A Very Brief Introduction To Clifford Algebra

Stephen Crowley

Email: stephen.crowley@hushmail.com

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Abstract. This article distills many of the essential definitions from the very thorough book, Clifford Algebras: An Introduction, by Dr D.J.H. Garling, with some minor additions.

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1. Definitions

The contents of this paper are distilled and in many instances copied verbatim from the very thorough [1].

1.1. Notation and Symbols.

The symbol $\mathbb{Z}$ denotes the set of integers. The symbol "$\forall$" represents the expressions "for all" or "for each". Likewise, the symbol "$\in$" means "is an element of" or "in". Additional symbols and notations will be introduced as needed.

1.2. Groups.

A *group* is a non-empty set G together with a *composition law*, a mapping $(g,h)\to gh$ from $G\times G$ to G which satisfies

1. $(gh)j = g(hj)\forall g,h,j\in G$ (associativity)
2. there exists e in G such that $eg = ge{=}g\forall g\in G$ and
3. $\forall g\in G$ there exists $g^{-1}\in G$ such that $gg^{-1}=g^{-1}g=e$

It follows that the *identity element* g is unique and that $\forall g\in G$ the *inverse* g^{-1} is also unique. A group G is said to be *abelian*(named after the Norwegian mathematician Niels Henrik Abel), or *commutative*, if $gh=hg\forall g,h\in G$. If G is commutative then the law of composition is often written as addition: $(g,h)\to g{+}h$ where in such case the identity is denoted by 0 and the inverse of g by $-g$.

1.2.1. Subgroups.

A non-empty subset H of a group G is a *subgroup* of G if $h_1h_2\in H$ and $h^{-1}\in H$ whenever $h\in H$ then H becomes a group under the law of composition inherited from G and this situation is denoted by $H\subseteq G$ if H is possibly equal to G and $H\subset G$ if H is strictly contained in and definately not equal to G.

If $A\subseteq G$ then there is a smallest subgroup $\mathrm{Gp}(A)$ of G which contains A called the *subgroup generated* by A. If $A=\{g\}$ is a singleton then we write $\mathrm{Gp}(g)$ for $\mathrm{Gp}(A)$. Then $\mathrm{Gp}(g)=\{g^n\colon n\in\mathbb{Z}\}$ where $g^0=e$, g^n is the product of n copies of g when $n>0$, and g^n is the product of $|n|$ copies of g^{-1} when $n<0$. A group G is *cyclic* if $G=\mathrm{Gp}(g)$ for some $g\in G$. If G has a finite number of elements then the *order* $o(G)$ of G is the number of elements of G. If $g\in G$ then the order $o(g)$ of g is the order of the group $o(\mathrm{Gp}(g))$.

A mapping $\theta\colon G\to H$ from a group G to a group H is a called a *homomorphism*, or simply a *morphism*, if

$$\theta(g_1\,g_2)=\theta(g_1)\,\theta(g_2)\forall g_1,g_2\in G$$

It follows that θ maps the identity in G to the identity in H and that $\theta(g^{-1})=\theta(g)^{-1}$. A bijective morphism is called an *isomorphism*, and an isomorphism $G\to G$ is called an *automorphism* of G. The set $\mathrm{Aut}(G)$ of automorphisms of G forms a group when the law of composition is taken as the composition of mappings.

It should be pointed out that a homomorphism is not the same as a homeomorphism. The Greek meaning of homomorphism is "to form alike" whereas a homeomorphism is a "continuous transformation".

1.2.2. Quotient Groups.

A subgroup K of a group G is a *normal*, or *self-conjugate*, subgroup if $g^{-1}hg\in K\forall g\in G, k\in K$. If θ: $G\rightarrow H$ is a homomorphism then the *kernel* $\ker(\theta)$ of θ, the set $\{g\in G\colon \theta_g=e_H\}$ is a normal subgroup of G where e_H is the identity in H.

Conversely, suppose K is a normal subgroup of G then the relation $g_1\sim g_2$ is an *equivalence relation* on G if $g_1^{-1}g_2\in K$. The equivalence classes are called the *cosets* of K in G. If C is a coset of K then C is of the form $\mathrm{Kg}=\{kg\colon k\in K\}$ and $Kg=gK=\{gk\colon k\in K\}$. If C_1 and C_2 are cosets of K in G then so is $C_1C_2=\{c_1c_2\colon c_1\in C_1, c_2\in C_2\}$; if $C_1=Kg_1$ and $C_2=Kg_2$ then $C_1C_2=Kg_1g_2$. With this composition law the set G/K of cosets is known as the *quotient group* having identity K and $(Kg)^{-1}=Kg^{-1}$. The quotient mapping defined by the equivalence relation $q\colon G\rightarrow G/K$ is then a morphism of G onto G/K, with kernel K, and $q(g)=Kg$.

1.2.3. Exact Sequences, Centers, and Centralizers.

A group G is *simple* if it has no normal subgroups other than $\{e\}$ and G. The group with one element is denoted by 1 if the composition law is multiplication, or 0 if the composition law is addition. Suppose that a sequence of groups $\{G_0=1,G_1,\ldots,G_k,G_{k+1}=1\}$ is given and that $\theta_j\colon G_j\rightarrow G_{j+1}$ is a morphism for $0\leqslant j\leqslant k$ then the diagram

$$1\xrightarrow{\theta_0}G_1\xrightarrow{\theta_1}G_2\xrightarrow{\theta_2}\ldots\xrightarrow{\theta_{k-1}}G_k\xrightarrow{\theta_k}1$$

is an *exact sequence* if $\theta_{j-1}(G_{j-1})$ is the kernel of $\theta_j\forall 1\leqslant j\leqslant k$. When $k=3$ the sequence is a *short exact sequence*. For example, if K is a normal subgroup of g and $q\colon G\rightarrow G/K$ is the quotient mapping then

$$1\longrightarrow K\xrightarrow{\subseteq}G\xrightarrow{q}G/K\longrightarrow 1$$

is a short exact sequence. If A is a subset of a group G then the *centralizer* $C_G(A)$ of A in G is defined as

$$C_G(A)=\{g\in G\colon ga=ag\forall a\in A\}$$

Similarly, the *center* $Z(G)=C_G(G)$ is defined as

$$Z(G)=\{g\in G\colon gh=hg\forall h\in G\}$$

and is a normal subgroup of G. The product of two groups $G_1\times G_2$ is a group when the composition law is defined by

$$(g_1,g_2)(h_1,h_2)=(g_1h_1,g_2h_2)$$

The subgroup $G_1\times\{e_2\}$ is identified with G_1 and the subgroup $\{e_1\}\times G_2$ with G_2.

1.2.4. Specific Instances.

The set of real numbers $\mathbb{R}$ forms a commutative group when addition is the composition law. The set of non-zero real numbers $\mathbb{R}^*$ is a group under multiplication. The set of integers $\mathbb{Z}$ is a subgroup of $\mathbb{R}$. Any two groups of order 2 are isomorphic. Denote the multiplicative subgroup $\{+1,-1\}\subset\mathbb{R}$ by D_2, and the additive group $\{0,1\}$ by D_2 which is isomorphic to the quotient group $\mathbb{Z}/2\mathbb{Z}$. Though they are small, these groups play fundamental roles in the theory of Clifford algebras and other branches of mathematics and physics. Suppse there is a short exact sequence

$$1\longrightarrow D_2\xrightarrow{j}G_1\xrightarrow{\theta}G_2\longrightarrow 1$$

Then $j(D_2)$ is a normal subgroup of G_1, from which it follows that $j(D_2)$ is contained in the center of G_1. If $g\in G$ then we write $-g$ for $j(-1)g$. Then $\theta(g)=\theta(-g)$ and if $h\in G_2$ then $\theta^{-1}\{h\}=\{-g,g\}$ for some $g\in G$ in which case we say that G_1 is a double cover of G_2. Double covers play fundamental roles in the theory of spin groups. The complex numbers $\mathbb{C}$ form a commutative group under addition and $\mathbb{R}$ is a subgroup of $\mathbb{C}$. The set $\mathbb{C}^*$ of non-zero complex numbers is a group under multiplication. The set $\mathbb{T}=\{z\in\mathbb{C}\colon|z|=1\}$ is a subgroup of $\mathbb{C}^*$. There is also the short exact sequence

$$0\longrightarrow\mathbb{Z}\xrightarrow{\subseteq}\mathbb{R}\xrightarrow{q}\mathbb{T}\longrightarrow 1$$

where $q(\theta)=e^{2\pi i\theta}$. The subset $\mathbb{T}_n=\{e^{2\pi i\frac{j}{n}}\colon 0\leqslant j\leqslant n\}=\{z\in\mathbb{C}\colon z^n=1\}$ is a cyclic subgroup of $\mathbb{T}$ of order $o(n)$. Conversely, if $G=Gp(g)$ is a cyclic group of order n then the mapping $g^k\rightarrow e^{2\pi i\frac{k}{n}}$ is an isomorphism of G onto $\mathbb{T}_n$.

