New exact solution of 3-bodies problem.

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Here is presented a system of equations of 3-bodies problem in well-known Lagrange’s form (describing a relative motions of 3-bodies). Analyzing of such a system, we obtain an exact solution in special case of constant ratios of relative distances between the bodies.

Above simplifying assumption reduces all equations of initial system to a proper unique form, which leads us to a final solution: initial triangle of bodies $m_1$, $m_2$, $m_3$ is moving as entire construction, simultaneously rotating over the common center of masses as well as increasing or decreasing of it’s size proportionally.
Let us consider the system of an ordinary differential equations for 3-bodies problem, at given initial conditions [1-3]:

\[
m_1q_1'' = -\gamma \left\{ \frac{m_1m_2(q_1 - q_2)}{|q_1 - q_2|^3} + \frac{m_1m_3(q_1 - q_3)}{|q_1 - q_3|^3} \right\},
\]

\[
m_2q_2'' = -\gamma \left\{ \frac{m_2m_1(q_2 - q_1)}{|q_2 - q_1|^3} + \frac{m_2m_3(q_2 - q_3)}{|q_2 - q_3|^3} \right\},
\]

\[
m_3q_3'' = -\gamma \left\{ \frac{m_3m_1(q_3 - q_1)}{|q_3 - q_1|^3} + \frac{m_3m_2(q_3 - q_2)}{|q_3 - q_2|^3} \right\}.
\]

- here \(q_1, q_2, q_3\) - means the radius-vector of bodies \(m_1, m_2, m_3\), accordingly.

For the purposes of exploring a relative motions of 3-bodies one to each other, let’s rewrite the system above as below (by linear transformation of initial equations):

\[
(q_1 - q_2)'' + \gamma (m_1 + m_2) \left( \frac{q_1 - q_2}{|q_1 - q_2|^3} \right) = \gamma m_1 \left\{ \frac{(q_3 - q_1)}{|q_3 - q_1|^3} + \frac{(q_2 - q_3)}{|q_2 - q_3|^3} \right\},
\]

\[
(q_2 - q_3)'' + \gamma (m_2 + m_3) \left( \frac{q_2 - q_3}{|q_2 - q_3|^3} \right) = \gamma m_2 \left\{ \frac{(q_3 - q_2)}{|q_3 - q_2|^3} + \frac{(q_1 - q_3)}{|q_1 - q_3|^3} \right\},
\]

\[
(q_3 - q_1)'' + \gamma (m_1 + m_3) \left( \frac{q_3 - q_1}{|q_3 - q_1|^3} \right) = \gamma m_3 \left\{ \frac{(q_2 - q_1)}{|q_2 - q_1|^3} + \frac{(q_3 - q_2)}{|q_3 - q_2|^3} \right\}.
\]

Let’s designate as below:
Above designating causes the transformation of a previous system to another form:

\[
R_{1,2} = (q_1 - q_2), \quad R_{2,3} = (q_2 - q_3), \quad R_{3,1} = (q_3 - q_1) \quad (**)
\]

\[
R_{1,2}''' + \gamma (m_1 + m_2) \frac{R_{1,2}'}{R_{1,2}^3} = \gamma m_2 \left\{ \frac{R_{3,1}'}{R_{3,1}^3} + \frac{R_{2,3}'}{R_{2,3}^3} \right\},
\]

\[
R_{2,3}''' + \gamma (m_2 + m_3) \frac{R_{2,3}'}{R_{2,3}^3} = \gamma m_1 \left\{ \frac{R_{1,2}'}{R_{1,2}^3} + \frac{R_{3,1}'}{R_{3,1}^3} \right\}, \quad (1.1)
\]

\[
R_{3,1}''' + \gamma (m_1 + m_3) \frac{R_{3,1}'}{R_{3,1}^3} = \gamma m_2 \left\{ \frac{R_{2,3}'}{R_{2,3}^3} + \frac{R_{1,2}'}{R_{1,2}^3} \right\}.
\]

Analysing system (1.1) we should note that if we sum all the above equations one to each other it would lead us to the result below:

\[
R_{1,2}''' + R_{2,3}''' + R_{3,1}''' = 0.
\]

If we also sum all the equalities (*) one to each other, we should obtain

\[
R_{1,2} + R_{2,1} + R_{3,1} = 0 \quad \text{[**]}
\]

Besides, if we substitute an expression for \( R_{3,1} \) - from 2-nd to 1-st equation of (1.1), then to the 3-d - we should obtain below:
So, the linear recombining of equations (1.1) let us define some vector function $F(t)$ which seems to be unique for all equations of (1.2). Otherwise, taking into consideration (**), we also obtain

\[
\begin{align*}
\left( R_{1,2}'' + \gamma(m_1 + m_2 + m_3) \frac{R_{1,2}}{R_{1,2}'} \right)^2 \frac{R_{1,2}}{m_3} &= F(t) \\
\left( R_{2,3}'' + \gamma(m_1 + m_2 + m_3) \frac{R_{2,3}}{R_{2,3}'} \right)^2 \frac{R_{2,3}}{m_3} &= F(t) \\
\left( R_{3,1}'' + \gamma(m_1 + m_2 + m_3) \frac{R_{3,1}}{R_{3,1}'} \right)^2 \frac{R_{3,1}}{m_2} &= F(t)
\end{align*}
\] (1.2)

It is well-known fact [1-3] that there are existing only 5 cases of exact (1.1) solutions (below $\mathbf{R}_i = (x_i, y_i, z_i)$, $i = 1, 2; 2, 3; 3, 1$):

- 3 Lagrange’s linear cases, when $\mathbf{R}_{1,2} \sim \mathbf{R}_{2,3} \sim \mathbf{R}_{3,1}$
2 Euler’s cases of equipotential triangle, when

\[
|R_{1,2}| = |R_{2,3}| = |R_{3,1}|, 
\]

\[\iff (** \implies F(t) = 0.\]

Let’s consider solutions of (1.2) for which is valid an assumption below

\[
\begin{align*}
\frac{|R_{2,3}|}{|R_{1,2}|} &= a, & \frac{|R_{2,1}|}{|R_{3,1}|} &= b, & \quad (***)
\end{align*}
\]

- then we obtain:

\[
F(t) = \frac{\gamma(m_1 + m_2 + m_3)}{|R_{2,3}|} \cdot \left( \frac{a^3 \cdot R_{1,2} + R_{2,3} + b^3 \cdot R_{3,1}}{a^2 \cdot m_3 + m_1 + b^2 \cdot m_2} \right), \quad R_{3,1} = -R_{2,3} - R_{1,2},
\]

\[
\implies F(t) = \frac{\gamma(m_1 + m_2 + m_3)}{a^2 \cdot m_3 + m_1 + b^2 \cdot m_2} \cdot \left( \frac{a^3 - b^3}{a} \cdot \frac{R_{1,2}}{|R_{1,2}|} + (1 - b^3) \cdot \frac{R_{2,3}}{|R_{2,3}|} \right).
\]

From equality above we could conclude that in case \( a = b = 1 \), we obtain Euler’s cases of equipotential triangle, but in case \( R_{1,2} \sim R_{2,3} \sim R_{3,1} \) all the equations of system (1.2) could be reduced to one of Lagrange’s linear cases [1].

Besides, let’s consider only solutions, for which the equalities below are valid:
In such a case, system of equations (1.2) could be reduced as below:

\[
\begin{align*}
\frac{|R_{12}|^2}{m_3} & = \frac{|R_{23}|^2}{m_1} \Rightarrow \frac{m_1}{m_3} = \frac{|R_{23}|^2}{|R_{12}|^2} = a^2 \\
\frac{|R_{31}|^2}{m_2} & = \frac{|R_{23}|^2}{m_1} \Rightarrow \frac{m_1}{m_2} = \frac{|R_{31}|^2}{|R_{23}|^2} = b^2
\end{align*}
\]

In such a case, system of equations (1.2) could be reduced as below:

\[
\begin{align*}
\left( \frac{R_{1,2}}{R_{1,2}} + \gamma(m_1 + m_2 + m_3) \frac{R_{1,2}}{|R_{1,2}|^2} \right) &= \tilde{F}(t) = \frac{m_3}{|R_{1,2}|^2} \cdot F(t) \\
\left( \frac{R_{2,3}}{R_{2,3}} + \gamma(m_1 + m_2 + m_3) \frac{R_{2,3}}{|R_{2,3}|^2} \right) &= \tilde{F}(t) = \frac{m_1}{|R_{2,3}|^2} \cdot F(t) \\
\left( \frac{R_{3,1}}{R_{3,1}} + \gamma(m_1 + m_2 + m_3) \frac{R_{3,1}}{|R_{3,1}|^2} \right) &= \tilde{F}(t) = \frac{m_2}{|R_{3,1}|^2} \cdot F(t)
\end{align*}
\]

So, analysing above (1.4) we can observe an identical character of equations for evolution of each of 3-bodies relative distances \( R_{1,2}, R_{2,3}, R_{3,1} \).

