Triangle exact solution of 3-bodies problem.

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Here is presented a system of equations of 3-bodies problem in well-known Lagrange’s form (describing a relative motions of 3-bodies). Analyzing of such a system, we obtain an exact solution in special case of constant ratios of relative distances between the bodies.

Above simplifying assumption reduces all equations of initial system to a proper similar form, which leads us to a final solution: initial triangle of bodies $m_1, m_2, m_3$ is moving as entire construction, simultaneously rotating over the common center of masses as well as increasing or decreasing of it’s size proportionally.
Let us consider the system of an ordinary differential equations for 3-bodies problem, at given initial conditions [1-3]:

\[ m_1 \ddot{q}_1 = -\gamma \left\{ \frac{m_1 m_2 (q_1 - q_2)}{\|q_1 - q_2\|^3} + \frac{m_1 m_3 (q_1 - q_3)}{\|q_1 - q_3\|^3} \right\}, \]

\[ m_2 \ddot{q}_2 = -\gamma \left\{ \frac{m_2 m_1 (q_2 - q_1)}{\|q_2 - q_1\|^3} + \frac{m_2 m_3 (q_2 - q_3)}{\|q_2 - q_3\|^3} \right\}, \]

\[ m_3 \ddot{q}_3 = -\gamma \left\{ \frac{m_3 m_1 (q_3 - q_1)}{\|q_3 - q_1\|^3} + \frac{m_3 m_2 (q_3 - q_2)}{\|q_3 - q_2\|^3} \right\}. \]

- here \( q_1, q_2, q_3 \) - means the radius-vector of bodies \( m_1, m_2, m_3 \), accordingly.

For the purposes of exploring a relative motions of 3-bodies one to each other, let’s rewrite the system above as below (by linear transformation of initial equations):

\[ (q_1 - q_2)^\prime\prime + \gamma (m_1 + m_2) \frac{(q_1 - q_2)}{\|q_1 - q_2\|^3} = \gamma m_3 \left\{ \frac{(q_3 - q_1)}{\|q_3 - q_1\|^3} + \frac{(q_2 - q_3)}{\|q_2 - q_3\|^3} \right\}, \]

\[ (q_2 - q_3)^\prime\prime + \gamma (m_2 + m_3) \frac{(q_2 - q_3)}{\|q_2 - q_3\|^3} = \gamma m_1 \left\{ \frac{(q_3 - q_1)}{\|q_3 - q_1\|^3} + \frac{(q_1 - q_2)}{\|q_1 - q_2\|^3} \right\}, \]

\[ (q_3 - q_1)^\prime\prime + \gamma (m_1 + m_3) \frac{(q_3 - q_1)}{\|q_3 - q_1\|^3} = \gamma m_2 \left\{ \frac{(q_1 - q_2)}{\|q_1 - q_2\|^3} + \frac{(q_2 - q_3)}{\|q_2 - q_3\|^3} \right\}. \]

Let’s designate as below:
\[ R_{1,2} = (q_1 - q_2), \quad R_{2,3} = (q_2 - q_3), \quad R_{3,1} = (q_3 - q_1) \quad (*) \]

Above designating causes the transformation of a previous system to another form:

\[ R_{1,2}'' + \gamma (m_1 + m_2) \frac{R_{1,2}}{R_{1,2}} = \gamma m_3 \left( \frac{R_{3,1}}{R_{3,1}} + \frac{R_{2,3}}{R_{2,3}} \right), \]

\[ R_{2,3}'' + \gamma (m_1 + m_2) \frac{R_{2,3}}{R_{2,3}} = \gamma m_1 \left( \frac{R_{1,2}}{R_{1,2}} + \frac{R_{3,1}}{R_{3,1}} \right), \quad (1.1) \]

\[ R_{3,1}'' + \gamma (m_1 + m_2) \frac{R_{3,1}}{R_{3,1}} = \gamma m_2 \left( \frac{R_{2,3}}{R_{2,3}} + \frac{R_{1,2}}{R_{1,2}} \right). \]

Analysing system (1.1) we should note that if we sum all the above equations one to each other it would lead us to the result below:

\[ R_{1,2}'' + R_{2,3}'' + R_{3,1}'' = 0. \]

If we also sum all the equalities (*) one to each other, we should obtain

\[ R_{1,2} + R_{2,3} + R_{3,1} = 0 \quad (**). \]

