Probabilistic Interpretation of Quantum Mechanics with Schrödinger Quantization Rule

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Abstract

Quantum theory is a probabilistic theory, where certain variables are hidden or non-accessible. It results in lack of representation of systems under study. However, I deduce system’s representation in probabilistic manner, introducing probability of existence $w$, and quantize it exploiting Schrödinger’s quantization rule. The formalism enriches probabilistic quantum theory, and enables systems’s representation in probabilistic manner.

keywords Schrödinger Operators • Probability • Hidden Variables

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1 Introduction

Classical physics is based on mechanistic perspective, where no contingencies appear [1, 2]. It results in a deterministic theory, where no chances appear, and systems are governed by mechanistic laws. On the contrary, quantum theory is a probabilistic theory [3, p. 260]. So is its interpretation [4]. Quantum theory is not based on mechanistic order [2]. Indeterminism, an ingredient part of the theory, appears due to some hidden variables [5, 6, 7]. In a non-deterministic (acausal) theory (like QM) certain variables are (hidden) non-accessible. It persists in lack of representation of the system under study.

However, we define system’s existence in probabilistic manner. We assign a probability ($w$) in order to define a system in isolation. For $w = 1$ system is in pure state and all its variables are accessible, for $w \in (0, 1)$ it is in mixed state as certain of variables are hidden or non-accessible (e.g. in presence of many type of interactions [8]). For $w = 0$ the system is in forbidden state and all its variables are hidden and system can be represented by none. We quantize this observable using Schrödinger’s quantization rule and obtain $\hat{w} = -i\hbar \partial/\partial s$. Exploiting usual formalism of QM [9, 10, 11] we further deduce quantum dynamical equations, based on non-commutativity between probability $w$ and dynamicals $A$.

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2 Probability Eigenvalue Formalism

We have a general form of Schrödinger’s wavefunction\(^1\) [10] belonging to system’s Hilbert space \(\mathbb{H}\), in generalized perspective

\[
\psi(s(q_i, t)) := R(q_i, t) \exp\left(\frac{i}{\hbar} s(q_i, t)\right), \quad i = 1, 2, 3, \ldots, f, \tag{2.1}
\]

which is orthonormalizable

\[
\langle \psi_\alpha | \psi_\beta \rangle = \int_{-\infty}^{+\infty} \psi_\alpha^*(s(q_i, t)) \psi_\beta(s(q_i, t)) \, d\tau = \delta_{\alpha\beta}, \tag{2.2}
\]

where \(d\tau = \prod_{i=1}^{f} h dq_i, \) \(h\) being scale factor and \(f\) is degrees of freedom) is generalized volume element of the configuration space. [The system has all these variables, except \(\psi\) (and tacitly its space \(\mathbb{H}\)) in Praxic perspective [12]]. Differentiate (2.1) partially w.r.t. \(A\)ction to obtain

\[
\frac{\partial \psi(s(q_i, t))}{\partial s(q_i, t)} = \frac{i}{\hbar} \psi(s(q_i, t)). \tag{2.3}
\]

We propose a unit (zero-order differential) operator that satisfies for an ordinary function \(f\) as well as for wavefunction (See Appendix A)

\[
\mathcal{I} f = f; \quad \mathcal{I} \psi(s(q_i, t)) = \psi(s(q_i, t)). \tag{2.4}
\]

Following deduction (2.4) for (2.3), we obtain

\[
\mathcal{I} \psi(s(q_i, t)) + i\hbar \frac{\partial \psi(s(q_i, t))}{\partial s(q_i, t)} = 0, \tag{2.5}
\]

which is in the form of eigenvalue equation. We deduce Schrödinger unit operator \(\hat{\mathcal{I}}\) [in the sense of Schrödinger’s quantization rule] satisfying unit eigenoperator equation [13, Dwivedi 2005]

\[
\hat{\mathcal{I}} \psi = \mathcal{I} \psi; \quad \hat{\mathcal{I}} = -i\hbar \frac{\partial}{\partial s}. \tag{2.6}
\]

Its expectation value is given by inner-product

\[
\langle \hat{\mathcal{I}} \rangle = \langle \psi | \hat{\mathcal{I}} | \psi \rangle = \int_{-\infty}^{+\infty} \psi^*(s(q_i, t)) \left( -i\hbar \frac{\partial \psi(s(q_i, t))}{\partial s(q_i, t)} \right) \, d\tau = \int_{-\infty}^{+\infty} |\psi(s(q_i, t))|^2 \, d\tau = \text{Prob.}(-\infty, +\infty). \tag{2.7}
\]

[It could also be obtained alternatively using (2.4) and (2.6) in inner-product (2.7).] The operator \(\hat{\mathcal{I}}\), having trace \(\text{Prob.}(-\infty, +\infty)\), entails properties of our probability operator \(\hat{w}\). For a system in isolation:

\[
\begin{aligned}
\text{Prob.}(-\infty, +\infty) &= w_{\text{pure}} = 1 \quad \text{for pure state;} \\
\text{Prob.}(-\infty, +\infty) &= w_{\text{mixed}} \in (0, 1) \quad \text{for mixed state;} \\
\text{Prob.}(-\infty, +\infty) &= w_{\text{forbidden}} = 0 \quad \text{for forbidden state.}
\end{aligned} \tag{2.8}
\]

Thus \(\hat{\mathcal{I}}\) is essentially \(\hat{w}\) that satisfies probability eigenvalue equation

\[
\hat{w} |\psi_w\rangle = w |\psi_w\rangle; \quad \hat{w} = -i\hbar \frac{\partial}{\partial s}, \tag{2.9}
\]

Or

\[
w \psi_w(s(q_i, t)) + i\hbar \frac{\partial \psi_w(s(q_i, t))}{\partial s(q_i, t)} = 0, \tag{2.10}
\]

\(^1\)It’s remarkable that it could be treated as function of \(q_i\) and \(t\) as well as function of \(s\) explicitly.
having solution
\[ \psi_w(s(q, t)) = A \exp \left( \frac{i}{\hbar} ws(q, t) \right). \] (2.11)

For now and later on we will treat \( \psi \) as function of \( s \) explicitly. For orthonormalization we have the inner-product,
\[ \langle \psi_w'|\psi_w \rangle = \int_{-\infty}^{+\infty} \psi_w^*(s) \psi_w(s) \, ds = |A|^2 \int_{-\infty}^{+\infty} \exp \left( \frac{i}{\hbar} (w - w') s \right) \, ds 
= |A|^2 2\pi \delta(w - w'). \] (2.12)

For \( A = 1/\sqrt{2\pi\hbar} \), we have
\[ \psi_w(s) = \frac{1}{\sqrt{2\pi\hbar}} \exp \left( \frac{i}{\hbar} ws \right) \] (2.13)
that follows Dirac orthonormality
\[ \langle \psi_w'|\psi_w \rangle = \delta(w - w'). \] (2.14)

However, these eigenfunctions form complete set \( (\psi = \sum_w c_w \psi_w) \). For (square-integrable) function \( \psi(s) \),
\[ \psi(s) = \int_0^1 c(w) \psi_w(s) \, dw = \frac{1}{\sqrt{2\pi\hbar}} \int_0^1 c(w) \exp \left( \frac{i}{\hbar} ws \right) \, dw. \] (2.15)
The expansion coefficient is obtained by Fourier’s trick
\[ \langle \psi_w'|\psi \rangle = \int_0^1 c(w) \langle \psi_w'|\psi_w \rangle \, dw = \int_0^1 c(w) \delta(w - w') \, dw = c(w'), \] (2.16)
or
\[ c(w) = \langle \psi_w'|\psi \rangle. \] (2.17)
Exploiting completeness (2.15), the amplitude \( R \) in (2.1) is obtained
\[ R = \frac{1}{\sqrt{2\pi\hbar}} \int_0^1 c(w) \exp \left( \frac{i}{\hbar} s(w - 1) \right) \, dw. \] (2.18)

3 Quantum Dynamical Equations

Dynamics is a law relating physical quantities in course of time (or some internal observables [15]). In Praxic theory Action is a fundamental physical entity [14]. However, it could often be customary to deduce dynamics in course of Action. Let differentiate the inner-product,
\[ \langle \hat{A} \rangle = \langle \psi|\hat{A}|\psi \rangle = \int_{-\infty}^{+\infty} \psi^* \hat{A} \psi \, d\tau, \] (3.1)
exactly \( w.r.t. \) Action with differential-integral rule
\[ \hat{f} g(\kappa) = \hat{f} \int_{-\infty}^{+\infty} \phi(\tau) K(\kappa, \tau) \, d\tau = \int_{-\infty}^{+\infty} \hat{f} \{ \phi(\tau) K(\kappa, \tau) \} \, d\tau, \] (3.2)
we obtain (using chain rule for \( \hat{f} := \frac{\partial}{\partial \kappa} \))
\[ \frac{\partial}{\partial s} \langle \hat{A} \rangle = \langle \frac{\partial \psi}{\partial s}|\hat{A}|\psi \rangle + \langle \psi|\frac{\partial \hat{A}}{\partial s}|\psi \rangle + \langle \psi|\hat{A}|\frac{\partial \psi}{\partial s} \rangle. \] (3.3)
Considering probability eigenvalue equations
\[ \frac{\partial \psi}{\partial s} = \frac{i}{\hbar} \hat{w} \psi, \quad \langle \psi|\frac{\partial \hat{A}}{\partial s}|\psi \rangle = -\frac{i}{\hbar} \langle \hat{w}^\dagger \psi | \]. (3.4)

As Probability does not exist in the domain \((-\infty, 0) \cup (1, +\infty)\), we have omitted integration over this domain. It does not create trouble in formalism.
we obtain
\[ \frac{\partial}{\partial s} \langle \hat{A} \rangle = \langle \frac{\partial \hat{A}}{\partial s} \rangle - \frac{i}{\hbar} \langle [\hat{w}^\dagger \hat{A}]^\dagger \psi \rangle - \langle \hat{A} \hat{w} \psi \rangle \].
\[(3.5)\]

Here \(\hat{A}\), defined by \(\hat{A} = \langle \psi | \hat{A} | \psi \rangle\), is a dynamical \([15]\) — an observable-valued-function of system’s variables \(-\hat{A}(q_1, t)\) as distinct from observables. Since probability is a real aspect of nature, i.e., in operator representation, it must be hermitian\(^3\),
\[ \langle \hat{w}^\dagger \hat{A} | \psi \rangle = \langle \hat{w} \hat{A} | \psi \rangle , \]
\[\text{which yields} \]
\[ \frac{\partial}{\partial s} \langle \hat{A} \rangle = \langle \frac{\partial \hat{A}}{\partial s} \rangle - \frac{i}{\hbar} [\hat{w}, \hat{A}]_{\lambda} . \]
\[(3.7)\]

This is first order \(\text{quantum dynamical equation}.\) Following the analogy, we further obtain second and third order quantum dynamical equations
\[ \frac{\partial^2}{\partial s^2} \langle \hat{A} \rangle = \langle \frac{\partial^2 \hat{A}}{\partial s^2} \rangle - \frac{i}{\hbar} \langle [\hat{w}^\dagger \hat{A}]^\dagger \hat{w} \hat{A} | \psi \rangle - \langle \hat{A} \hat{w} \hat{A} | \psi \rangle \]
\[ - \langle \hat{A} \hat{w} \hat{A} \rangle \]
\[\text{for operators} (\frac{\partial^2 \hat{A}}{\partial s^2}, n = 0, 1, 2, ...) \text{compatible with} \hat{w}, \text{these equations follow Ehrenfest’s theorem} \]
\[ \frac{\partial^n \langle \hat{A} \rangle}{\partial s^n} = \langle \frac{\partial^n \hat{A}}{\partial s^n} \rangle . \]
\[\text{(3.10)}\]

It holds good for observables having simultaneous eigenstates with probability \(\hat{w}\).

**Appendix**

**A \hspace{1em} Unit Operator**

Unit operator (eigenoperator), analogous to \(\text{idemtity matrix}\), is deduced as a zero-order (ordinary or partial) differential operator (irrespective of with respect to what) defined as
\[ \mathcal{I} := \partial^0_x = \frac{\partial^0}{\partial x^n} ; \quad (x = q, p, t, ...) . \]
\[\text{(A.1)}\]

We have observed in \(\text{mathematical analysis}\) that a zero-order differential operator does not change the function to which it is applied which leads to deduce it unit operator satisfying \(\mathcal{I} f = f\). For example, in Ostrogodsky transformation, zero-order prime of generalized co-ordinate \(q^n, (n = 0, 1, 2, 3, ...)\) for \(n = 0\) is given by \(q\). It may be extended to \(q^n = \mathcal{I} q = q\) for \(n = 0\) with \(\mathcal{I} := \partial^0_q\). The deduction is less applicable in mathematical analysis but is very important to deal with quantum problems. Unit operator is quantized to \(\hat{\mathcal{I}} := -i \hbar \frac{\partial}{\partial q}\) satisfying unit eigenoperator equation \(\hat{\mathcal{I}} |\psi\rangle = |\psi\rangle\) while treating quantum problems. For example, a quantum transformation with \(\psi, (n = 0, 1, 2, 3, ...)\) (being \(n\)-th order partial derivative of \(\psi\) w.r.t. any variable \(x\)) is extended for \(n = 0\), \(\psi = \mathcal{I} \psi = \psi\) with \(\mathcal{I} := \partial^0_x\). This is a quantum problem and we quantize \(\mathcal{I}\) to \(\hat{\mathcal{I}}\) which yields \(\psi + i \hbar \frac{\partial \psi}{\partial s} = 0\), for \(n = 0\).

\[^3\text{It also follows from counter-intuitive behavior of probability operator} \hat{w}.\]
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