Analytical Derivation of the Drag Curve $C_D = C_D(R)$

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Through a convenient mathematical approach for the Navier-Stokes equation, we obtain the quadratic dependence $v^2$ of the drag force $F_D$ on a falling sphere, and the drag coefficient, $C_D$, as a function of the Reynolds number. Viscosity effects related to the turbulent boundary layer under transition, from laminar to turbulent, lead to the tensorial integration related to the flux of linear momentum through a conveniently chosen control surface in the falling reference frame. This approach turns out to provide an efficient route for the drag force calculation, since the drag force turns out to be a field of a non-inertial reference frame, allowing an arbitrary and convenient control surface, finally leading to the quadratic term for the drag force.

DEFINING THE MATHEMATICAL PROBLEM

Regarding the application of the Newton second law to a small closed subsystem $\sigma$ with boundary $\partial(\delta\sigma)$ and volume $\delta\sigma$ of a continuum fluid in an inertial reference frame, one obtains the Navier-Stokes equation:

$$\int_{\sigma} d\vec{F}_{ext} = \int_{\delta\sigma} \rho(\vec{r}, t) \vec{f}(\vec{r}, t)d\sigma + \int_{\partial(\delta\sigma)} \mathbf{T} \cdot \hat{n} dS, \quad (1)$$

where $\vec{f}(\vec{r}, t)$ is a locally external acceleration field, $\rho(\vec{r}, t)$ the scalar density field, and $\mathbf{T}$ is the most general tensor due to the effects of the surrounding fluid on $\sigma$, being given by:

$$T_{ik} = -p\delta_{ik} + \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x_l}, \quad (2)$$

where $p$ is the local thermodynamic pressure field, being $\mathbf{T}$ written in terms of its components under the summation convention on repeated indices and where $\mathbf{T}$ was obtained from the combination of effects due to strain and shear:

$$\mathbf{T} = \alpha(\nabla \vec{v})_{ls} + \beta(\nabla \vec{v})_{s} = \alpha \left[ (\nabla \vec{v})_s - \frac{1}{3} \nabla \cdot \vec{v} \mathbf{1} \right] + \zeta \nabla \cdot \vec{v} \mathbf{1}, \quad (3)$$

from which one defines the viscosity coefficients, $\alpha = 2\eta$ (this latter relation following from the coupling to the planar flow case, in which one defines the dynamical viscosity $\eta$) and $\zeta$, under an isotropic assumption. Back to the Eq. (1), one obtains the Navier-Stokes equation:

$$\rho(\vec{r}, t) \vec{u}(\vec{r}, t) - \rho(\vec{r}, t) \vec{f}(\vec{r}, t) + \nabla p(\vec{r}, t) - \eta \nabla^2 \vec{u}(\vec{r}, t) +$$

$$- \left( \frac{1}{3} \eta + \zeta \right) \nabla \left( \nabla \cdot \vec{u}(\vec{r}, t) \right) = \vec{0}. \quad (4)$$

Under a divergence-free hypothesis for the velocity field (constant density turns out to be a sufficient condition), one has got, hence, in the ground reference frame, the following mathematical problem:

$$\left\{ \begin{array}{l}
\rho \vec{v} - \rho \vec{g} + \nabla p - \eta \nabla^2 \vec{v} = \vec{0}, \quad \nabla \cdot \vec{v} = 0; \\
\lim_{|\vec{r}| \to \infty} \vec{v} = \vec{0}, \quad \vec{v}(\partial \text{ sphere}) = \dot{h}(t)\hat{e}_z \text{ nonslip},
\end{array} \right. \quad (5)$$

where $\vec{g}$ is the local gravitational field and $\dot{h}(t)$ is the scalar velocity of the center of a falling sphere within the fluid.

GEDANKENEXPERIMENT

One measures the local gravitational field in the non-inertial frame attached to the falling sphere from the following gedankenexperiment: hollow sphere having got mass $m$, with an internal weighing apparatus (with negligible mass) to measure the normal force $\vec{N}$ that the ground of the hollow sphere exerts on a proof mass $m_0$. By isolating the system $m + m_0$, and, subsequently, by isolating the system $m_0$, one obtains:

$$\frac{\vec{N}}{m_0} = \frac{\vec{F}_{\text{drag}}}{(m + m_0)} + \delta \vec{F}_{\text{drag}}(m_0), \quad (6)$$

where $\vec{F}_{\text{drag}}$ is the force the fluid exerts on the hollow sphere, without the proof mass $m_0$, and $\delta \vec{F}_{\text{drag}}(m_0)$ is the increment to this force - due to the consideration of the internal proof mass $m_0$. Hence, the gravitational field $\vec{g}_0$ within the hollow sphere is given by:

$$\vec{g}_0 = \lim_{m_0 \to 0} \frac{-\vec{N}}{m_0} = \lim_{m_0 \to 0} \frac{\vec{F}_{\text{drag}} + \delta \vec{F}_{\text{drag}}(m_0)}{(m + m_0)} = -\frac{\vec{F}_{\text{drag}}}{m}, \quad (7)$$

from which the force we want to calculate turns out to be a property of the non-inertial reference frame attached to the sphere. Adopting this falling reference frame, we
have got the mathematical problem:
\[
\begin{align*}
\rho \ddot{\vec{v}} + \frac{\vec{F}_{\text{drag}}}{m} + \nabla p - \eta \nabla^2 \vec{v} &= \vec{0}, \quad \nabla \cdot \vec{v} = \vec{0}; \\
\lim_{|r| \to \infty} \vec{v} &= -\dot{h}(t) \hat{e}_z, \quad \vec{v}(\partial \text{sphere}) = \vec{0} \text{ nonslip.}
\end{align*}
\]
Comparing the Eqs. (5) and (8), one infers the force we want to calculate (divided by the sphere mass \(m\)) turns out to be a field at each point of the fluid in the falling reference frame, given by the Eq. (7). This turns out to be an acceleration field in the adopted reference frame, given by the Eq. (7). This turns out to be an acceleration field in the adopted reference frame, given by the Eq. (7).

\[\frac{\partial}{\partial t} (\vec{v} \cdot \vec{v}) \approx 0, \text{ in relation to } \frac{\partial}{\partial t} \rho = 0, \text{ in virtue of the inter-}
\]

**CALCULATING \(\bar{g}_0\)**

Applying the continuity equation in its most general form, calculating the instantaneous time rate of linear momentum variation within an arbitrary control volume, fixed and undeformable, one reaches the expression for the calculation of \(\bar{g}_0\):

\[
\bar{g}_0 = \frac{\vec{F}_{\text{drag}}}{m} = \frac{1}{\int \rho dV} \left( \int \Pi \cdot \hat{n} dS - \frac{\partial}{\partial t} \int \rho \vec{v} dV \right),
\]

\[\Pi = [-1 p + \rho (\vec{v} \otimes \vec{v})],\]

since \(\bar{g}_0\) does not depend on the spatial coordinates within the fluid, once this field equally permeates each point of the fluid in the falling reference frame at any given instant \(t\).

### OBTAINING THE DRAG FORCE \(F_D\) AND THE DRAG COEFFICIENT \(C_D\)

Applying the Eqs. (9) and (10) to the control region \(FGBAF\) depicted in the Fig. 1, at the stationary flow regime \(t \to \infty\), one obtains:

\[
\bar{F}^\infty = \frac{m}{m + m_{BL}} \left\{ - \int_{FG} + \int_{GB} + \int_{BA} + \int_{AF} \right\} \frac{1}{\rho_{\infty}} \cdot \hat{n} dS - \int_{FG} \rho (\vec{v}_\infty \otimes \vec{v}_\infty) \cdot \hat{n} dS, \tag{11}
\]

where \(m_{BL}\) is the mass of the boundary layer attached to the sphere. The pressure field on \(FG\) can be obtained, since this chosen surface \((FG)\) does not violate the lami-

\[
\text{\ldots}
\]

where \(\varphi^\infty\) is a scalar field due to the vanishing rotational of the force the fluid exerts on the sphere. The pressure field on \(AB\) (\(AB\) touching the wake, the rear region of the flow) is obtained from the condition of broken equi-

\[
\bar{p}_\infty^S = -\frac{\rho}{m} \varphi^\infty_{GBAF} + \bar{p}_\infty^0 - \frac{9}{16} \rho \left(\dot{h}^\infty(t)\right)^2 \sin^2 \theta, \tag{13}
\]

Hence:

\[
\bar{p}_\infty^{FG} = -\frac{\rho}{m} \varphi^\infty_{FG} + \rho_\infty^0 - \frac{9}{8} \rho \left(\dot{h}^\infty(t)\right)^2 \sin^2 \theta, \tag{12}
\]

\[
\left(\langle \vec{v} \otimes \vec{v}\rangle_{FG} \cdot \hat{n}\right)_t = \left(\vec{v}_\infty \otimes \vec{v}_\infty\right) \cdot \hat{n} =
\]

Fig. 1: Figure for the integration.
\[
\frac{9}{16} \left( \hat{h}^\infty(t) \right)^2 \sin^2 \theta \hat{e}_r,
\]
where \(\theta_S\) is the separation angle, depicted in the Fig. 2.

Using these results within the Eq. (11), one obtains the quadratic contribution for the drag force via straightforward integration:

\[
\left(1 + \frac{m_{BL}}{m}\right) \vec{F}^\infty = \text{Buoyancy} + \frac{m_{BL}}{m} \vec{F}^\infty + \frac{9\pi}{32} \rho \left( \hat{h}^\infty(t) \right)^2 R^2 \sin^4 \theta_S \hat{e}_z \Rightarrow \vec{F}_D = \frac{9\pi}{32} \rho \left( \hat{h}^\infty(t) \right)^2 R^2 \sin^4 \theta_S \hat{e}_z.
\]

Renaming \(\hat{h}^\infty(t) \equiv v\), knowing that the drag force points along the \(\hat{e}_z\) direction, we simply write for the quadratic drag force contribution, the quadratic scalar component:

\[
F_D = \frac{9\pi}{32} \rho v^2 R^2 \sin^4 \theta_S.
\]

One should notice this contribution arises from our consideration regarding the turbulent profile within the boundary layer, from which we see there is no any linear contribution arising at this flow regime. Writing the drag force as a series on \(v\):

\[
F_D(v) = \sum_{k=0}^{\infty} a_k v^k,
\]

we know from the low Reynolds number regime that the linear contribution is given by the Stokes force [4]:

\[
a_0 = 0, \quad a_1 v = 6\pi \eta Rv.
\]

Hence, up to the drag crisis, the drag force reads:

\[
F_D = 6\pi \eta Rv + \frac{9\pi}{32} \rho \left( \sin^4 \theta_S \right) R^2 v^2.
\]

The drag coefficient, \(C_D\), and the Reynolds number, \(R\), are defined by:

\[
C_D = \frac{2 F_D}{\pi \rho \eta R^2 v^2}, \quad R = \frac{2 \rho Rv}{\eta}.
\]

Hence, from the Eqs. (19) and (20), one obtains the drag coefficient as a function of the Reynolds number, up to the drag crisis:

\[
C_D(R) = \frac{24}{R} + \frac{9}{16} \sin^4 \theta_S.
\]

Fig. 3 shows the graph for the Eq. (21), for \(\theta_S = 70.4^\circ\). This is the separation angle obtained from the Froessling method [1]. One sees this dependence on the Reynolds number agrees with the experimental one over the entire range of Reynolds numbers up to the drag crisis, as one verify, e.g., in [2] and [3].

\[\text{Fig. 3: Drag coefficient vs. Reynolds number, Eq. (21).}\]

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[4] The Oseen force will turn out to be within the quadratic contribution we obtained, viz., included within the Eq. (16). One may infer the Oseen force is a quadratic contribution.