It could be possible only if such a triangle of bodies \( m_1, m_2, m_3 \) is moving as entire construction, rotating over the center of masses as well as increasing or decreasing the lengths of sides of such a triangle proportionally.

In according with our previous assumption (***) it means that absolute meanings of proportions between the relative distances \( R_{1,2}, R_{2,3}, R_{3,1} \) should be the same all the time & should be equal to the initial proportions (given by special initial conditions).
For the reason that entire triangle \( m_1, m_2, m_3 \) are proved to be moving in identical way, we could make a conclusion from (1.4) that every set of points of such a triangle shall also be moving in the same way (expression for \( F \) is given above):

\[
F'' + \gamma(m_1 + m_2 + m_3) \frac{F}{|F|^3} = \frac{m_3}{|R_{1,2}|^2} F(t), \quad |F| = \text{const} = C,
\]

\[
\Rightarrow F'' - \left( \frac{m_3}{|R_{1,2}|^2} - \frac{\gamma(m_1 + m_2 + m_3)}{C^3} \right) F = 0,
\]

- here \( F \) - is a vector, which does not depend on relative distances between masses \( m_1, m_2 \) or \( m_3 \). Besides, above equality describes a harmonic character of triangle \( m_1, m_2, m_3 \) rotation over the center of masses (\( \omega \) – is the angle velocity of triangle rotation):

\[
\Rightarrow F = F_0 \sin (\omega t + \varphi_0), \quad \omega = \sqrt{\frac{m_3}{|R_{1,2}|^2} - \frac{\gamma(m_1 + m_2 + m_3)}{C^3}},
\]

\[
C = |F(t)| = \frac{\gamma(m_1 + m_2 + m_3)}{(a^2 \cdot m_3 + m_1 + b^2 \cdot m_2)} \left| \frac{(a^3 - b^3)}{a} \frac{R_{1,2}}{|R_{1,2}|} + (1 - b^3) \frac{R_{2,3}}{|R_{2,3}|} \right| = \frac{\gamma}{3} \left( 1 + \frac{1}{b^2} + \frac{1}{a^2} \right) \cdot \sqrt{\left( \frac{(a^3 - b^3)}{a} \right)^2 + 2 \frac{(a^3 - b^3)}{a} (1 - b^3) \cdot \cos \left( \frac{R_{1,2}}{R_{2,3}} \right) + (1 - b^3)^2}.
\]

We could obtain a proper restriction for the regime of triangle \( m_1, m_2, m_3 \) rotation (from expression above for angle velocity \( \omega \)):

\[
\frac{m_3}{|R_{1,2}|^2} - \frac{\gamma(m_1 + m_2 + m_3)}{C^3} \geq 0, \quad \Rightarrow \left| R_{1,2} \right|^2 \leq \frac{C^3}{\gamma \left( \frac{1}{a^2} + \frac{a^2}{b^2} + 1 \right)}.
\]
Finally, the general solution of (1.4) should be factorized as below ($R_0 = R(t_0)$):

$$R = \frac{R_0}{R_0} \cdot R(t) \cdot \sin (\omega t + \varphi_0) ,$$

- here $R = R_i$ – is a vector of general motion, which describes an identical character of evolution for each of 3-bodies relative distances $R_{1,2}, R_{2,3}, R_{3,1}$ ($i = 1, 2, 3, 1, 2$), besides:

$$R(t) = |R| \cdot \max \{\sin (\omega t + \varphi_0)\}$$

– is the scale factor or measure of appropriate relative distances between the bodies.

Thus, we obtain:

$$\frac{R_0}{R_0} \cdot \left\{ (R \cdot \sin (\omega t + \varphi_0))'' + \frac{\gamma (m_1 + m_2 + m_3)}{(R \cdot \sin (\omega t + \varphi_0))^2} \right\} = \left( \frac{m_1}{R^2 \cdot \sin^2 \omega t} \right) \cdot F_0 \cdot \sin (\omega t + \varphi_0) ,$$

$$\Rightarrow (R \cdot \sin (\omega t + \varphi_0))'' + \frac{\gamma (m_1 + m_2 + m_3)}{(R \cdot \sin (\omega t + \varphi_0))^2} = \left( \frac{\alpha \cdot m_1}{R^2 \cdot \sin^2 \omega t} \right) \cdot \sin (\omega t + \varphi_0) ,$$

$$\Rightarrow (R \cdot \sin (\omega t + \varphi_0))'' + \left\{ \gamma (m_1 + m_2 + m_3) - \alpha \cdot m_1 \cdot \sin (\omega t + \varphi_0) \right\} \cdot \frac{1}{(R \cdot \sin (\omega t + \varphi_0))^2} = 0 ,$$

- where $\alpha$ – are the coefficients of proportionality between the proper coordinates of vectors $R_{1,2} / |R_{1,2}|, R_{2,3} / |R_{2,3}|$. We also should take into consideration the above expression for angle velocity $\omega$:

$$\omega = \sqrt{\frac{m_1}{R^2} - \frac{\gamma (m_1 + m_2 + m_3)}{C^3}} .$$
If we designate \( y(t) = R(t) \cdot \sin(\omega t + \varphi_0) \), the last equation could be represented as below:

\[
y'' + \left[ \gamma (m_1 + m_2 + m_3) - \alpha_i \cdot m_i \cdot \sin (\omega t + \varphi_0) \right] \cdot \frac{1}{y^2} = 0 ,
\]

- but such an ordinary differential equation for finding the function \( R(t) \) is very complicated to solve by analytical methods, so it should be solved by numerical math methods.

Besides, according to the Bruns theorem [4], we know that there is no other invariants except well-known 10 integrals for 3-bodies problem (including integral of energy, momentum, etc.).

Let’s summarise:

First of all, we represent the equations of 3-bodies problem in appropriate Lagrange form (1.2), describing a relative motions of 3-bodies. Then we consider a solutions of (1.2) for which is valid an assumption (1.3):

\[
\frac{m_1}{m_3} = \left| \frac{R_{2,3}}{R_{1,3}} \right|^2 = a^2 , \quad \frac{m_1}{m_2} = \left| \frac{R_{2,3}}{R_{3,1}} \right|^2 = b^2
\]

For such a kind of solutions we obtain that equations (1.4) describe a motions of identical character for evolution each of 3-bodies relative distances \( R_{1,2}, R_{2,3}, R_{3,1} \).

Besides, we obtain that triangle of 3-bodies \( m_1, m_2, m_3 \) is proved to be rotating on circle orbit around the common center of masses as well as increasing or decreasing the size of above triangle proportionally. Size (radius) of such an orbit is determined
by masses $m_1, m_2, m_3$ as well as by parameters $a, b$. It means that absolute meanings of proportions between the relative distances $R_{1,2}, R_{1,3}, R_{3,1}$ should be the same all the time & should be equal to the initial proportions (given by special initial conditions).

Thus, the general solution of (1.4) should be factorized as below ($R_0 = R(t_0)$):

$$R = \left[ \frac{R_0}{R_0} \right] \cdot R(t) \cdot \sin (\omega t + \phi) ,$$

- here $R = R_i$ – is a vector of general motion, which describes the identical character of evolution for each of 3-bodies relative distances $R_{1,2}, R_{2,3}, R_{3,1}$ ($i = 1, 2, 3, 3, 1$), where:

$$R(t) = ||R|| \cdot \max \{\sin (\omega t + \phi)\}$$

– is the scale factor or measure for appropriate relative distances between the bodies.

Besides, here:

$$\omega = \sqrt{\frac{m_i}{R^2} - \frac{\gamma (m_1 + m_2 + m_3)}{C^3}} ,$$

- where expression for $C$ is given above.

It means that angle velocity $\omega$ depends on the radius of circle orbit of 3-bodies triangle rotation: - the larger radius, the less an angle velocity $\omega$; - the less radius, the larger an angle velocity $\omega$, but such a regime of rotation is valid only up to the appropriate meaning of radius below:

$$R \leq \sqrt{\frac{C^3}{\gamma \left( \frac{1}{a^2} + \frac{a^2}{b^2} + 1 \right)}} .$$
Finally, all vector equations (1.4) could be reduced only to one ODE below:

\[(R \cdot \sin(\omega t + \varphi_0))^{\prime} + \frac{\gamma (m_1 + m_2 + m_3) - \alpha_i \cdot m_i \cdot \sin(\omega t + \varphi_0)}{(R \cdot \sin(\omega t + \varphi_0))^2} = 0,\]

- where \(\alpha_i\) – are the coefficients of proportionality between the proper coordinates of initial vectors \(R_{1,2}/|R_{1,2}|, R_{2,3}/|R_{2,3}|\).

The last ordinary differential equation - in regard to the function \(R(t)\) - is very complicated to solve by analytical methods, but it could be solved properly by numerical math methods.

References: