$$

$$A = F^{-1}AG \tag{28}$$

$$B=GBF^{-1}$$

Which imply：

$$kI=BA-G=GBF^{-2}AG-G \tag{29}$$

$$I=GS$$

(Where $S=\frac{1}{k}(BF^{-2}AG-I)$)
Next, we select $T=BF^{-2}AG$
Then as above, T_k can be chosen so that for every k, $\|T_k\|\le k$
We induce that ：

$$A_1=\frac{1}{1}(T_1-I)$$

$$A_2=\frac{1}{2}(T_2-T_1-I)$$

……

$$A_k=\frac{1}{k}(T_k-T_{k-1}-I) \tag{30}$$

If $k>2$，it follows as above that: ($A_k<1$)

$$T_k<\frac{k(k+3)}{2}$$

$$T_{k-1}<\frac{(k-1)(k+2)}{2}$$

$$T_k-T_{k-1}<k-1 \tag{31}$$

We substitute them to (27)(28)(29),then:

$$(B_k - B_{k-1})F^{-2}AG < k-1 \tag{32}$$

$$\pi_K - \pi_{K-1} = B_K - B_{K-1} \tag{33}$$

$$\begin{aligned}
&< (k-1)G^{-1}A^{-1}F^2 \\
&= (k-1)G^{-1}\psi_n{}^{-1}F^2 \\
&= (k-1)\ \psi_n{}^{-1}F \\
&= (k-1)(B_n - kA^{-1})
\end{aligned}$$

Now for $k \in N$, $n \geq N_k$,we use theorem 3.2 with (31)(32)(33) to deduce that:

$$\left\|(\pi_k - \pi_{k-1})\psi_n\right\| < (k-1)(B_n A - k) \tag{34}$$

$$= (k-1)G$$

$$\left\|\psi_n(\pi_k - \pi_{k-1})\right\| < (k-1)(AB_n - k) \tag{35}$$

$$= (k-1)F$$

$$\left\|(\pi_k - \pi_{k-1})\psi_n(\pi_k - \pi_{k-1})\right\| \tag{36}$$

$$\begin{aligned}
&< (k-1)^2 G^2 A^{-1} \\
&= (k-1)^2 A^{-1} F^2
\end{aligned}$$

Taking adjoints with theorem 3.1 leads to,

$$(k-1)G + (k-1)F + \frac{(k-1)^2}{2}(G^2A^{-1} + A^{-1}F^2) > p - 3\varepsilon \tag{37}$$

The key of this method is to estimate the upper bound of projection operator π_k. From the deduction above, we transform $\pi_k - \pi_{k-1}$'s upper bound to π_k's upper bound. And then, we use the results of (24) in section (2) weak fixed point and error estimate, to omit q. Combining it with the continuous integrals: $0<\frac{\sqrt{k}}{k}<\frac{\sqrt{k-1}}{k-1}<.........<\frac{\sqrt{3}}{3}<\frac{\sqrt{2}}{2}<1$ together. Finally, we assume that $\pi_k < m$. since $\psi_n = A > 1$, we deduce from (37) as follow:

$$[m(r-q)-k](k-1)+1>\sqrt{p+1} \tag{38}$$

We omit $r-q$, set the k in (37) equivalent with the k in (21)
Where we use the formula: $q=\frac{4r^2+4r\ \ p+3p^2}{6p+4r}$, and also select $r=\frac{3}{2}p$
So that we can obtain the last results in this section:

$$k^4-2k^3-k^2+2k>p \tag{39}$$

4. Weakly admissible compact convex sets and schauder conjecture

In this section, we study a problem about schauder fixed point .Firstly, we introduce some theorems first:
Theorem 4.1:

let X be an infinite dimensional convex set in a linear metric space and let A_i be finite subsets of X and $\varepsilon > 0$.then there exist disjoint finite sets $B_i \subset X$, $i = 1, \ldots n$,such that

1. $B = \bigcup Bi$ is a linear independent subset of X

2.to every $x \in convBi$, $\|x - convAi\| < \varepsilon$

3. there exist an affine map $h: convA \to convB$,where $A = \bigcup_{i=1}^{n} A_i$, so that $\|x - h(x)\| < \varepsilon$,for every $x \in convA$

Theorem 4.2：

For every i，there exists a continuous map gi from $f_i(Xi)$ into a convex Polyhedron $Fi \subset X$, so that $\sum_1^n \|gi(y) - y\| < 2^{-4}\varepsilon$

Theorem 4.3：

Let X be an n-dimensional closed convex subset in a linear metric space E. Then there is a retraction r: $E \to X$ such that

$\|r(x) - x\| \le 2(n+1)\|x - X\|$ for every $x \in X$

Theorem 4.4：

If $e = (1,1, \ldots\ldots)$ and $e_1, e_2, \ldots$are its elemental vectors，then $\{ e, e_1, e_2, \ldots\ldots \}$ is its schauder base.