Besides, if we substitute an expression for \( R_{1,2}/|R_{1,2}|^{3} \) - from 2-nd to 1-st equation of (1.1), then \( R_{1,2}/|R_{1,2}|^{3} \) to the 3-d - we should obtain below:
So, the linear recombining of equations (1.1) let us define some vector function $F(t)$ which seems to be unique for all equations of (1.2). Otherwise, taking into consideration (**), we also obtain

\[
\begin{align*}
R_{1,2}'' + \gamma (m_1 + m_2 + m_3) & \left[ \frac{R_{1,2}}{|R_{1,2}|} \right] \cdot \frac{1}{m_3} = F(t), \\
R_{2,3}'' + \gamma (m_1 + m_2 + m_3) & \left[ \frac{R_{2,3}}{|R_{2,3}|} \right] \cdot \frac{1}{m_1} = F(t), \quad (1.2) \\
R_{3,1}'' + \gamma (m_1 + m_2 + m_3) & \left[ \frac{R_{3,1}}{|R_{3,1}|} \right] \cdot \frac{1}{m_2} = F(t).
\end{align*}
\]

It is well-known fact [1-3] that there are existing only 5 cases of exact (1.1) solutions (below $R_i = (x_i, y_i, z_i)$, $i = 1, 2, 3, 3, 1$):

- **3 Lagrange’s linear cases**, when $R_{1,2} \sim R_{2,3} \sim R_{3,1}$

- **2 Euler’s cases of equipotential triangle**, when

\[
|R_{1,2}| = |R_{2,3}| = |R_{3,1}| \iff (** ) \iff F(t) = 0.
\]
Let’s consider a solutions of (1.2) for which is valid an assumption below

$$\frac{|R_{2,3}|}{|R_{1,2}|} = a = \text{const}, \quad \frac{|R_{2,3}|}{|R_{3,1}|} = b = \text{const}$$

It means that proportions of absolute meanings of relative distances $R_{1,2}, R_{2,3}, R_{3,1}$ between the bodies should be the same all the time & should be equal to the initial proportions (which are given by special initial conditions).

It could be possible only if such a triangle of bodies $m_1, m_2, m_3$ is moving as entire construction, rotating over the center of masses as well as increasing or decreasing the lengths of sides of such a triangle proportionally.

Besides, we obtain:

$$F(t) = \gamma \cdot \left( \frac{a^3 \cdot R_{1,2} + b^3 \cdot R_{3,1}}{|R_{2,3}|^3} \right),$$

$$R_{3,1} = -R_{2,3} - R_{1,2} \quad .$$

$$\Rightarrow F(t) = \gamma \cdot \left( \frac{(a^3 - b^3) \cdot R_{1,2} + (1 - b^3) \cdot R_{2,3}}{|R_{2,3}|^3} \right).$$

So, in case $a = b = 1$ we obtain Euler’s cases of equipotential triangle, but in the case $R_{1,2} \sim R_{2,3} \sim R_{3,1}$ all the equations of system (1.2) could be reduced to one of Lagrange’s linear cases [1].
In accordance with assumption (***) above, such a solution of (1.2) should be factorized as below \((R_0 = R(t_0))\):

\[
R = \frac{R_0}{|\vec{R}_0|} \cdot \vec{R}(t) \cdot \sin(\omega t + \varphi_0),
\]

- here \(\vec{R} = \vec{R}_i\) – is a vector of general motion, which describes an identical character of evolution for each of 3-bodies relative distances \(R_{1,2}, R_{2,3}, R_{3,1}\) \((i = 1, 2, 3, 2, 3)\), besides \((\varphi_0 = \varphi(t_0) – the initial angle when triangle of bodies began to rotate)\):

\[
R(t) = |\vec{R}| \cdot \max \{\sin (\omega t + \varphi_0)\}
\]

– is the scale factor or measure of appropriate relative distances between the bodies.

As for the integral of momentum, for such a case it describes a harmonic character of rotation of triangle \(m_1, m_2, m_3\) over the center of masses (\(\omega – is the angle velocity of triangle rotation, I – is the proper moment of inertia of such a triangle)):

\[
I \cdot \omega = \text{const}, \quad I \sim R^2, \quad \Rightarrow \quad \omega = \frac{C}{R^2}, \quad C = \text{const}
\]  \(1.3\)

Thus, if an assumption below is valid

\[
\frac{|R_{2,3}|}{m_1} = \frac{|R_{1,2}|}{m_3}, \quad \frac{|R_{2,3}|}{m_1} = \frac{|R_{3,1}|}{m_2}
\]

\[
\Rightarrow \quad \frac{m_1}{m_3} = a, \quad \frac{m_1}{m_2} = b.
\]
above (1.2) system of vector equations could be reduced to only one ODE below:

\[
\left( R(t) \cdot \sin(\omega t + \varphi_0) \right)'' + \gamma \left( m_1 + m_2 + m_3 \cdot \frac{b^3 + (b^3 - 1) \cdot \alpha_{xi}}{a^3} \right) \frac{\text{sign} \left[ \sin(\omega t + \varphi_0) \right]}{\left( R(t) \cdot \sin(\omega t + \varphi_0) \right)^2} = 0,
\]

where \( \alpha_{xi} \) – are the coefficients of proportionality between the initial coordinates of vectors \( R_{2,3} / R_{1,2} \parallel R_{1,2} / R_{1,1} \). We also should take into consideration the expression (1.3) for angle velocity: \( \omega = C / R^2 \), \( C = \text{const} \).

Such an ordinary differential equation for finding the function \( R(t) \) is very complicated to solve by analytical methods, so it should be solved by numerical math methods.

Besides, according to the Bruns theorem [4], we know that there is no other invariants except well-known 10 integrals for 3-bodies problem (including integral of energy, momentum, etc.).

Let’s summarise:

First of all, we represent the equations of 3-bodies problem in appropriate Lagrange form (1.2), describing a relative motions of 3-bodies. Such a system is proved to describe a motions of similar character for evolution of each of 3-bodies relative distances \( R_{1,2}, R_{2,3}, R_{3,1} \).

Then we consider a solutions of (1.2) for which is valid an assumption below:

\[
\frac{m_1}{m_3} = \frac{R_{2,3}}{R_{1,2}} = a, \quad \frac{m_1}{m_2} = \frac{R_{2,3}}{R_{3,1}} = b.
\]
Besides, we assume that triangle of 3-bodies $m_1, m_2, m_3$ is rotating on circle orbit around the common center of masses as well as increasing or decreasing the size of above triangle proportionally. Size (radius) of such an orbit is determined by masses $m_1, m_2, m_3$ as well as by parameters $a, b$. It means that proportions of absolute meanings of the relative distances $R_{1,2}, R_{2,3}, R_{3,1}$ should be the same all the time & should be equal to the initial proportions (which are given by special initial conditions).

Thus, such a solution of (1.2) should be factorized as below ($R_o = R(t_0)$):

$$R = \frac{R_o}{\|R_o\|} \cdot R(t) \cdot \sin(\omega t + \varphi_0),$$

- here $\omega = C / R^2, \varphi_0 = \varphi (t_0)$ – the initial angle when triangle of bodies began to rotate, $R = R_i$ – is a vector of general motion, which describes the identical character of evolution for each of 3-bodies relative distances $R_{1,2}, R_{2,3}, R_{3,1}$ ($i = 1,2, 2,3, 3,1$),

where:

$$R(t) = |R| \cdot \max \{\sin(\omega t + \varphi_0)\}$$

– is the scale factor or measure for appropriate relative distances between the bodies.

Finally, all vector equations of (1.2) could be reduced only to one ODE below:

$$
\left( R(t) \cdot \sin(\omega t + \varphi_0) \right)'' + \gamma \left( m_1 + m_2 + m_3 \cdot \left( \frac{b^4 + (b^2 - 1) \cdot \alpha_{xi}}{a^3} \right) \right) \cdot \text{sign} \left[ \sin(\omega t + \varphi_0) \right] \frac{\sin(\omega t + \varphi_0)}{|R(t) \cdot \sin(\omega t + \varphi_0)|^2} = 0,
$$

- where $\alpha_{xi}$ – are the coefficients of proportionality between the proper coordinates of initial vectors $R_{2,3} / |R_{2,3}|, R_{1,2} / |R_{1,2}|$. 

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The last ordinary differential equation - *in regard to the function* $R(t)$ - is very complicated to solve by analytical methods, but it could be solved properly only by numerical math methods.

References: