Logistic equation of population growth or exhaustion of main resources: generalization to the case of reactive environment, reduction to *Abel* ODE, asymptotic solution for final Human population prognosis.

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Keywords: Logistic equation, evolution of population, Human population prognosis, exhaustion of main resources, reactive environment, catastrophe theory, *Riccati* type of ordinary differential equations, *Abel* ordinary differential equation.

Here are presented a key points of new universal model for population evolution in reactive environment: 1) generalization of the Logistic equation in the case of reactive environment for population dynamics model in biology (also, for the model of exhaustion of main resources in geology, or for filling of an ecological niches in ecology, or for modeling of capacities of a proper markets in economics), 2) new type of asymptotic solution for such an equation (which is tested on human population growth), 3) reduction of such an equation to *Abel* ordinary differential equation in general case.

Due to a very special character of Riccati's type equation, it's general solution is proved to have *a proper gap* of components of such a solution.

It means an existence of continuous general solution only at some definite, *restricted* range of parameter of time, or a possibility of sudden gradient catastrophe in regard to the components of solution (population growth or exhaustion of main resources), at definite moment of time-parameter.

In accordance with [1-3], logistic equation describes how population evolves over time. Such an equation actually determines a *linear* dependence of *self-similar rate* of evolution process (or dynamics of population) in regard to the proper residual capacity *of non-filled part of niche* (which is known as "potential difference", defining a proper rate of population dynamics to be), as below [4]:

$$\left(\frac{d\,N\,/\,d\,t}{N} \right) = b \cdot \left(K - N \right) \,,$$

- here t – is time-parameter, N – is the population total (*or a proper level of niche saturation, or the total of main resources*), $N = N(t)$; $b = b\,(t)$ – is the Malthusian parameter (*rate of maximum population growth*); K – is the so-called carrying capacity (*i.e., the maximum sustainable population at total saturation of a proper niche*), $K = K\,(t)$.

Besides [4], a key point in modeling of such a population dynamics processes - is to take into consideration the moment of *reactive environment*. In this case, as for resistance of environment:

$$R_{\,active} + R_{\,reactive} \,,$$

- where $R_{\,active}$ – is the active, constant resistance of environment, or a resistance of environment due to saturation of niche by extraneous elements (*or in modeling a process of an exhaustion of main resources, it is an exhaustion of the niche in regard to it's own elements*); $R_{\,reactive}$ - is the reactive, non-constant resistance of environment, i.e. a proper resistance of environment as reaction in regard to increasing of extraneous elements, incorporated into this population $N(t)$ (*or into non-filled part of a proper niche*), where $R_{\,reactive} = R_{\,reactive}\,(N)$.

Such an active resistance $R_{\,active}$ does not depend on the amounts of elements (i.e., it has a stable, constant value). For example, in the case of human population dynamics, such an active resistance $R_{\,active}$ means the world-wide accidents which unfortunately take place every year in the world ~ like aviation-accidents or catastrophes of technic, natural or other character (*as a result, we have a decreasing of human population by millions every year*).

We should also take into consideration that the level of saturation of a proper niche is known to be determined by the level of demographic pressure [4]:

$$P(t) = N(t)/K(t),$$

- besides, we know about the principle “counteraction is to be like action”:

$$R_{reactive}\,(N) \sim N(t),$$

- the total of environment resistance should be defined as below:

$$R(t) \sim R_0\,(t) + (N(t)/K(t))\ .$$

Summarizing all the assumptions above, we could formulate below a new universal principle which is to define the dynamics of such a processes of evolution:

Self-similar rate of evolution process is to be directly proportional to the residual capacity of non-filled part of a proper niche, besides it is simultaneously to be in inverse proportion to resistance of environment.

Taking into consideration the above universal principle, we should represent logistic evolution equation in a new form below:

$$\left(\frac{d\,N\,/\,d\,t}{N}\right) \;=\; b\cdot\left(\frac{1\;-\;(N\,/\,K)}{R\;+(N\,/\,K)}\right),$$

- here besides previous notations, R – is a function of active, constant resistance of environment $R_0\,(t)$, but we will denote it as $R\,(t)$ here & below.

Representing last equation in a form below:

$$(\,K\cdot R\;+N\,)\cdot N' \;=\; b\cdot(\,K\cdot N\;-\;N^{\,2}\,) \qquad (1.1)$$

- we obtain a proper *Abel* ordinary differential equation of the 2-nd kind in regard to the function *N(t)* (*besides, Abel ODE is known to be a Riccati type equation* [5]) .

Due to a very special character of such an equations, it’s general solution is known to have *a proper gap* of function *N(t)* at some moment *t* [6]. It means an existence of continuous solution only at some definite, *restricted* range of parameter *t*, or possibility of sudden gradient catastrophe [7] at some moment *t* .

Let's make a proper replacement of variables $N(t) + K \cdot R = 1/y(t)$ in equation (*1.1*), then we obtain a proper *Abel* ordinary differential equation of the 1-st kind [5]:

$$y' = b \cdot \left\{ K^2 \cdot R \cdot (1+R) \cdot y^3 - \left(K \cdot (1+2R) + \frac{(K \cdot R)'}{b} \right) \cdot y^2 + y \right\}, \qquad (1.2)$$

If we assume: $K(t) = const = K$, $b(t) = const = b$, $R(t) = const = R$, above equation (*1.2*) could be reduced as below:

$$y' = b \cdot \left\{ K^2 \cdot R \cdot (1+R) \cdot y^3 - K \cdot (1+2R) \cdot y^2 + y \right\}.$$

- where variables are proved to be separated [5].

Besides, if we assume $R = 1/K$ (*i.e., it is a case when the level of environment resistance is to be directly proportional to the carrying capacity*), such an assumption should properly simplify the right part of above equation (*1.2*):

$$y' = b \cdot \left\{ \frac{1}{R} \cdot (1+R) \cdot y^3 - \frac{1}{R} \cdot (1+2R) \cdot y^2 + y \right\},$$

- or, in other form:

$$\left(\frac{R}{1+R} \right) \cdot \int \frac{dy}{[y-1] \cdot y \cdot \left[y - \left(\frac{R}{R+1} \right) \right]} = b \cdot \int dt,$$

- but by making a proper replacement of variables $\ln y = z$, above equation could be easily solved:

$$\frac{y \cdot (y-1)^R}{\left(y - \frac{R}{1+R} \right)^{(1+R)}} = e^{b \cdot \Delta t},$$

- where, by expressing of function $y(t) = 1/(N(t) + 1)$, we obtain:

$$\left[\frac{N(t)}{R \cdot N(t) - 1}\right]^{(1+R)} \times \left(-\frac{1}{N(t)}\right) = \frac{e^{b \cdot \Delta t}}{(1+R)^{(1+R)}}$$

If we represent the last equality in other form ($R = 1/K$):

$$\left[\frac{1}{K} - \frac{1}{N(t)}\right]^{\left(1+\frac{1}{K}\right)} \times (-N(t)) = \left(1 + \frac{1}{K}\right)^{\left(1+\frac{1}{K}\right)} \times e^{-b \cdot \Delta t}$$

- we obtain a proper expression for $N(t)$.

Besides, if we take into consideration that carrying capacity K in the case of modeling of human population is large enough ($K \sim$ *18 billions of persons* [4]), last equation could be easily simplified under the condition $(1 + 1/K) \to 1$:

$$N(t) = \left(1 - e^{-b \cdot \Delta t}\right) \cdot K \quad .$$

Thus, we obtain a simple asymptotic solution for final Human population prognosis (*besides, for population dynamics in biology, or for exhaustion of main resources in geology, or filling of an ecological niches in ecology, or for modeling of markets capacities in economics*).

At final, let's note an interesting case of equation (*1.2*) exact solution, where variables are proved to be separated under conditions $R = 1/K = const$, $b(t) \neq const$, $(1/K) \to 1$:

$$N(t) = \left(1 - e^{-\int b(t)\,dt}\right) \cdot K \quad .$$

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