We start with theorem 4.1 ,by using theorem 4.4:

Observe that

$$x \in c, \lim x(n) = \lambda \tag{40}$$

We assume that :

$$x_n = \lambda e + \sum (x(j) - \lambda) e_j \tag{41}$$

Then to every $1 \le j \le n$, we obtain ：

$$x_n(j) = \lambda + x(j) - \lambda = x(j) \tag{42}$$

For every $j \ge n$, $x_n(j) = \lambda$

Next from theorem 4.1 and 4.2,

$$\|x - y\| = \left\|\sum a_j (h(v_j) - v_j)\right\| \le \sum \left\|h(v_j) - v_j\right\| \le \varepsilon \tag{43}$$

Therefore,

$$\|x_n\| = \|\lambda e + x - y\| \le \lambda \sqrt{n} + \varepsilon \tag{44}$$

It means that when $1 \le j \le n$,

$$\lambda h(v_j) / v_j \le \lambda \sqrt{n} + \varepsilon \tag{45}$$

$$h(v_j) \le (\sqrt{n} + \frac{\varepsilon}{\lambda}) v_j$$

Set

$$Y_i = span\{u_1, ..., u_{i-1}, u_{i+1}, ..., u_n\} \tag{46}$$

$$c_i = d(u_i, Y_i), \quad c = \min c_i$$

Then Y_i is an finite dimensional closed space and $u_i \notin Y_i$

Also set

$$y_j = -\sum_{i \ne j} (\frac{k_i}{k_j}) u_i \tag{47}$$

Consequently, it follows from theorem 4.4 immediately that:

$$\left\|\sum k_i u_i\right\| = \left\|k_j u_j - k_j y_j\right\| \geq |k_j| \|c\| \tag{48}$$

Associate with (45)，we get

$$\|h(v_j)\| \leq \sqrt{n} + \frac{\varepsilon}{\lambda} \tag{49}$$

So,for every $x \in convA$, $A = \bigcup_{i=1}^{n} A_i$

$$\left\|\sum a_j h(v_j)\right\| \leq \sqrt{n} + \frac{\varepsilon}{\lambda} \quad \text{, where } \sum a_j = 1 \tag{50}$$

From (46)(49)(50)，we obtain

$$|a_j| \|c\| < \sqrt{n} + \frac{\varepsilon}{\lambda} \tag{51}$$

$$|c| < \frac{\sqrt{n} + \frac{\varepsilon}{\lambda}}{|a_j|}$$

We also note that the theorem 4.3:

$$\|r - x\| \leq 2(n+1)\|x - X\| \tag{52}$$

Substitute the results above to theorem 4.3，

$$\frac{\sqrt{n} + \frac{\varepsilon}{\lambda}}{|a_j|} > \frac{\|r - x\|}{2(n+1)} \tag{53}$$

$$2(\sqrt{n} + \frac{\varepsilon}{\lambda})(n+1) > \|r - x\| |a_j| \tag{54}$$

From ε's arbitary，we get：

$$2\sqrt{n}(n+1) / |a_j| > \|r - x\| \tag{55}$$

Now let us discuss the inequality $\|x_n\| \le \lambda\sqrt{n}+\varepsilon$ and the equality $x(j) = \lambda h(v_j)/v_j = x_n(j)$ above . From the results in (54) and (55) ,we can write out: $2\sqrt{n}(n+1)/|a_j|>\|r-x\|$, Where $i\neq j$. Next ，we consider the problem of $a_j(i\neq j)$, is that the inequality must suit the condition that $|c|>\|x-X\|$. So, we should select the a_i， which ensure $c_i=d(u_i,Y_i)$ and $c=\min c_i$ (46) .This also means that i is on the unique dimension which makes $c=\min d(u_i,Y_i)$'s, and a_j can be any a which is different from this unique a_i. Which guarantee the inequality we get last is established..

part Ⅱ schauder fixed point's characterization for compact groups

We recall the first part of this paper. In that part ,we get some considerable results in weak fixed point and also in schauder conjecture .Here, we try to apply the properties in weak fixed point to the schauder conjecture. Then, we classified the schauder fixed points, and characterization the compact groups.

Firstly ， we introduce the main thoughts of the first part ：

1. summary the first part

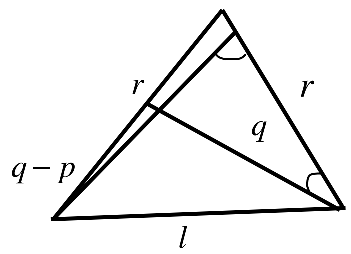
In Part one section (2) weak fixed point and error estimate, we assume that $\varepsilon < d(x,y)$, then select $r\in Q$， such that $d(x,y)-\varepsilon < r < d(x,y)$

We will ask why we can assume in this way? Because it is compact group, so it is seperable. Then we can use ε net, which is also mentioned in theorem 2.1 (Part one): let G be a compact group, and let $\{D_a : a \in \Lambda\}$ be a decreasing net of bounded subsets of $B(G)$,and ϕ_m be a weak convergent sequence with weak limit ϕ.

In addition, we use an inequality in number theory as below:

Because $\sqrt{2}$ is an irrational number , so $\left|2q^2 - p^2\right| > 1$

And $\left|q/p - \frac{\sqrt{2}}{2}\right| > \frac{1}{4p^2}$



We recall figure 1 and formula (1)(2)(3) in Part one (section one),

$$\|\phi_{m1}\| > q - \varepsilon/4$$

$$\|\phi_{m1} - \psi_n\| < r + \varepsilon/4$$

$$\|\psi_n\| > r - q + p - \varepsilon/4$$

It means that there exists a triangle, whose three lines are $q, r, r-q+p$

Then, we extend the triangle's one line to make it an isosceles triangle. Since there exists a distance less than ε,so we can use sin and cos theorem to calcular q .(Part one (24)) Because ψ_n is compact , so we can extend it. In addition, one line's length is r,and another line's is

$r + r^{'}$. so it is an isosceles triangle when $r^{'}$'s length is less than ε.

Figure 2:

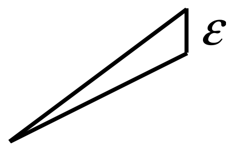
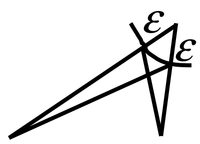


Figure 3:



For instance ：we can make the two triangles' measure of areas equal，to estimate the errors in weak fixed point type problems..In Part one section (3) projection operator and the neumann algebra. We select that $r = \frac{3}{2}p$， to ensure $r - q = 0$. Then we can omit $m(r - q)$,which lead to the result below : $k^4 - 2k^3 - k^2 + 2k > p$ .Also note that in Part one section (4) weakly admissible compact convex sets and schauder conjecture，we use theorem1.1 to assume that $u_1, u_2, \ldots\ldots, u_n$ are linear independent vectors in norm space X , so that when the constant c>0 , for every k_i we have $\left\|\sum k_i u_i\right\| > ck_j$.

2. schauder fixed point's characterization

Next ， we consider how to characterize the compact group. Firstly We recall some results we get in Part one section 4. weakly admissible compact convex sets and schauder conjecture .

From the inequality (45)(46),we get ：

$$\lambda h(v_j)/v_j \le \lambda\sqrt{n}+\varepsilon \tag{1}$$

We can write out：

$$\lambda h(v_j)/v_j \le \lambda+\lambda\varepsilon+\varepsilon \tag{2}$$

Then， we ge the final results. When it is on the unique dimension which make $c=\min d(u_i,Y_i)$ to be the minimum value， we get the inequality ：

$$2(n+1)/|a_j|>\|r-x\| \tag{3}$$

The condition of this inequality is compact metric space. Because metric spaces are all hausdroff spaces. So, metric space is group, associate with the condition of compact groups, we can use weak fixed point properties to study the inequality (3).

Now, we discuss the following theorems:

Theorem 2.1：let X be a linear metric space and let a be a nonzero point of X .then there is a retraction map ra:X$\to[0,a]$ so that:

$$\|x-r_a(x)\|\le 4\|x-[0,a]\|$$

Theorem 2.2: let G be a compact group, and let $\{D_a : a\in\Lambda\}$ be a decreasing net of bounded subsets of $B(G)$,and φ_μ be a weak convergent bounded net with weak limit φ_μ, then

$$r(\varphi)+\limsup\|\varphi_\mu-\varphi\|=\limsup r(\varphi_\mu)$$

i.e.:

$$\limsup\{\|\varphi-\psi\|\psi\in D_a\}+\limsup\|\varphi_\mu-\varphi\|=\limsup\limsup\{\|\varphi_\mu-\psi\|\psi\in D_a\}$$

Theorem 2.3：
assume that X is a norm space， $a\in X$,and k is a nonempty number, then the map ： $x\to x+a$, $x\to kx$ is X to itself's homeomorphism .
Theorem 2.4:
If X is norm space,then it is homeomorphism to an open ball.

$$U(0,r)=\{x\in X,\|x\|<r\}$$

It follows from theorem 2.1 that :

$$[x,y]=\{tx+(1-t)y:t\in[0,1]\} \tag{4}$$

It can be seen that，

$$\|x-r_a(x)\|\le 4\|x-(1-t)a\| \tag{5}$$

$$\|x-r_a(x)\|\le 4\|x-a+t_j a\|$$

Then (3)can be written in the form:

$$t_j a=a_j\le\frac{2(n+1)}{\|r-x\|} \tag{6}$$

So that ，

$$\|x-r_a(x)\|\le 4(\|x-a\|+\frac{2(n+1)}{\|r-x\|}) \tag{7}$$

Next ,we apply theorem 2.3 and 2.4 to further discuss the schauder fixed point .since the weak fixed point in compact groups have banach algebra. So our assumptions are reasonable.
Consider the inequality (7)：

$$\|x-r_a(x)\| \le 4(\|x-a\| + \frac{2(n+1)}{\|r-x\|})$$

Here , we consider the functions ϕ and ψ 's properties under the condition of weak fixed point property for left reversible semigroups . Firstly, from theorem 2.3 and 2.4 ,we can assume:

$$\varphi(y)=\frac{ry}{1+\|y\|}\to q \tag{8}$$

$$\psi(x-a)=\frac{x-a}{r-\|x-a\|}\to r-q+p \tag{9}$$

With the limit of q and $r-q+p$

Since,

$$\varphi(y)=\frac{ry}{1+\|y\|}=x\le r \tag{10}$$

From the homeomorphism of X, we have:

$$r-\|x-a\|=\frac{r}{1+\|y\|} \tag{11}$$

$$\|x-a\|=r-\frac{r}{1+\|y\|}<r-\frac{r}{1+r-q+p} \tag{12}$$

From the homeomorphism property and the weak fixed point property for left reversible semigroups.we can also assume that:

$$\frac{1}{\|r-x\|}=\frac{x-a}{r-\|x-a\|}\to\frac{x}{r-\|x\|}=\|y\| \tag{13}$$

Please note that :we use the following properties in theorem 2.3

$$T_{-a}T_a(x)=T_{-a}(x+a)=(x+a)-a=x \tag{14}$$

Then,

$$\|r-x\|=\frac{r}{(1+\|y\|)(x-a)}=\frac{r}{(1+\|y\|)\varphi(y)} \tag{15}$$

observe that,

$$\frac{1}{\|r-x\|}=\frac{(1+\|y\|)\varphi(y)}{r}\leq\frac{(1+r-q+p)q}{r} \tag{16}$$

we substitute it into (7) to obtain:

$$\|x-r_a(x)\|\leq 4[(r-\frac{r}{1+r-q+p})+2(n+1)\frac{(1+r-q+p)q}{r}] \tag{17}$$

We select $r=\frac{3}{2}p$, where $r-q=0$, so that:

$$\|x-r_a(x)\|\leq 4[r(1-\frac{1}{1+p})+2(n+1)\frac{(1+p)\frac{3}{2}p}{r}] \tag{18}$$

Since theorem 2.1, we have:

Because :

$$r_a(x)=\sup\{\|x-y\|, y\in D_a\} \tag{19}$$

Therefore,

$$\|x-r_a(x)\|=\|x-\sup\|x-y\|\|\geq\|x\|-\sup\|x-y\| \tag{20}$$

Then from theorem 2.2 and select the limit form of (18), we get:

$$\lim\|\psi\|\leq 4[r(1-\frac{1}{1+p})+2(n+1)\frac{(1+p)\frac{3}{2}p}{r}] \tag{21}$$

We also know that $\psi\in D_a$ is decreasing net, so we can use the property of net ,also combining with theorem 2.2 , we have:

$$\|\psi_\beta\|>\sum\|\psi_\beta(i)\|\geq k(p-\varepsilon)/2 \tag{22}$$

$$k^{'}(p-\varepsilon)/2 \le 4[r(1-\frac{1}{1+p})+2(n+1)\frac{(1+p)\frac{3}{2}p}{r} \quad (23)$$

Where we select the subnet as ：

$$\lim_{(a,n)}\left\|\psi_{a,n}\right\|=\lim_{\beta}\left\|\psi_{\beta}\right\| \quad (24)$$

Then we can omit ε to obtain：

$$\frac{k^{'}}{2}p \le 4[\frac{3}{2}p(1-\frac{1}{1+p})+2(n+1)(1+p)] \quad (25)$$

From (39) in Part one,we can write out the following formula:

$$f(k)=k^4-2k^3-k^2+2k>p \quad (26)$$

Then, turn to the inequality (26) above,we induce that：

Case $p=0$， n can be any dimension

Case $p=1$， $n \ge [(\frac{k^{'}}{8}-\frac{3}{4})/4-1]$

Case $p=2$， $n \ge [(\frac{k^{'}}{4}-2)/6-1]$

Case $p=3$，

$$n \ge [(\frac{3k^{'}}{8}-\frac{27}{8})/8-1] \quad (27)$$

From the deduce above，we can select $p=n$, so that:

$$n \ge [(\frac{nk^{'}}{8}-a)/(2n+2)-1] \quad (28)$$

Combining with (26),we can pass to the following inequality:

$$f(k)=k^4-2k^3-k^2+2k>p \quad (29)$$

Substitute it to (28),we get:

$$p=n\geq [(\frac{nk'}{8}-a)/(2n+2)-1]\geq [(\frac{nk}{8}-a)/(2n+2)-1]\ (k'>k) \tag{30}$$

It therefore suffices to prove the inequality:

$$[(\frac{nk}{8}-a)/(2n+2)-1] \tag{31}$$

Because， $f(k)\geq 0$

So， we can select $\frac{nk}{8}-a>2n+2$

To obtain，

$$n\geq \frac{16+8a}{k-16} \tag{32}$$

Next， we assume that：

$$n_1=[(\frac{nk'}{8}-a)/(2n+2)-1]\ (k'>k) \tag{33}$$

substitute it into (29)， we obtain:

$$f(k)\geq n_1,$$

that is:

$$k^4-2k^3-k^2+2k-n_1\geq 0 \tag{34}$$

This is a fourth degree equation， we match an item to it as follows：

$$k^4-2k^3+k^2-k^2-k^2+2k-n_1\geq 0 \tag{35}$$

$$(k^2-k)^2-2(k^2-k)-n_1\geq 0$$

Work out this equation， we get：

$$k^2-k\geq 1+\sqrt{1+n_1} \tag{36}$$

so，

$$n_1 \leq (k^2 - k - 1)^2 - 1 \tag{37}$$

if we transform this equation $f(k) \geq n_1$ to third degree form，such that：

$$k^3 - 2k^2 - k + 2 - \frac{n_1}{k} \geq 0 \tag{38}$$

Applying the Shengjin's Formulas：

$$k^3 = 2k^2 + k - 2 + \frac{n_1}{k} \tag{39}$$

$$A = b^2 - 3ac$$

$$B = bc - 9ad$$

$$C = c^2 - 3bd$$

$$\Delta = B^2 - 4AC \tag{40}$$

$$[2 - 9(2 - \frac{n_1}{k})]^2 = 28[1 + 6(2 - \frac{n_1}{k})] \tag{41}$$

Immediate calculation shows that ： $\frac{n_1}{k} = \frac{19}{9}$

then，

$$n_1 = \frac{19}{9}k \tag{42}$$

3. Conclusion ：

we will ask a technical problem again that how to use the results we get above ? For instance we can assume $n = 3$，that is a three dimension topology space. We substitute the results we get into (33), which leads to:

$$n_1 = [(\frac{nk'}{8} - a)/(2n+2) - 1] > \frac{19}{9}k \quad (43)$$

$$\frac{3}{8}k' \geq 8(\frac{19}{9}k + 1) + a \quad (44)$$

$$k' \geq \frac{64}{3}(\frac{19}{9}k + 1) + \frac{8}{3}a$$

From (32)， we have

$$n = 3 \geq \frac{16+8a}{k-16} \quad (45)$$

Then，

$$k \geq \frac{64}{3} + \frac{8a}{3} \quad (46)$$

Therefore，

$$k' \geq \frac{64}{3}(\frac{19}{9}k+1) + \frac{8}{3}a \geq \frac{64}{3}(\frac{19}{9}(\frac{64}{3} + \frac{8a}{3}) + 1) + \frac{8}{3}a \quad (47)$$

This result gives the conclusion of the whole paper.

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