1.2.5. Mappings.

A bijective mapping of a set X onto itself is called a *permutation*. The set Σ_X of permutations of X is a group under the composition of mappings. The permutation set Σ_X is noncommutative if X contains at least 3 elements. The group of permutations of the set $\{1, ..., n\}$ is denoted by Σ_n. A *transposition* is a permutation which fixes all but 2 of the elements. The permutation group Σ_n has a normal subgroup A_n of order $\frac{1}{2}n!$ composed of all the permutations that can be expressed as the product of an even number of transpositions. Thus, we have the short exact sequence

$$1 \longrightarrow A_n \xrightarrow{\subseteq} \Sigma_n \xrightarrow{\epsilon} D_2 \longrightarrow 1$$

The *signature* is defined by

$$\epsilon(\sigma) = \begin{cases} +1 & \sigma \in A_n \\ -1 & \text{otherwise} \end{cases}$$

1.2.6. Dihedral Groups.

The *full dihedral group* D is the group of isometries of the complex plane $\mathbb{C}$ which fix the origin

$$D = \{g: \mathbb{C} \to \mathbb{C}: g(0) = 0 \,\text{and}\, |g(z) - g(w)| = |z - w| \forall z, w \in \mathbb{C}\}$$

An element of D is either a *rotation* $R_\theta = e^{i\theta} z$ or a *reflection* $S_\theta = e^{i\theta}\bar{z}$. The set Rot of rotations is a subgroup of D and the mapping $R{:}\, e^{i\theta} \to R_\theta$ is an isomorphism of $\mathbb{T}$ onto Rot. Particularly, we see that $R_\pi(z) = -z$ since

$$S_\theta^2(z) = e^{i\theta}\overline{(e^{i\theta}\bar{z}\,)} = e^{i\theta}\, e^{-i\theta} z = z$$

A similiar calculation shows

$$S_\theta^{-1} R_\phi S_\theta = R_{-\phi}$$

so that R is a normal subgroup of D. We have an exact sequence

$$1 \longrightarrow \mathbb{T} \xrightarrow{R} D \xrightarrow{\delta} D_2 \longrightarrow 1$$

where $\delta(R_\theta) = 1$ and $\delta(S_\theta) = -1 \forall \theta \in [0, 2\pi)$. If we let $R_n = R(\mathbb{T}_n)$ and then $D_{2n} = R_n \cup R_n S_0$ is a subgroup of D called the *dihedral group of order* $2n$(some authors use the notation D_n). The symbol "$\cup$" means "union". There is an equivalence relation $D_4 \cong D_2 \times D_2$ so we see that D_4 is commutative. If $n \geqslant 3$ then D_{2n} is the symmetry group of a regular polygon with n vertices with the center at the origin. The subgroup D_{2n} of D is a noncommutative subgroup of order $2n$. If $n = 2k$ is even then $Z(D_{2n}) = \{1, r_{-1}\}$ and we have the short exact sequence

$$1 \longrightarrow D_2 \longrightarrow D_{2n} \longrightarrow D_{2k} \longrightarrow 1$$

The group D_{2n} is a double cover of D_{2k}. If $n = 2k + 1$ is odd then $Z(D_{2n}) = \{1\}$. Particularly, the dihedral group D_8 is the noncommutative group of symmetries of the square with center at the origin. Let's set $\alpha = r_i$, $\beta = \sigma_1$, $\gamma = \sigma_i$ then $D_8 = \{\pm 1, \pm\alpha, \pm\beta, \pm\gamma\}$ where $-x$ denotes $r_{-1}x = x r_{-1}$ and

$$\begin{array}{ccc} \alpha\beta = \gamma & \beta\gamma = \alpha & \gamma\alpha = \beta \\ \beta\alpha = -\gamma & \gamma\beta = -\alpha & \alpha\gamma = -\beta \\ \alpha^2 = -1 & \beta^2 = 1 & \gamma^2 = 1 \end{array}$$

There is the short exact sequence

$$1 \longrightarrow D_2 \longrightarrow D_8 \longrightarrow D_4 \longrightarrow 1$$

1.2.7. Quaternions.

The *quaternionic group* $\mathcal{Q}$ is a group of order $o(8)$ with elements $\{\pm 1, \pm i, \pm j, \pm k\}$ having identity element 1 and the composition law defined by

$$\begin{array}{ccc} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = j \\ i^2 = -1 & j^2 = -1 & k^2 = -1 \end{array}$$

where $(-1)x=x(-1)=-x$, $(-1)(-x)=(-x)(-1)=x\forall x\in\{1,i,j,k\}$. The center of $\mathcal{Q}$ is $Z(\mathcal{Q})=\{+1,-1\}$ and there is the short exact sequence

$$1\longrightarrow D_2\longrightarrow\mathcal{Q}\longrightarrow D_4\longrightarrow 1$$

The group of quaternions $\mathcal{Q}$ is a double cover of the commutative dihedral group D_4. The groups D_8 and $\mathcal{Q}$ are particularly important in the study of Clifford algebras and both provide double covers of D_4 but they are not isomorphic to each other. $\mathcal{Q}$ has 6 elements of order $o(4)$, 1 of order $o(2)$, and 1 of order $o(1)$. D_8 has 2 elements of order $o(4)$, 5 of order $o(2)$, and 1 of order $o(1)$.

Both $\mathbb{R}$ and $\mathbb{C}$ are real finite-dimensional algebras, with real dimension 1 and 2 respectively, and both are also *fields* since every non-zero element within them have multiplicative inverses. A *division algebra* is a noncommutative field. The algebra $\mathbb{H}$ of *quaternions* provides an example of a noncommutative finite-dimensional real division algebra. The space of 2×2 complex matrices $M_2(\mathbb{C})$ can be considered as an 8-dimensional real algebra. The associate Pauli matrices $\tau_{0,x,y,z}$, to be introduced in a later section, form a linearly independent subset of $M_2(\mathbb{C})$ and so their real linear span is a 4-dimensional subspace of $\mathbb{H}$. If we let $h=a\tau_0+b\tau_x+c\tau_y+d\tau_z\in\mathbb{H}$ then

$$h=\begin{pmatrix} a+id & ib+c\\ ib-c & a-id\end{pmatrix}$$

and consequently

$$\mathbb{H}=\begin{pmatrix} z & w\\ -\bar{w} & \bar{z}\end{pmatrix}\forall z,w\in\mathbb{C}$$

$\mathbb{H}$ is a 4-dimensiional unital subalgebra of the 8-dimensional real algebra $M_2(\mathbb{C})$ since composition of the associate Pauli matrices generates the group $\pi(\mathcal{Q})$. Elements $x\in\operatorname{span}(1)$ are called *real quaternions* iff x is in the center $Z(\mathbb{H})$ of $\mathbb{H}$. Elements of $\operatorname{span}(i,j,k)$ are called *pure quaternions*, the space of which is denoted by $\operatorname{Pu}(\mathbb{H})$, iff x^2 is real and non-positive.

1.3. Vector Spaces.

1.3.1. Linear Subspaces.

Let K denote the field of either the real $\mathbb{R}$ or complex $\mathbb{C}$ numbers. A *vector space* E over K is a commutative additive group $(E,+)$, together with a mapping, scalar multiplication, $(\lambda,x)\to\lambda x$ of $K\times E$ into E which satisfies

- $1x=x$
- $(\lambda+\mu)(x)=\lambda x+\mu x$
- $\lambda(\mu x)=(\lambda\mu)x$
- $\lambda(x+y)=\lambda x+\lambda y$

$\forall\lambda,\mu\in K$ and $x,y\in E$. The elements of E are called *vectors* and the elements of K are called *scalars*. It follows that $0x=0$ and $\lambda 0=0\forall x\in E,\lambda\in K$. Note that the same symbol 0 is used for the additive identity element in E and the zero element in K. A non-empty subset F of a vector space E is a *linear subspace* of E if it is a subgroup of E and $\lambda x\in U\forall\lambda\in K,x\in F$. A linear subspace is then a vector space with operations inherited from E. If A is a subset of E then the intersection of all the linear subspaces containing A is a linear subspace known as the subspace $\operatorname{span}(A)$ *spanned* by A. If E is spanned by a finite set then E is said to be *finite-dimensional*. By convention, all considered vector spaces will be finite-dimensional unless otherwise stated.

A subset $B\subseteq E$ is *linearly independent* if whenever $\lambda_1,\ldots,\lambda_k$ are scalars and $b_1,\ldots,b_k$ are distinct elements of B for which $\lambda_1b_1+\cdots\lambda_kb_k=0$ then $\lambda_1=\cdots=\lambda_k=0$. A linearly independent finite subset $B\subseteq E$ which spans E is called a *basis* and is denoted by $(b_1,\ldots,b_d)$. If $(b_1,\ldots,b_d)$ is a basis for E then every element $x\in E$ can be written uniquely as $x=x_1b_1+\cdots+x_db_d=0$ where $x_1,\ldots,x_d$ are scalars. All finite-dimensional vector spaces have a basis. If A is a linearly independent subset of E contained in a subset $C\subseteq E$ which spans E then there is a basis B for E such that $A\subseteq B\subseteq C$. Any pair of bases have the same number of elements which is the *dimension* of E denoted $\dim(E)$. For example, let $E=K^d$ be the product of d copies of K with coordinatewise addition and with scalar multiplication

$$\lambda(x_1,\ldots,x_d)=(\lambda x_1,\ldots,\lambda x_d)$$

If $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is a vector with 1 in the jth position then K^d is a vector space having basis $(e_1, \ldots, e_d)$ called the *standard basis*. More generally, let $M_{m,n} = M_{m,n}(K)$ denote the set of all K-valued functions on $\{1, \ldots, m\} \times \{1, \ldots, n\}$ then $M_{m,n}$ becomes a vector space over K when addition and multiplication are defined coordinatewise. The elements of $M_{m,n}$ are the familiar *matrices*. The matrix taking the value 1 at (i, j) and 0 elsewhere is denoted $E_{i,j}$. The set of matrices $\{E_{ij}: 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ forms a basis for $M_{m,n}$ so that $\dim(M_{m,n}) = mn$.

If E_1 and E_2 are vector spaces then the product $E_1 \times E_2$ is also a vector space with addition and scalar multiplication defined by

$$\begin{aligned}(x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2)\\ \lambda(x_1, x_2) &= (\lambda x_1, \lambda x_2)\end{aligned}$$

where $\dim(E_1 \times E_2) = \dim(E_1) + \dim(E_2)$. A mapping $T: E \rightarrow F$ where E and F are vector spaces over the same field K is *linear* if

$$\begin{array}{c}T(x + y) = T(x) + T(y)\\ T(\lambda x) = \lambda T(x)\end{array} \quad \forall \lambda \in K, x, y \in E$$

The *image* $T(E)$ of E is a linear subspace of F and the *null-space* defined by $N(T) = \{x \in E: T(x) = 0\}$ is a linear subspace of E. The *rank* of T is $\mathrm{rank}(T) = \dim(T(E))$ and the *nullity* of T is $n(T) = \dim(N(T))$. The fundamental *rank-nullity* formula is then $\mathrm{rank}(T) + n(T) = \dim(E)$.

A bijective linear mapping $J: E \rightarrow F$ is called an *isomorphism* iff(if and only if) $N(J) = \{0\}$ and $J(E) = F$ then $\dim(E) = \dim(F)$. The topology of a vector space can be considered by noting that K^d is a complete metric space having the usual Euclidean distance metric

$$d(x, y) = \sqrt{\sum_{j=1}^{d} |x_j - y_j|^2}$$

We can then define a norm $\|.\|$ on F by setting $\|J(x)\| = \|x\|$ which depends on the choice of the basis, however any pair of norms are equivalent and define the same topology. If F_1 and F_2 are linear subspaces of E for which the linear mapping $(x_1, x_2) \rightarrow x_1 + x_2: F_1 \times F_2 \rightarrow E$ is an isomorphism then E is the *direct sum* of F_1 and F_2 denoted by $F_1 \otimes F_2$. This happens iff $F_1 \cap F_2 = \{0\}$, $E = \mathrm{span}(F_1 \cup F_2)$ and every element $x \in E$ can be written uniquely as $x = x_1 + x_2 \forall x_1 \in F_1, x_2 \in F_2$. Suppose that $(e_1, \ldots, e_d)$ is a standard basis for E and that $y_1, \ldots, y_d$ are the elements of a vector space F. If $x = \lambda_1 e_1 + \ldots + \lambda_d e_d \in E$ and $T(x) = \lambda_1 y_1 + \cdots + \lambda_d y_d$ then T is the unique linear mapping of E into F for which $T(e_j) = y_j \forall 1 \leqslant j \leqslant d$. This process of constructing T is called *extension by linearity*.

1.3.2. Endomorphisms.

The set $L(E, F)$ of linear mappings from E to F is a vector space defined by

$$\begin{array}{c}(S + T)(x) = S(x) + T(x)\\ (\lambda s)(x) = \lambda(S(x))\end{array} \quad \forall S, T \in L(E, F), x \in E, \lambda \in K$$

where $\dim(L(E, F)) = \dim(E) \dim(F)$. The elements of $L(E) = L(E, E)$ are the *endmorphisms* of E. The Greek meaning of the word endomorphism is "to form inside". The composition of $T \in L(E, F)$ and $S \in L(F, G)$ is $ST = S \circ T \subseteq L(E, G)$. Suppose that $(e_1, \ldots, e_d)$ is a standard basis for E, $(f_1, \ldots, f_d)$ is a basis for F and that $T \in L(E, F)$. If we let $T(e_j) = \Sigma_{i=1}^{c} t_{ij} f_i$ and $x = \Sigma_{j=1}^{d} x_j e_j$ then $T(x) = \Sigma_{i=1}^{c} (\Sigma_{j=1}^{d} t_{ij} x_j) f_i$. The mapping $T \rightarrow (t_{ij})$ is an isomorphism of $L(E, F)$ onto $M_{c,d}$ so that $\dim(L(E, F)) = cd = \dim(E) \dim(F)$. T is said to be represented by the $c \times d$ matrix (t_{ij}). If $(g_1, \ldots, g_d)$ is a basis for G and $S \in L(F, G)$ is represented by the matrix (s_{hi}) then the product $R = ST \in L(E, G)$ which defines matrix multiplication is represented by the matrix (r_{hj}) where $r_{hj} = \Sigma_{i=1}^{c} s_{hi} t_{ij}$. If $(e_1, \ldots, e_d)$ is the standard basis for E then an element $T \in L(E)$ of the endmorphisms of E can be represented by a matrix (t_{ij}), and the mapping $T \rightarrow (t_{ij})$ is an algebra isomorphism of $L(E)$ onto the algebra $M_d(K)$ of $d \times d$ matrices, where composition is defined as matrix multiplication.

1.3.3. Duality of Vector Spaces.

$\mathbb{R}$ is a 1-dimensional vector space over K whereas $\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$ with basis $\{1, i\}$. The space $L(E, K)$ is called the *dual* or *dual space* of E and is denoted by E', the elements of which are known as *linear functionals* on E. Suppose that $(e_1, \ldots, e_d)$ is a basis for E. If we let $x = \Sigma_{i=1}^d x_i e_i$ and $\phi_i(x) = x_i \forall 1 \leqslant i \leqslant d$ then $\phi_i \in E'$ and $(\phi_1, \ldots, \phi_d)$ is a basis for E' known as the *dual basis* of $(e_1, \ldots, e_d)$. Thus $\dim(E) = \dim(E')$. If $x \in E$, $\phi \in E'$ and $j(x)(\phi) = \phi(x)$ then $j\colon E \rightarrow E''$ is an isomorphism of E onto E'', the dual of E' known as the *bidual* of E.

Suppose that we let $T \in L(E, F), \psi \in F', x \in E$ and $(T'(\psi))(x) = \psi(T(x))$ then $T(\psi) \in E'$ and T' is a linear mapping of F' into E' known as the *transposed* mapping of T. If A is a subset of E then the *annihilator* $A^{\perp}$ in E' of A is the set

$$A^{\perp} = \{\phi \in E' \colon \phi(a) = 0 \forall a \in A\}$$

which is a linear subspace of E'. Similiarly, if B is a subset of E' then the annihilator $B^{\perp}$ in E of B is the set

$$B^{\perp} = \{x \in E \colon \phi(x) = 0 \forall \phi \in B\}$$

It then follows that $A^{\perp\perp} = \operatorname{span}(A)$ and $B^{\perp\perp} = \operatorname{span}(B)$. If F is a linear subspace of E then $\dim(F) + \dim(F^{\perp}) = \dim(E)$. If $T \in L(E, F)$ then $T(E)^{\perp} = N(T')$ and $N(T)^{\perp} = T'(F')$.

1.4. Algebras, Representations, and Modules.

A finite-dimensional associative *algebra* A over K is a finite-dimensional vector space over K equipped with a composition law $(a, b) \rightarrow ab$ from $A \times A \rightarrow A$ which satisfies

- $(ab)c = a(bc)$ (associativity)
- $a(b + c) = ab + ac$
- $(a+b)c = ac + bc$
- $\lambda(ab) = (\lambda a)b = a(\lambda b)$

$\forall \lambda \in K, a, b, c \in A$ where, as usual, multiplication takes precedence over addition. An algebra A is said to be *unital* if there exists $1 \in A$, the *identity element*, such that $1a = a1 = a \forall a \in A$. Most algebras to be considered will be unital. An algebra is said to be *commutative* if $ab = ba \forall a, b \in A$. A mapping ϕ from an algebra A over K to an algebra B over K is said to be an *algebra morphism* if it is linear and $\phi(ab) = \phi(a)\phi(b) \forall a, b \in A$. If A and B are unital and $\phi(1_A) = 1_B$ where 1_A is the identity element of A and 1_B is the identity element of B then ϕ is said to be a *unital morphism*. An algebra morphism of an algebra into itself is called an *endomorphism*.

Suppose that G is a finite group with identity element e then K^G is a finite-dimensional vector space with basis $\{\delta_g \colon g \in G\}$ known as the *group algebra* which is commutative iff G is commutative. Multiplication on K^G is defined by letting $a = \Sigma_{g \in G} a_g \delta_g$ and $b = \sum_{g \in G} b_g \delta_g$ then setting $ab = \sum_{g \in G} c_g \delta_g$ where

$$c_g = \sum_{hj = g} a_h b_j = \sum_{h \in G} a_h b_{h^{-1}g} = \sum_{j \in G} a_{gj^{-1}b_j}$$

If A is an algebra then the *opposite algebra* A^{opp} is obtained by keeping addition and scalar multiplication the same and defining a new composition law by reversing the original so that $a * b = ba$. A linear subspace B of an algebra A is a *subalgebra* of A if $b_1 b_2 \in B \forall b_1, b_2 \in B$. If A is unital then a subalgebra B is a unital subalgebra if the identity element of A belongs to B. For example, if A is a unital subalgebra then the set $\operatorname{End}(A)$ of unital endomorphisms of A is a unital subalgebra of $L(A)$. The *centralizer* $C_A(B)$ of a subset B of an algebra A is

$$C_A(B) = \{a \in A \colon ab = ba \forall b \in B\}$$

and the *center* $Z(A)$ of A is the centralizer $C_A(A)$ which is a commutative subalgebra if A is a unital algebra

$$Z(A) = \{a \in A \colon ab = ba \forall b \in A\}$$

A unital algebra is said to be *central* if $Z(A)$ is the 1-dimensional subspace span(1). An element p of an algebra A is an *idempotent* if $p^2 = p$. If p is an idempotent of a unital algebra A then $1 - p$ is also an idempotent and $A = pA \otimes (1 - p)A$ is the direct sum of linear subspaces of A. If additionally $p \in Z(A)$ then pA and $(1 - p)A$ are subalgebras of A with identity elements p and $1 - p$ respecively. The subalgebras pA and $(1 - p)A$ are unital if $p \in \{0, 1\}$ in which case the mapping $a \rightarrow pa$ is a unital algebra morphism of A onto pA. An idempotent in $L(E)$ or $M_d(K)$ is called a *projection*.

1.4.1. Super-algebras.

An element j of a unital algebra A is an *involution* if $j^2 = 1$ from which it follows that $\frac{1+j}{2}$ is an idempotent. In a similiar way, an endomorphism θ of an algebra A is an involution if $\theta^2 = I$. If θ is an involution of a unital algebra A and $p = \frac{I+\theta}{2}$ then we can write $A = A^+ \otimes A^-$ where $A^+ = p(A)$ and $A^- = (I-p)(A)$ then $p(A)$ is a subalgebra of A such that

$$\begin{aligned} A^+ = &\ \{a \in A : \theta(a) = a\} \\ A^- = &\ \{a \in A : \theta(a) = -a\} \end{aligned}$$

and

$$\begin{aligned} A^+.A^+ \subseteq &\ A^+ \\ A^-.A^- \subseteq &\ A^+ \\ A^-.A^+ \subseteq &\ A^- \\ A^+.A^- \subseteq &\ A^- \end{aligned} \tag{1}$$

Conversely, if $A = A^+ \otimes A^-$ is a direct sum decomposition for which (1) holds, then the mapping which sends $a^+ + a^-$ to $a^+ - a^-$ is an involution of A known as the *Z_2-graded algebra* or *super-algebra*. The elements of A^+ are called the *even* elements and the non-zero elements of A^- are called the *odd* elements. Any element a of A can be decomposed as $a = a^+ + a^-$, the sum of the even and odd parts of a. If $a \in A^+ \cup A^-$ then a is said to be *homogeneous*. The real algebra $\mathbb{C}$ is a super-algebra when the involution j is defined as $j(x+iy) = x - iy$.

1.4.2. Ideals.

If a is an element of a unital algebra A then b is a *left inverse* of a if $ba = 1$. An element b is similiarly said to be a *right inverse* of a if $ab = 1$. a is *invertable* if it has both a left and right inverse in which case they are unique and equal. This unique element is called the *inverse* of a and is denoted by a^{-1}. The set of invertible elements is denoted by $G(A)$; it is a group under functional composition. $G(A)$ is called the *general linear group* of A when A is the endomorphisms of a vector space $L(E)$. Similiarly, we denote $G(M_d(K))$ by $GL_d(K)$.

There is a unique mapping known as the *determinant*, $\det : M_d(K) \to K$ satisfying

$$\begin{aligned} \det(ST) = &\ \det(S).\det(T) \\ \det(I) = &\ I \qquad\qquad\qquad \forall S, T \in M_d(K), \lambda \in K \\ \det(\lambda T) = &\ \lambda^d \det(T) \end{aligned}$$

where $T \in GL_d(K)$ iff $\det(T) \neq 0$. A subalgebra J of an algebra A is a *left ideal* if $aj \in J \forall a \in A, j \in J$; similarly, J is a *right ideal* if $ja \in J \forall a \in A, j \in J$. A subalgebra which is both a left ideal and a right ideal is called a *two-sided ideal*, or simply an *ideal*. If ϕ: $A \to B$ is an algebra morphism then the null-space $N(\phi) = \{a \in A : \phi(a) = 0\}$ is an ideal in A. An ideal J in a unital algebra is said to be a *proper ideal* iff J is a proper subset of A which is the case iff $1_A \notin J$. An algebra A is a *simple algebra* if the only proper ideal in A is the *trivial ideal* $\{0\}$.

1.4.3. Representations.

A morphism π from a unital algebra A into a $d \times d$ symmetric matrix $M_d(K)$ or a set of endomorphisms of a vector space $L(E)$ is called a *representation* of A. A linear subspace F of E is π-invariant if $\pi(a)(F) \subseteq F \forall a \in A$. The representation π is an *irreducible representation* if $\{0\}$ and F are the only π-invariant subspaces of E. If π is one-to-one then it is said to be *faithful*. A faithful representation of A is therefore a unital isomorphism of A onto a subalgebra of the space of real and complex $d \times d$ symmetric matrices denoted by $M_d(K)$; the elelements of A are represented as matrices. An important example of a faithful representation is the *left regular representation* l: $A \to L(A)$ of a unital algebra A which is given by setting $l(a)(b) = ab$ for which $l_a(1) = a$ so we see that l is faithful.

1.4.4. The Exponential Function.

Suppose that A is a finite-dimensional real unital algebra and $|.|$ is a norm, or equivalently a distance metric, on A. Another norm on A can be defined by setting $\|a\| = \sup\{|ab| : |b| \leqslant 1\}$, where sup in simple terms means "maximum value"; then $\|I\| = 1$ and $\|ab\| \leqslant \|a\|.\|b\|$. It then follows that if $a \in A$ then the sum $e^a = \sum_{j=0}^{\infty} \frac{a^j}{j!}$ converges. The mapping $a \to e^a$ from A to A is called the *exponential function*.

Proposition 1. *Suppose that A is a finite-dimensional real unital algebra and that $a, b \in A$ then*

i. *The exponential function is continuous*

ii. *If $ab = ba$ then $e^{a+b} = e^a e^b$*

iii. *e^a is invertible with inverse e^{-a}*

iv. *The mapping $t \to e^t$ is a continuous morphism of the group $(\mathbb{R}, +)$ into $G(A)$*

v. *If $ab = -ba$ then $e^a b = b e^{-a}$*

vi. *If $a^2 = -1$ then $e^{ta} = \cos(t) + \sin(ta)$, the mapping $e^{it} \to e^{at}$ is a homeomorphism of $\mathbb{T}$ onto a compact subgroup of $G(A)$, and the mapping $t \to e^{ta}$ from $\left[0, \frac{\pi}{2}\right]$ into $G(A)$ is a continuous path from I to a.*

vii. *Suppose that $a \neq I$ and that $a^2 = I$, then $e^{ta} = \cosh(t) + \sinh(ta)$, and the mapping $t \to e^{at}$ is a homeomorphism of $\mathbb{R}$ onto an unbounded subgroup of $G(A)$.*

1.4.5. Group Representations.

The representation of groups can also be considered. A morphism from a group G into $GL_d(K)$ is called a *representation* of G, and π is said to be *faithful* if it is injective(one-to-one). A faithful representation of G is therefore an isomorphism of G onto a subgroup of $GL_d(K)$; the elements of G are represented as invertable matrices. For example, the mapping $\pi: D \to M_2(\mathbb{R})$ defined by

$$\begin{aligned} \pi(R_\theta) &= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \\ \pi(S_\theta) &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \end{aligned}$$

is a faithful representation of the full dihedral group as a group of reflections and rotations of $\mathbb{R}^2$. If this mapping is restricted to D_8 we have a faithful representation

$$\begin{array}{llll} \pi(1) = I & \pi\left(R_{\frac{\pi}{2}}\right) = J & \pi(S_0) = U & \pi\left(S_{\frac{\pi}{2}}\right) = Q \\ \pi(R_\pi) = -I & \pi\left(R_{\frac{3}{2}\pi}\right) = -J & \pi(S_\pi) = -U & \pi\left(S_{\frac{3}{2}\pi}\right) = -Q \end{array}$$

where the matrices and their actions are described by the table

Name	Matrix	Action
I	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Identity
$-I$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	Rotation by π
J	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	Rotation by $\frac{\pi}{2}$
$-J$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	Rotation by $-\frac{\pi}{2}$
U	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Reflection in the direction $(0,1)$ with mirror $y = 0$
$-U$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	Reflection in the direction $(1,0)$ with mirror $x = 0$
Q	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Reflection in the direction $(1,-1)$ with mirror $y = x$
$-Q$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	Reflection in the direction $(1,1)$ with mirror $y = -x$

The matrices Q and U play symmetric roles. If we let

$$W=\pi\left(R_{\frac{\pi}{4}}\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$$

then the mapping $g\to W^{-1}gW$ is an automorphism of $\pi(D_8)$ and we have

$$\begin{aligned}W^{-1}QW&=U\\W^{-1}UW&=-Q\\W^{-1}JW&=J\end{aligned}$$

so we see that the matrices Q and U can be interchanged with the appropriate sign changes. These fundamental matrices satisify the relations

$$\begin{gathered}Q^2=U=J^4=I\\QU=-UQ=J\\QJ=-JQ=U\\JU=-UJ=Q\end{gathered}$$

An equally important set of matrices is the faithful representation of the quaternionic group $\mathcal{Q}$ in $M_2(\mathbb{C})$ which is defined using the *Pauli spin matrices* $\sigma_0,\sigma_x,\sigma_y,\sigma_z$ and *associate Pauli matrices* $\tau_0,\tau_x,\tau_y,\tau_z$

Name	Matrix
$\sigma_0=I$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$\sigma_x=Q$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$\sigma_y=iJ$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$
$\sigma_z=U$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
$\tau_0=I$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$\tau_x=iQ$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$
$\tau_y=-J$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$
$\tau_z=iU$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$

(2)

and setting

$$\begin{array}{llll}\pi(1)=+\tau_0 & \pi(i)=+\tau_x & \pi(j)=+\tau_y & \pi(k)=+\tau_z\\ \pi(-1)=-\tau_0 & \pi(-i)=-\tau_x & \pi(-j)=-\tau_y & \pi(k)=-\tau_z\end{array}$$

1.4.6. Modules, Submodules, and Direct Sums.

Let A be a unital algebra over the real field $\mathbb{R}$. A *left A-module* M is a real vector space M together with a multiplication mapping $(a,m)\to am$ from $A\times M\to M$ which is *bilinear*

$$\left.\begin{aligned}(\lambda_1a_1+\lambda_2a_2)\,m&=\lambda_1(a_1m)+\lambda_2(a_2m)\\a\,(\mu_1m_1+\mu_2m_2)&=\mu_1(am_1)+\mu_2(am_2)\end{aligned}\right.\ \forall a_1,a_2,\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$$

and satisfies

$$\left.\begin{aligned}(ab)\,m&=a\,(bm)\\1_Am&=m\end{aligned}\right.\ \forall a,b\in A$$

A *right A-module* is similiarly defined with $m(ab)=(ma)b$. A right A^{opp}-module is the same as a left A-module and vice versa. A representation of A in $L(M)$ is given by $\theta(a)(m)=am$. Conversely, if θ is a representation of A in $L(E)$ then E is a left A-module when $ax=\theta(a)(x)$. Thus there exists a one-to-one correspondence between left A-modules and representations of A.

The *direct sum* of left A-modules can be formed; if M_1 and M_2 are left A-modules then the vector space sum $M_1 \otimes M_2$ becomes a left A-module when $a(m_1, m_2) = (a m_1, a m_2)$. A linear subspace N of M is a *left A-submodule* if $a n \in N \forall a \in A, n \in N$ then N is a left A-module in a natural way. If N_1 and N_2 are left A-submodules then so is $N_1 + N_2$. The intersection of left A-submodules is again a left A-submodule. If C is a non-empty subset of a left A-module M then there is a smallest left A-submodule containing C defined by

$$[C]_A = \{a_1 c_1 + \cdots + a_k c_k : k \geqslant 0, a_i \in A, c_i \in C\}$$

if $C = \{c_1, \ldots, c_n\}$ we write $[c_1, \ldots, c_n]_A$ then

$$[c_1, \ldots, c_n]_A = \{a_1 c_1 + \cdots + a_n c_n : a_i \in A\}$$

If $M = [c_1, \ldots, c_n]_A$ is a left A-module then we say that M is a *finitely generated* left A-module. If $M = [c]_A = \{a c : a \in A\}$ then M is a *cyclic left A-module* and c is called a *cyclic vector* for M. Suppose that M_1 and M_2 are both left A-modules; then a linear mapping T from M_1 to M_2 is called a *module homomorphism*, or A-homomorphism or simply *module morphism* or A-morphism. Suppose that $N = N_1 \otimes \cdots \otimes N_n$ and $M = M_1 \otimes \cdots \otimes M_m$ are direct sums of left A-modules. A linear mapping $\theta : N \rightarrow M$ is an A-morphism iff there is a set of A-morphisms $\theta_{ij} : N_j \rightarrow M_i$ such that

$$\theta(x) = \left(\sum_{j=1}^n \theta_{ij} x_j \right)_{i=1}^m \forall (x_1, \ldots, x_n) \in N$$

Particularly, if $M_i = N_j = L$ then $\mathrm{Hom}_A(N, M)$ is isomorphic to $M_{m,n}(\mathrm{End}_A(L))$, the space of $m \times n$ matrices with entries in $\mathrm{End}_A(L)$. The space $\mathrm{End}_A(A)$ is naturally isomorphic as an algebra to A^{opp}.

1.4.7. Simple Modules.

A left A-module M is said to be simple if it is not $\{0\}$ and has no left A-submodules other than $\{0\}$ and M itself. Thus A is simple iff the representation $\pi : A \rightarrow L(M)$ defined by $\pi(a)(x) = a x$ is irreducible which is the case iff every non-zero element of M is a cyclic vector. A unital algebra A is a simple left A-module iff A is a division algebra; that is, every element of A has a two-sided inverse.

Theorem 2. *Schur's Lemma*

Suppose that $T \in \mathrm{Hom}_A(M_1, M_2)$ where M_1 and M_2 are simple left A-modules. If $T \neq 0$ then T is an A-isomorphism of M_1 onto M_2 and T^{-1} is also an A-isomorphism.

1.4.8. Semi-simple modules.

A left A-module is *semi-simple* if it can be written as a direct sum of simple left A-modules $M = M_1 \otimes \cdots \otimes M_d$ which is the case only when M is the span of its simple left A-submodules. A semi-simple left A-module is finitely generated since each M_i is cyclic. If N is a non-zero simple submodule of M then N is isomorphic as a left A-module to M_i for some $1 \leqslant i \leqslant k$. A finite-dimensional simple unital algebra A is semi-simple.

Corollary 3. *Wedderburn's Theorem*

If A is a finite-dimensional simple real unital algebra then there exists $k \in \mathbb{N}$ and a division algebra $D = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ such that $A \cong M_k(D)$ and thus $A \cong \mathrm{End}_D(D^k)$

Any irreducible unital representation of $M_k(D)$ is essentially ther same as the natural representation of $M_k(D)$ as $\mathrm{Hom}_D(D^k)$, and any finite-dimensional representation is a direct sum of irreducible representations.

Theorem 4.

Let $D = \mathbb{R}, \mathbb{C}, \text{or } \mathbb{H}$ and suppose that W is a finite-dimensional real vector space and that $\pi : M_k(D) \rightarrow L(W)$ is a real unital representation. Then the mapping $(\lambda, w) \rightarrow \pi(\lambda I)(w)$ from $D \times W$ to W makes W a left D-module, $\dim_D(W) = r k$ for some r, and there is a π-invariant decomposition

$$W = W_1 \otimes \cdots \otimes W_r$$

where $\dim_D(W_s) = k \forall 1 \leqslant s \leqslant r$. For each $1 \leqslant s \leqslant r$ there is a basis $(e_1, \ldots, e_k)$ of W_s and an isomorphism π_s: $A \rightarrow M_k(D)$ such that

$$\pi(A) \sum_{j=1}^k x_j e_j = \sum_{i=1}^k e_i \sum_{j=1}^k a_{ij} x_j \text{ where } (a_{ij}) = \pi_s(A)$$

1.5. Multilinear Algebra.

1.5.1. Multilinear Mappings.

1.5.2. Tensor Products.

1.5.3. The Trace.

Suppose that E is a d-dimensional vector space over K with basis $(e_1, \ldots, e_d)$ and dual basis $(\phi_1, \ldots, \phi_d)$. Consider the tensor product $E \otimes E'$ which has as its basis $\{e_i \otimes \phi_j : 1 \leqslant i, j \leqslant d\}$. The mapping $(x, \psi) \to \psi(x)$: $E \times E' \to K$ is bilinear and so there exists $t \in (E \otimes E')'$ called the *contraction* of the elementary tensor $x \otimes y$ such that $t(x \otimes y) = \psi(x) \forall x \in E,\ \psi \in E'$. There exists a natural isomorphism λ of $E \otimes E'$ onto $L(E)$. If $T \in L(E)$ then the *trace* of T is defined $\tau(T) = t(\lambda^{-1}(T))$. If T is represented by the matrix (t_{ij}) then

$$\tau(T) = t\left(\sum_{i,j} t_{ij}\, e_i \otimes \phi_j\right) = \sum_{i,j} t_{ij}\, \phi_j(e_i) = \sum_{i=1}^{d} t_{ii}$$

It is often more convienent to work with the *normalized trace*

$$\tau_n(T) = \frac{\tau(T)}{\dim(E)}$$

If $T_1 \in L(E_1)$ and $T_2 \in L(E_2)$ then it is true that

$$\tau(T_1 \otimes T_2) = \tau(T_1)\, \tau(T_2)$$

Suppose that A is a finite-dimensional unital algebra. The left regular representation l is an algebra isomorphism of A into $L(A)$. Define the trace and normalized trace, $\tau(a) = \tau(l_a)$ and $\tau_n(a) = \tau_n(l_a)$. Then $\tau(ab) = \tau(ba)$ and $\tau_n(ab) = \tau_n(ba) \forall a, b \in A$. Since l_1 is the identity mapping on A, $\tau_n(1) = 1$. Elements of the null space of τ_n are called *pure elements*, and elements of span(1) are called *scalars*. Let $\mathrm{Pu}(x) = x - \tau_n(x) 1 \forall x \in A$; then Pu is a projection of A onto the subspace $\mathrm{Pu}(A)$ of pure elements of A and $A = \mathrm{span}(1) \otimes \mathrm{Pu}(A)$. If $x = a1 + bi + cj + dk \in \mathbb{H}$ then l_x is represented by the matrix

$$l_x \cong \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

with respect to the basis $(1, i, j, k)$ so that $\tau(x) = 4a$ and $\tau_n(x) = a$.

1.5.4. Alternating Mappings and the Exterior Algebra: Fermionic Fock Spaces.

Suppose that E is a d-dimensional vector space over K with basis $(e_1, \ldots, e_d)$ and dual basis $(\phi_1, \ldots, \phi_d)$ and that F is a vector space over K. A k-linear mapping $a: E^k \to F$ is *alternating* if $a(x_1, \ldots, x_k) = 0$ whenever there exists distinct indices i and j for which $x_i = x_j$.

Proposition 5. *Suppose that $m \in M^k(E, F)$. Then the following statements mean the same thing*

i. m is alternating

ii. $m(x_1, \ldots, x_k) = \varepsilon(\sigma)\, m(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \forall \sigma \in \Sigma_k$ (where $\varepsilon(\sigma)$ is the signature of the permutation σ)

iii. If i, j are distinct elements of $\{1, \ldots, k\}$ and if $x_i' = x_j$, $x_j' = x_i$ and $x_l' = x_l$ for all other indices l then $m(x_1, \ldots, x_k) = -m(x_1', \ldots, x_k')$

The set of alternating k-linear mappings of E^k into F is a linear subspace of $M^k(E, F)$ denoted by $A^k(E, F)$. $A^k(E)$ will be written for $A^k(E, K)$ and the dual of $A^k(E)$ by $\bigwedge^k (E)$. Suppose that $(x_1, \ldots, x_k) \in E^k$. The evaluation mapping $a \to a(x_1, \ldots, x_k)$ from $A^k(E)$ to K is called the *wedge product* or *alternating product* of $x_1, \ldots, x_k$ and is denoted by $x_1 \wedge \ldots \wedge x_k$. The wedge product is a linear functional on $A^k(E)$ and so are elements of $\bigwedge^k (E)$. The alternating k-linear mapping $(x_1, \ldots, x_k): E^k \to \bigwedge^k (E)$ is denoted by $\wedge^k$. It is the case that

$$\begin{aligned} \bigwedge^k (E) &= \mathrm{span}\{x_1 \wedge \ldots \wedge x_k : (x_1, \ldots, x_k) \in E^k\} \\ &= \mathrm{span}\{e_{j_1} \wedge \ldots \wedge e_{j_k} : 1 \leqslant j_1 < \cdots < j_k \leqslant d\} \end{aligned}$$

To prove the fact, suppose that $a \in \left(\bigwedge^k (E)\right)^{\perp}$, then $a(x_1, \ldots, x_k) = (x_1 \wedge \ldots \wedge x_k)(a) = 0 \forall (x_1, \ldots, x_k) \in E^k$ so we see that $a = 0$. We expand $x_1, \ldots, x_k$ in terms of the basis and it follows from the fact that $\wedge^k$ is an alternating k-linear mapping that $(x_1, \ldots, x_k) \in \operatorname{span}(e_{j_1} \wedge \ldots \wedge e_{j_k}) \forall 1 \leqslant j_1 < \cdots < j_k \leqslant d$. The dimension of the wedge product is the binomial coefficient "d choose k".

$$\dim\left(\bigwedge^k (E)\right) = \binom{d}{k} = \frac{d!}{k!(d-k)!}$$

The *exterior algebra* $\bigwedge^* (E)$ of E is a unital associative algebra of dimension 2^d with multiplication denoted by $\wedge$. It is a super-algebra (1.4.1) if we set

$$\bigwedge^+ (E) = \left\{ \operatorname{span}\left(\bigwedge^k (E)\right) : k \text{ even} \right\}$$

$$\bigwedge^- (E) = \left\{ \operatorname{span}\left(\bigwedge^k (E)\right) : k \text{ odd} \right\}$$

If $T \in L(E, F)$ and $1 \leqslant k \leqslant d$ then T defines an element $T^{(k)} \colon A^{(k)}(F) \to A^{(k)}(E)$ by

$$T^{(k)}(a)(x_1, \ldots, x_d) = a(T(x_1), \ldots, T(x_k))$$

The transpose mapping of T in $L\left(\bigwedge^k (E), \bigwedge^k (F)\right)$ is denoted by $\wedge^k(T)$. Particulary, if $E = F$ and $k = d$ then $\wedge^d(T)$ is a linear mapping from the 1-dimensional space $\bigwedge^d (E)$ into itself, and so there exists an element $\det(T) \in K$ such that $\wedge^d(T)(x_1, \ldots, x_d) = (\det(T)(x_1, \ldots, x_d))$. In the field of quantum physics, exterior algebras are known as *fermionic Fock spaces*. The structures of Hilbert state spaces associated with free fields are called Fock spaces, in honor of Vladimir Fock(1898-1974). [3, p.112][2, 2.2.2][1, 3.5]

1.5.5. The Symmetric Tensor Algebra: Bosonic Fock Spaces.

Suppose that E is a d-dimensional vector space over K, with basis $(e_1, \ldots, e_d)$. A k-linear mapping $s\colon E^k \to F$ is *symmetric* if

$$s(x_1, \ldots, x_k) = s(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \forall \sigma \in \Sigma_k$$

The set $S^k(E, F)$ of symmetric k-linear mappings $s\colon E^k \to F$ is a linear subspace of $M^k(E; F)$. We will denote $S^k(E, K)$ by $S^k(E)$ and the evaluation mapping $s \to s(x_1, \ldots, x_k)$ from $S^k(E)$ to K by the *symmetric tensor products*

$$x_1 \otimes_s \cdots \otimes_s x_k \in S^k(E)'$$

The span of the symmetric k-linear tensor product mappings $\otimes_s$ is denoted by

$$\begin{aligned} \otimes_s^k(E) &= \operatorname{span}\{e_{j_1} \otimes_s \cdots \otimes_s e_{j_k} \colon 1 \leqslant j_1 \leqslant \cdots \leqslant j_k \leqslant d\} \\ &= S^k(E)' \end{aligned}$$

The dimension of $\otimes s(E)$ is given by

$$\dim(\otimes_s(E)) = \binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!(k-1)!}$$

In quantum physics, symmetric tensor algebras are called bosonic Fock spaces.

1.5.6. Creation and Annihilation Operators.

Let

$$(x_1, \ldots, x_{k+1})$$

denote a sequence with $k+1$ terms, and let

$$(x_1, \ldots, \hat{x}_j, \ldots, x_{k+1})$$

denote a sequence with k terms, obtained by deleting the jth term. Similiarly, let

$$(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1})$$

denote a sequence with $k-1$ terms, obtained by deleting the ith and jth terms. The following pair of operators feature prominently in the field of quantum physics. If $a, b \in \bigwedge^* (E)$, let $l_a(b) = a \wedge b$ then the mapping l: $a \to l_a$ is the left regular representation of the algebra $\bigwedge^* (E)$ in $L(\bigwedge^* (E))$; it is a unital isomorphism of $\bigwedge^* (E)$ onto a subalgebra of $L(\bigwedge^* (E))$. In particular, if $x \in E$ then we rename the operator l_x to m_x and call it the *creation operator*

$$m_x\colon \bigwedge^k (E) \to \bigwedge^{k+1} (E)$$

Suppose that $\phi \in E'$ and $a \in A^k(E)$. Define $l_\phi(a) \in M^{k+1}(E)$ by setting

$$l_\phi(a)(x_1, \ldots, x_{k+1}) = \phi(x_1) a(x_2, \ldots, x_{k+1})$$

and note that $l_\phi(a)$ is not alternating. Let

$$\begin{aligned} P_\phi(a)(x_1, \ldots, x_k) &= \frac{1}{(k+1)!} \sum_{\sigma \in \Sigma_{k+1}} \varepsilon(\sigma) \phi(x_{\sigma(1)}) a(x_{\sigma(2)}, \ldots, x_{\sigma(k+1)}) \\ &= \frac{1}{k+1} \sum_{j=1}^{k+1} (-1)^{j-1} \phi(x_j) a(x_1, \ldots, \hat{x}_j, \ldots, x_{k+1}) \end{aligned}$$

then $P_\phi(a) \in A^{k+1}(E)$ and $P_\phi \in L(A^k(E), A^{k+1}(E))$. The *annihilation operator* is defined by the transpose mapping

$$\delta_\phi\colon \bigwedge^{k+1} (E) \to \bigwedge^{k} (E)$$

and we have $\delta_\phi(\lambda) = 0 \forall \lambda \in K$. If we let k vary and consider δ_ϕ as an element of $L(\bigwedge^* (E))$ it can be shown that

$$\begin{aligned} \delta_\phi(x_1 \wedge \cdots \wedge x_k) &= \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \varepsilon(\sigma) \phi(x_{\sigma(1)}) (x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(k)}) \\ &= \frac{1}{k} \sum_{j=1}^{k} (-1)^{j-1} \phi(x_j) a(x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_k) \end{aligned}$$

and that

$$m_x^2 = \delta_\phi^2 = 0$$

and

$$m_x \delta_\phi + \delta_\phi m_x = \phi(x)$$

The creation and annihilation operators are also known as raising and lowering operators.

1.5.7. Tensor Products of Algebras.

1.5.8. Tensor Products of Super-Algebras.

1.6. Quadratic Forms.

1.6.1. Real Quadratic Forms.

Suppose that E is a real vector space. A real-valued function q on E is called a *quadratic form* on E if there exists a symmetric bilinear form b on E such that $q(x) = b(x, x) \forall x \in E$, and a vector space E equipped with a quadratic form q is called a *quadratic space* (E, q). Thus each symmetric bilinear form on E defines a quadratic form on E. The set $Q(E)$ of quadratic forms on E is a linear subspace of the vector space of all real-valued functions on E. Distinct symmetric bilinear forms define distinct quadratic forms. Every linear subspace of E is regular iff q is positive definite or negative definite

1.6.2. Orthogonality.

As in the previous section, let (E, q) be a quadratic space with associated bilinear form b. We say that x and y are orthogonal and write $x \perp y$ or equivalently, due to symmetry, $y \perp x$, if $b(x, y) = 0$ for x and y in E. If $x \perp y$ then $q(x+y) = q(x) + q(y)$. If A is a subset of E, we define the *orthogonal set* $A^\perp$ by

$$A^\perp = \{x\colon x \perp a \forall a \in A\}$$

Proposition 6. *Suppose A is a subset of a regular quadratic space E, and that F is a linear subspace of E.*

i. $A^{\perp}$ *is a linear subspace of* E

ii. *If* $A \subseteq B$ *then* $A^{\perp} \supseteq B^{\perp}$

iii. $A^{\perp\perp} \supseteq A$ *and* $A^{\perp\perp\perp} = A^{\perp}$

iv. $A^{\perp\perp}$=*span(A)*

v. F=$F^{\perp\perp}$

vi. $\dim(F) + \dim(F^{\perp}) = \dim(E)$

Proposition 7. *Suppose F is a linear subspace of a regular quadratic space (E, q), then the following are equivalent.*

i. (F, q) *is regular*

ii. $F \cap F^{\perp} = \{0\}$

iii. $E = F \otimes F^{\perp}$

iv. $(F^{\perp}, q)$ *is regular*

1.6.3. Diagonalization.

Theorem 8. *Suppose that b is a symmetric bilinear form on a vector space E and that there exists a basis $(e_1, \ldots, e_d)$ and a pair of non-negative integers with $p + m = r$, the rank of b, such that if B is represented by the matrix $B = (b_{ij})$ then*

$$b_{ij} = \begin{cases} 1 & j = i \text{ and } 1 \leqslant i \leqslant p \\ -1 & j = i \text{ and } p + 1 \leqslant i \leqslant p + m \\ 0 & \text{otherwise} \end{cases}$$

A basis which satisfies the conclusions of Theorem 8 is called a *standard orthogonal basis*. If (E, q) is a Euclidean space and b is the associated bilinear form then $b_{ii} = 1 \forall 1 \leqslant i \leqslant d$ and the basis is called an *orthonormal basis*. If (e_i) is a standard orthogonal basis and if $x = \sum_{i=1}^{d} x_i e_i$ and $y = \sum_{i=1}^{d} y_i e_i$ then

$$b(x, y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+m} x_i y_i$$

and

$$q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+m} x_i^2$$

For more general fields there exists a basis and set of scalars for which

$$q(x) = \sum_{j=1}^{d} \lambda_j x_j^2 \forall x = \sum_{j=1}^{d} x_j e_j \in E$$

Theorem 9. *Sylvester's Law of Inertia*

Suppose that $(e_1, \ldots, e_d)$ and $(f_1, \ldots, f_d)$ are standard orthogonal bases for a quadratic space (E, q) with corresponding parameters (p, m) and (p', m'). Then $p = p'$ and $m = m'$. The letters p and m represent plus and minus respecitvely.

The pair (p, m) is called the *signature* of q so that we have

$$Q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2$$

If (E_1, q_1) and (E_2, q_2) are quadratic spaces with signatures (p_1, m_1) and (p_2, m_2) then the direct sum $E_1 \otimes E_2$ also becomes a quadratic space when we define $q(x_1 \otimes x_2) = q_1(x_1) + q_2(x_2)$. If $(E, q) = (E_1, q_1) \otimes (E_2, q_2)$ then the signature of (E, q) is $(p_1 + p_2, m_1 + m_2)$. The *Witt index* w of (E, q) is defined to be $w = \min(p, m)$. If $w > 0$ then (E, q) is a *Minkowski space*. A regular quadratic space with $m = 1$ is called a *Lorentz space*. Another special case occurs when $p = m$ and $p + m = 2p = 2m$ in which case (E, q) is called a *hyperbolic space*.

Proposition 10. *Suppose that (E, q) is a hyperbolic space of dimension $p + m = 2p$ then there exists a basis $(f_1, \ldots, f_{p+m})$ such that*

$$b(f_{2i}, f_{2i-1}) = b(f_{2i-1}, f_{2i}) = 1 \forall 1 \leqslant i \leqslant p$$

and $b_{ij} = 0$.

If $(e_1, \ldots, e_d)$ is a standard orthogonal basis for E then

$$f_{2i-1} = \frac{e_i + e_{p+i}}{\sqrt{2}}$$

$$f_{2i} = \frac{e_i - e_{p+i}}{\sqrt{2}}$$

is called a *hyperbolic space* and if $x = \Sigma_{i=1}^{d} x_i f_i$ and $y = \Sigma_{i=1}^{d} y_i f_i$ then

$$b(x, y) = (x_1 y_2 + x_2 y_1) + \cdots + (x_{d-1} y_d + x_d y_{d-1})$$

$$q(x) = 2(x_1 x_2 + x_3 x_4 + \cdots + x_{d-1} x_d)$$

1.6.4. Adjoint Mappings.

Suppose (E, q) is a regular quadratic space with bilinear form b. Then l_b is an injective linear mapping of E into E' which enables us to define the adjoint of a linear mapping from E into a quadratic space

Theorem 11. *Suppose that T is a linear mapping from a regular quadratic space (E_1, q_1) into a quadratic space (E_2, q_2). Then there exists a unique linear mapping T^a called the* adjoint *from E_2 to E_1 such that*

$$b_2(T(x), y) = b_1(x, T^a(y)) \forall x \in E_1, y \in E_2$$

where b_1 and b_2 are the associated bilinear forms.

Proposition 12. *Suppose that (E_1, q_1) and (E_2, q_2) are regular quadratic spaces with standard orthogonal basis $(e_1, \ldots, e_d)$ and $(f_1, \ldots, f_d)$ respectively, $T \in L(E_1, E_2)$ and that T is represented by the matrix (t_{ij}) with respect to these bases. Then T^a is represented by the* adjoint matrix *(t_{ji}^a) where*

$$t_{ji}^a = q_1(e_j) q_2(f_i) t_{ij}$$

As a corollary, if $T \in L(E)$ then $\det(T^a) = \det(T)$.

1.6.5. Complex Inner-Product Spaces.

An inner product on a complex vector space E is a mapping from $E \times E$ into $\mathbb{C}$ which satisfies

i. $\begin{aligned} \langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle &= \alpha_1 \langle x_1, y\rangle + \alpha_2 \langle x_2, y\rangle \\ \langle x, \beta_1 y_1 + \beta_2 y_2\rangle &= \bar{\beta}_1 \langle x, y_1\rangle + \bar{\beta}_2 \langle x, y_2\rangle \end{aligned} \quad \forall x, x_1, x_2, y, y_1, y_2 \in E, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$

ii. $\langle y, x\rangle = \overline{\langle x, y\rangle} \forall x, y \in E$

iii. $\langle x, x\rangle > 0 \forall \{x \in E : x \neq 0\}$

For example, if $E = \mathbb{C}^d$ then the usual inner product is given by

$$\langle z, w\rangle = \sum_{j=1}^{d} z_j \bar{w}_j$$

A complex vector space E equipped with an inner product is called an *inner-product space*. An inner product is *sesquilinear* if it is linear in the first variable and conjugate-linear in the second variable. This is the convention that mathematicians use; physicists use the reverse convention. The quantity

$$\|x\| = \sqrt{\langle x, x\rangle}$$

is a norm on E and the function $d: E \times E \rightarrow \mathbb{R}$ defined by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}$$

is a metric on E. If E is complete under this metric then E is called a *Hilbert space* in honor of David Hilbert(1862-1943). Any finite-dimensional inner product space is complete and is known as a *Hermitian space* in honor French mathematician Charles Hermite. It follows that if $(E, \langle ., .\rangle)$ is an inner-product space then there exists an orthonormal basis $<e_1, ..., e_d>$ for E such that

$$\langle e_i, e_j\rangle = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array} \right. \forall 1 \leqslant i, j \leqslant d$$

If T is a linear mapping from a Hermitian space E to a Hermitian space F then there is a unique linear mapping $T^{*}: F \rightarrow E$ called the *adjoint* such that

$$\langle T(x), y\rangle = \langle x, T^{*}(y)\rangle \forall x \in E, y \in F$$

The group of linear isometries of $(E, \langle ., .\rangle)$ is denoted by $U(E, \langle ., .\rangle)$ and is called the *unitary group*. If $\det(T) = 1$ then T is in the *special unitary group* $\mathrm{SU}(d)$.

1.7. Clifford Algebras.

1.7.1. Universality.

Suppose that (E, q) is a d-dimensional real vector space E with quadratic form q, associated bilinear form b and standard orthogonal basis $(e_1, ..., e_d)$ and that A is a unital algebra. A *Clifford mapping* is an injective linear mapping $j: E \rightarrow A$ such that $1 \notin j(E)$ and $j(x)^2 = -q(x)1 \forall x \in E$. If furthermore $j(E)$ generates A then A together with the mapping j is called a *Clifford algebra* for (E, q). If j is a Clifford mapping, and $x, y \in E$ then

$$\begin{aligned} j(x)j(y) + j(y)j(x) &= j(x + y)^2 - j(x)^2 - j(y)^2 \\ &= -(q(x + y) + q(x) + q(y))1 \\ &= -2b(x, y)1 \end{aligned}$$

If $x \perp y$ then $xy = -yx$.

Theorem 13. *Suppose that $a_1, ..., a_d$ are elements of a unital algebra A. Then there exsists a unique Clifford mapping $j: (E, q) \rightarrow A$ satisfying $j(e_1) = a_i \forall 1 \leqslant i \leqslant d$ iff*

$$\begin{array}{rll} a_i^2 & = -q(e_i) & \forall 1 \leqslant i \leqslant d \\ a_i a_j + a_j a_i & = 0 & \forall 1 \leqslant i < j \leqslant d \\ 1 & \notin \mathrm{span}(a_1, ..., a_d) & \end{array}$$

A Clifford algebra $\mathcal{A}(E, q)$ is said to be *universal* if whenever $T \in L(E, F)$ is an isometry of (E, q) into (F, r) and $\mathcal{B}(F, r)$ is a Clifford algebra for (F, r) then T extends to an algebra morphism $\tilde{T}: \mathcal{A}(E, q) \rightarrow \mathcal{B}(F, r)$

$$E \xrightarrow{T} F \xrightarrow{\subset} \mathcal{B}(F, r) \xleftarrow{\tilde{T}} \mathcal{A}(E, q) \xleftarrow{\supset} E$$

Let $\Omega = \Omega_d = \{1, ..., d\}$ and $C = \{i_1, ..., i_k\}$ where $1 \leqslant i_1 < \cdots < i_k \leqslant d$ then define the element e_C of A to be the product $e_{i_1} ... e_{i_k}$ with $e_{\emptyset} = 1$. If $|C| > 1$ then e_C depends on the ordering of the set $\{1, ..., d\}$. The element $e_{\Omega} = e_1 ... e_d$ will be particularly important. Suppose that $A = \mathrm{span}(P)$ is a Clifford algebra for (E, q) where $P = \{e_C: C \subseteq \Omega\}$. If P is a linearly independent basis for A then A is universal. If (E, q) is a quadratic space then there always exists a universal Clifford algebra $\mathcal{A}(E, q)$.

1.7.2. Representation of $\mathcal{A}_{0,3}$.

The Clifford algebra $\boldsymbol{A}_{0,3}$ is isomorphic to $M_2(\mathbb{C})$. The Pauli spin matrices (1.4.5) are used to obtain an explicit representation. Let us define j, a Clifford mapping of $\mathbb{R}_{0,3}$ into $M_2(\mathbb{C})$ which extends to an isomorphism of $\mathcal{A}_{0,3}$.

$$\begin{aligned} j(x, y, z) &= x\sigma_x + y\sigma_y + z\sigma_z \\ &= xQ + iyJ + zU \\ &= \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \end{aligned}$$

The reason for σ_y being complex is that that the center of $\mathcal{A}_{0,3}$ is $Z(\mathcal{A}_{0,3}) = \text{span}(1,\ e_\Omega) \cong \mathbb{C}$ where $e_\Omega = Q(iJ)U = i$. Thus $\mathcal{A}_{0,3}$ can be considered as a 4-dimensional complex algebra where multiplication by i becomes multiplication by e_Ω. We note that $j(\mathbb{R}_{0,3})$ is the 3-dimensional real subspace of $M_2(\mathbb{C})$ consisting of all Hermitian matrices with zero trace:

$$j(\mathbb{R}_{0,3}) = \{T \in M_2(\mathbb{C})\colon T = T^*, \tau(T) = 0\}$$

The group $\mathcal{A}_{0,3}^+$ is generated by the elements

$$\begin{aligned} f_x &= e_3\, e_2 \\ f_y &= e_1\, e_3 \\ f_z &= e_2\, e_1 \end{aligned}$$

and we have $j(f_x) = \tau_x, j(f_y) = \tau_y, \text{and}\, j(f_z) = \tau_z$ where $\tau_x, \tau_y, \text{and}\, \tau_z$ are the associate Pauli matrices. Thus

$$j(\mathcal{A}_{0,3}^+) = \mathbb{H}$$

1.7.3. Spin(8).

This section is copied nearly verbatim from [2, Appendix 5.B]. The description of Dirac matrices for Spin(8) requires a Clifford algebra with 8 anticommuting matrices. Because of their importance, and the fact that they are useful building blocks for 10-dimensional Dirac matrices, they will be described explicitly. The Dirac algebra of Spin(8) requires 16-dimensional matrices corresponding to the reducible $8_s + 8_c$ representation of Spin(8). This notation can be confusing due to its usage as representing the operation of taking powers, but this is how it is in the literature. These matrices can be written in block form

$$\gamma^i = \begin{pmatrix} 0 & \gamma_{a\dot{a}} \\ \gamma^i_{\dot{b}b} & \end{pmatrix}$$

where $\gamma_{\dot{a}a}$ is the transpose of $\gamma_{a\dot{a}}$. We also see that the equations $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ sare satisfied if

$$\begin{aligned} \gamma^i_{a\dot{a}}\, \gamma^j_{\dot{a}b} + \gamma^j_{a\dot{a}} \gamma^i_{\dot{a}b} &= 2\delta^{ij}\delta_{ab} \\ \gamma^i_{\dot{\mathrm{a}}\mathrm{a}}\, \gamma^j_{a\dot{b}} + \gamma^j_{\dot{a}a} \gamma^i_{a\dot{b}} &= 2\delta^{ij}\delta_{ab} \end{aligned} \quad \forall i, j = 1 \ldots 8$$

A specific set of matrcies $\gamma^i_{a\dot{a}}$ that satisfy these equations, expressed as direct products of 2×2 blocks is

$$\begin{aligned} \gamma^1 &= i\tau_y \times i\tau_y \times i\tau_y \\ \gamma^2 &= 1 \times \tau_x \times i\tau_y \\ \gamma^3 &= 1 \times \tau_z \times i\tau_y \\ \gamma^4 &= \tau_x \times i\tau_y \times 1 \\ \gamma^5 &= \tau_z \times i\tau_y \times 1 \\ \gamma^6 &= i\tau_y \times 1 \times \tau_x \\ \gamma^7 &= i\tau_y \times 1 \times \tau_z \\ \gamma^8 &= 1 \times 1 \times 1 \end{aligned}$$

where τ_x, τ_y, and τ_z are the associate Pauli matrices (1.4.5). We define

$$\gamma^{ij}_{\mathrm{ab}} = \frac{1}{2}(\gamma^i_{a\dot{a}}\, \gamma^j_{\dot{a}b} - \gamma^j_{a\dot{a}} \gamma^i_{\dot{a}b})$$

and similiarly for

$$\gamma^{ij}_{\dot{a}\dot{b}} = \frac{1}{2}(\gamma^i_{\dot{a}a}\, \gamma^j_{a\dot{b}} - \gamma^j_{\dot{a}a} \gamma^i_{a\dot{b}})$$

This representation has connections to 10-dimensional Majorana-Weyl spinors. [2, p.220]

2. MODULAR FORMS

2.1. Modular Transformations.

2.1.1. Definitions.

A *modular transformation* is one of the form

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \forall \{a, b, c, d \in \mathbb{Z}: ad - bc = 1\}$$

This set of transformations forms a group called the *modular group* which is isomorphic to $SL(2, \mathbb{Z})$. A function $G(\tau)$ is called a *modular form of weight 2k* if

$$G(\tau') = (c\tau + d)^{2k} G(\tau)$$

is true for all modular group transformations. Examples of modular forms of weight $2k$ are the Eisenstein series(NOT Einstein series...although that might make a good joke)

$$\begin{aligned} G_{2k}(\tau) &= \sum_{(m,n)\neq(0,0)} (m + n\tau)^{-2k} \\ &= 2\zeta(2k)\left(1 + \frac{(2\pi i)^{2k}}{(2k-1)!\zeta(2k)} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) e^{2\pi i m\tau}\right) \end{aligned}$$

which converges for $1 < k \in \mathbb{Z}$ where $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and

$$\sigma_\alpha(k) = \sum_{d|k} d^\alpha$$

which runs over all positive integer divisors of k. The idea is that the sum goes over all the lattice points in the complex plane whose structure is determined in terms of the complex number τ. Modular transforms map this lattice into itself so $G_{2k}(\tau)$ transforms simply under modular transformations. A basic theorem of modular forms states that an arbitrary holomorphic modular form of weight $2k$ can be expressed as a polynomial in G_4 and G_6. The only modular form of weight 8 is G_4^2 since the weights of modular forms are additive under multiplication. The smallest weight for which there is more than one independent modular form is 12 for which there are two independent modular forms, G_4^3 and G_6^2. [2, Appendix 6.B]

3. Physics

3.1. Particles with Spin $\frac{1}{2}$.

The Pauli spin matrices (1.4.5) were introduced by Wolfgang Pauli to represent the internal angular momentum of particles which have spin $\frac{1}{2}$. In quantum mechanics, an *observable* corresponds to a Hermitian linear operator T on a Hilbert space H and when the possible values of the observable are discrete, these are the possible eigenvalues of T. The Stern-Gerlach experiment showed that elementary particles have an intrinsic angular momentum, or *spin*. If x is a unit vector in $\mathbb{R}^3$ then the component J_x of the spin in the direction x is an observable. In a non-relativistic setting, this leads to the consideration of a linear mapping $x \to J_x$ from the Euclidean space $V = \mathbb{R}^3$ into the space $L_h(H)$ of Hermitian operators on an appropriate state space H. Particles are either *bosons*, in which case the eigenvalues of J_x are integers, or *fermions*, in which case the eigenvalues of J_x are of the form $\frac{2k+1}{2}$. In the case where the particle has spin $\frac{1}{2}$, each of the operators J_x has just two eigenvalues, namely $\frac{1}{2}$ (*spin up)*, and $-\frac{1}{2}$ (*spin down*). Consequently, $J_x^2 = I$. Now, take the negative-definite quadratic form $q(x) = -\|x\|^2$ on $\mathbb{R}^3$ and consider $\mathbb{R}_{0,3}$ as a subspace of $\mathcal{A}_{0,3}$. Let $j: \mathcal{A}_{0,3} \to M_2(\mathbb{C})$ be the isomorphism defined in (1.7.2) and let $J_i = \frac{1}{2} j(e_i) = \frac{1}{2}\sigma_i$ for $i = 1, 2, 3$. Thus

$$\begin{aligned} J_1 J_2 &= \frac{i}{2} J_3 \\ J_2 J_3 &= \frac{i}{2} J_1 \\ J_3 J_1 &= \frac{i}{2} J_2 \end{aligned}$$

The Pauli spin matrices are Hermitian, and identifying $\mathcal{A}_{0,3}$ with $M_2(\mathbb{C})$ we see that we can take the state space H to be the spinor space $\mathbb{C}^2$. If $v = (x, y, z) \in \mathbb{R}_{0,3}$ and $q(v) = -1$ then

$$J_v = \frac{1}{2}\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

is a Hermitian matrix with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ and corresponding eigenvectors.

$$\begin{pmatrix} x - iy \\ 1 - z \end{pmatrix} \text{and} \begin{pmatrix} x - iy \\ -1 - z \end{pmatrix}$$

respectively.

3.2. The Dirac Operator.

3.2.1. The Laplacian.

Let U be an open subset of a finite-dimensional real vector space and F is a finite-dimensional vector space then define $C(U, F)$ to be the vector space of all continuous F-valued functions defined on U, and, for $k > 0$, we define $C^k(U, F)$ to be the vector space of all k-times continuously differentiable F-valued functions defined on U. Suppose that U is an open subset of Euclidean space $\mathbb{R}_d$ and that F is a finite-dimensional vector space and that $f \in C^2(U, F)$. Then the *Laplacian* is a second-order linear differential operator Δ: $C^2(U, F) \to C(U, F)$ defined as

$$\Delta f = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}$$

A *harmonic* function is one which satisfies $\Delta f = 0$. Suppose that $\mathbb{R}_{p,m}$ is the standard regular quadratic space with signature (p, m). Then the corresponding Laplacian Δ_q is defined as

$$\begin{aligned} \Delta_q &= \sum_{j=1}^{d} q(e_j) \frac{\partial^2 f}{\partial x_j^2} \\ &= \sum_{j=1}^{p} \frac{\partial^2 f}{\partial x_j^2} - \sum_{j=p+1}^{m} \frac{\partial^2 f}{\partial x_j^2} \end{aligned} \tag{3}$$

Then a function f is *q-harmonic* if $\Delta_q f = 0$. Paul Dirac noticed that the 2nd order linear differential operator $-\Delta_q$ can be written as the square of a 1st-order linear differential operator in a noncommutative setting.

Suppose that U is an open subset of $\mathbb{R}_{p,m}$ and that F is a finite-dimensional left $\mathcal{A}_{p,m}$-module and that $f \in C^1(U, F)$ then we define the *(standard) Dirac operator* as

$$D_q = \sum_{j=1}^{d} q(e_j) e_j \frac{\partial f}{\partial x_j^2}$$

and say that f is *Clifford analytic* if $D_q f = 0$.

Theorem 14. *Suppose that (ε_i) is any orthogonal basis for $\mathbb{R}_{p,m}$ and (y_i) denotes the corresponding coordinates. Then*

$$D_q = \sum_{i=1}^{d} q(\varepsilon_i) \varepsilon_i \frac{\partial}{\partial y_i}$$

since each vector ε_i can be expressed in terms of the basis (e_j) as

$$\varepsilon_i = \sum_{j=1}^{d} a_{ij} e_j$$

and each vector e_j can be expressed in terms of the basis (ε_i) as

$$e_j = \sum_{i=1}^{d} b_{ji} \varepsilon_i$$

then

$$a_{ij} = b(\varepsilon_i, e_j) = b(e_j, \varepsilon_i) = b_{ji}$$

thus

$$\sum_{i=1}^{d} q(\varepsilon_i) a_{ij} a_{ik} = b(e_j, \varepsilon_k)$$

since

$$\frac{\partial f}{\partial y_i}(x) = \lim_{t\to 0} \frac{f\left(x + \sum_{j=1}^{d} t a_{ij} e_j\right) - f(x)}{t}$$

and it follows that

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{d} a_{ij} \frac{\partial}{\partial x_j}$$

Now consider D_q^2 as a mapping from $C^2(U, F)$ into $C(U, F)$:

$$\begin{aligned} D_q^2 &= \sum_{i=1}^{d} \sum_{j=1}^{d} q(e_i) q(e_j) e_i e_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \\ &= -\sum_{i=1}^{d} q(e_i) \frac{\partial^2}{\partial x_i^2} \\ &= -\Delta_q \end{aligned}$$

where Δ_q is the Laplacian (3). Thus, $-\Delta_q$ has been written as the product of two 1st-order differential operators but with the added complexity of working in a noncommutative setting. Thus if we consider

$$C^2(U, F) \xrightarrow{D_q} C^1(U, F) \xrightarrow{D_q} C(U, F)$$

then

$$\begin{aligned} \ker(D_q) &= \{f \in C^1(U, F) \colon f \text{ is Clifford analytic}\} \\ \ker(D_q^2) &= \{f \in C^2(U, F) \colon f \text{ is } q-\text{harmonic}\} \end{aligned}$$

Furthermore, if f is q-harmonic then $D_q f$ is Clifford analytic. Also see [2, 4.1].

3.3. Maxwell's Equations for an Electromagnetic Field.

Maxwell's equations for an electromagentic field can be expressed as a single equation involving the standard Dirac operator. The simplest case of electric and magnetic fields in a vacuum varying in space and time will be considered. Take an open subset U of $\mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$. The points of U will be denoted by (t, x, y, z) where t is a time variable and $x, y,$ and z are space variables. We begin by considering the *electric field* $E = (E_1, E_2, E_3)$ and the *magnetic field* $B = (B_1, B_2, B_3)$ as continuously differentiable vector-valued functions defined on U and taking values on $\mathbb{R}^3$. Given these, there exists a continuous vector-valued function J defined on U called the *current density*, and a continuous scalar-valued function ρ defined on U called the *charge density*. Then, with a suitable choice of units, *Maxwell's equations* are

∇.electric field	$\boldsymbol{\nabla . E}$	$\boldsymbol{=\rho}$	charge density
$\nabla \times \text{magnetic field} - \frac{\partial \text{electric field}}{\mathrm{d}t}$	$\boldsymbol{\nabla \times B - \frac{\partial E}{\partial t}}$	$\boldsymbol{=J}$	current density
$\frac{\partial \text{magnetic field}}{\partial t} + \nabla \times \text{electric field}$	$\boldsymbol{\frac{\partial B}{\partial t} + \nabla \times E}$	$\boldsymbol{=0}$	
∇.magnetic field	$\boldsymbol{\nabla . B}$	$\boldsymbol{=0}$	

These equations are invariant under Lorentz transformations rather than Euclidean co-ordinate changes. This is one of the main things that led Einstein to the theory of special relativity. Thus, instead of $\mathbb{R}^4$ we must consider the space $\mathbb{R}_{1,3}$, where $(1, 3)$ is its corresponding signature(9), with quadratic form

$$q(t, x, y, z) = t^2 - (x^2 + y^2 + z^2)$$

with orthogonal basis $(e_{0\ldots3}) = (e_0, e_1, e_2, e_3)$ and therefore consider U to be an open subset of $\mathbb{R}_{1,3}$. [1, p.159]

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