Constructing Legendrian Links from Chiral Reeb Chords

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Abstract

Some theorems on the construction of links (defined herein) from Reeb chords are established and proved. We relate these concepts to physics by constructing an arbitrary spacetime from a background knot using Reidemeister moves.

The dream of string-theorists would seem not to be to unify the fundamental forces, but to envision a brave new physics. In this new physics, objects possess the intrinsically topological property that they are glued at their ends to some higher dimensional object. Further, we are allowed to generalize the properties of the relationships these objects possess by inductively extending the number of dimensions, until we become bored or reach a convenient number of such.

We call these objects branes. However, unbeknownst to the physicists, the mathematicians have had their own toy for quite some time: Reeb chords. Simply put, a Reeb chord is a string that is forced to end on another string. The "strings" that these chords end on are actually a special type of submanifold, called a Legendrian submanifold.

We shall begin our tale with the very definition of a Legendrian (sub)manifold. Our story is woven out of these, and they will be our bread and butter:

Definition 0.1. *The standard contact structure on $\mathbb{R}^3$ is the two-plane field $\xi_0 = ker(\alpha_0)$, where $\alpha_0 = dz - ydx$ with respect to the standard co-ordinates (x, y, z). A 1-dimensional submanifold $\Lambda \subset \mathbb{R}^3$ is Legendrian if it is everywhere tangent to ξ_0.*

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1 Prologue

Write $Mo_{x,f,\varepsilon,\delta}(F)$ for the Morse group of F. ε and δ are here sufficiently small (accessible) cardinals, f is a functor and $x \in X$ is some value, where X is a pre-specified domain.

Notation 1.1. *We will write $\mathcal{R}_{ijk}$ for any permutation of Reidemeister moves acting on a link.*

Definition 1.1. *A cotangent vector $(x, \xi) \in T^*M$ is called characteristic with respect to F if, for some stratified Morse f with $df_x = \xi$, the Morse group $Mo_{x,f}(F)$ is non-zero.*

Lemma 1.1 (Shende-Treumann-Zaslow)**.** *Suppose $f : M \to \mathbb{R}$ is a smooth function such that, for all $x \in f^{-1}([a, b])$, the cotangent vector $(x, df_x) \notin SS(F)$, where $SS(F)$ is a conic Lagrangian. Suppose additionally that f is proper on the support of F. Then the restriction map*

$$\Gamma(f^{-1}(-\infty, b); F) \to \Gamma(f^{-1}(-\infty, a); F)$$

is a quasi-ismorphism.

Proof. See [2, Corollary 5.4.19]. □

Proposition 1.1. *Legendrian knots may be modelled on $S_t \times \mathbb{R}$, where S_t is Tate's circle.*

Proof. See [1, sec. 3.1.3]. We simply make the substitution

$$S^1 \longrightarrow S_t$$

Form the following category

$$\begin{tikzcd} S^1 \arrow[r, "\cong"] \arrow[d, hook, "i"'] & S_t \arrow[dl, "i'"] \\ D^1 & \end{tikzcd}$$

where $\{i, i'\} = \gamma(S_t) \vee \gamma(S^1)$, where γ is the closure operator and $\vee$ is the typical logical 'or'. Since the above diagram commutes, we obtain an embedding into the unit disc. Here, we use the formula $D^1 = S_t \times \mathbb{A}^1 = \mathbb{A}^1 \times S^1$ to give us the isomorphism $i \cong i'$, which is a commutative multiplication by the affine line. □

Using the terminology of [1], the Tate circle consists of a single arc and a single region.

Definition 1.2 (Arcs and regions)**.** *An arc is a 1-dimensional stratum of a maximal smooth subspace of $\Phi = \pi(\Lambda) = \phi(U)$. A region is a two-dimensional stratum. The component $\pi(\Lambda)$ is referred to as the front projection of a Legendrian knot.*

1.1 Spacetime Backgrounds

Let $\mathcal{S}$ be a spacetime and let $\mathcal{S}^*$ be its background. Our choice of $\mathcal{S}^*$ corresponds to a system of pre-defined links and crossings. Let us first specify that there is an equivalence

$$fib^n(X) \overset{\sim}{\twoheadrightarrow} \Lambda$$

or, in other words, Legendrian manifolds are restricted n-links. We will use this knowledge later to construct $n > 1$-links out of chiral Reeb chords.

Definition 1.3. *A link (or n-link) is an n-dimensional fiber which contacts two objects at opposing ends at at least a single point.*

Definition 1.4. *The endpoints of a link are called* contact points*. A contact point has a contact structure, $C(*, A)$, where A is the ambient space and $* \in \bar{\ell}$ is a point in the closure of a link.*

Remark 1.1. *1-links that begin and end on Lagrangian submanifolds are called* Reeb chords*; see, e.g. [4]. Write $[\gamma]$ for the set of all Reeb chords.*

Definition 1.5. *Fix a line λ_0 at height zero and fix a polygon P beneath it. Once the height of the polygon is increased such that some height$(x \in P)$ is greater than height(λ_0), the intersection $P \cap \lambda_0 = C$ is the set of crossings of links.*

We define a map

$$\Phi \longrightarrow \mathcal{S} = \mathcal{R}_{ijk}(\mathcal{S}^*)$$

which equates the spacetime with a combination of Reidemeister moves on links in the spacetime background. Recall from [1] and [3] the three primary types of Reidemeister moves: crossing, flipping, and twisting. Visual illustrations of all of these are given in [1], figures [2.3.1-3].

Example 1.1. *The most trivial example of a spacetime background constructed in this fashion is the empty background:*

$$\mathcal{S} \neq \emptyset \quad \mathcal{S}^* = \emptyset$$

$$\mathcal{R}_{ijk}(\mathcal{S}) \simeq Cr_x(f)$$

where Cr_x is a natural transformation

$$\begin{tikzcd}
\mathcal{S}^* \arrow[r, "f"] & \mathcal{S} \\
\emptyset \arrow[r, "x"'] \arrow[u, phantom, shift left=4, "\Uparrow Cr_x(f)"] & \neg\emptyset
\end{tikzcd}$$

1.2 Chord chirality

Fix a point $\gamma_0^0 e \in \gamma_0$, and a map $p : \gamma_0 \to \mathfrak{Posets}$, with the strong ordering $x < y < z$ for all $xyz \in \gamma_0/\sim$. We will say that a vector $\vec{v}$ on a Reeb chord is right-directed if it sends scalars $x < y$ to y by $x \mapsto y$, and left-directed if it sends scalars $y > x$ to x via $y \mapsto x$. This is just saying that monotone functors have a chirality. We say a point has *positive parity* (and write $*^+$) if $x(*) > x(\gamma_0^0 e)$, *negative parity* (and write $*^-$) if and only if $x(*) < x(\gamma_0^0 e)$ where x is the abscissa. Otherwise, we say the point has *nullparity*, in which case it is the barycenter of a Reeb chord.

Write ${}_L f([\gamma])$ for the set of all right-directed actions on Reeb chords, and $f([\gamma])_R$ for the set of all left-directed actions on Reeb chords. This is, effectively, the set of actions from (respectively) the right parity points, and left parity points. We have

$$f([\gamma])_R \simeq Hom(*^+, *) \text{ and } {}_L f([\gamma]) \simeq Hom(*^-, *)$$

which is an equality up to equivariant renormalization with respect to a nullparity element. That is to say, for all $*$ in the equivalence class $\gamma_0^0/\sim$, this is a strong equality.

Proposition 1.2. *Let F be a Reeb chord, and $Mo(F) = Mo_{x,f,\varepsilon,\delta}(F)$ be its Morse group. Select some $v \in k$ for an Archimedean ground field $\mathbb{K}$. Then*

$$Hom(v, v') \simeq Hom(Hom(f([\gamma])_R, *), Hom(*,_L f([\gamma]))$$

and this can be shown by setting

Proof.

$$Hom(f([\gamma])_R, v) = kx \text{ s.t. } (k, x) < v$$

$$Hom({}_L f([\gamma]), v) = \frac{x}{k} \text{ s.t. } x > v \,\&\, k < v$$

By triangle inequality, $kx \geq k + x$, so $\frac{x}{k} \leq x + k$. This follows automatically if $\mathbb{K}$ is a local ring. In this case, $\mathbb{K}$ is also a polarized abelian variety. This means the point $kx = \frac{x}{k}$ where $x = 1$ is the point γ_0^0.

By setting $* = v = \gamma_0^0$, we have proven this result, since $v \in \gamma_0^0/\sim$ by a previous declaration. □

Keep in mind, the left-directed and right-directed actions are actually actions of the Morse group we are working with. So, we could equivalently say that $Mo(F) \times \mathbb{K} =_L f([\gamma])$ and $\mathbb{K} \times Mo(F) = f([\gamma])_R$.

Proposition 1.3. *The Morse group of a chiral link contains a polarizable Abelian variety.*

This basically follows from the fact that for every $|v| \in Pic^0(F)$, there exists a reduced normal cone, $\mathfrak{Cone}(v) \supset [v \pm \varepsilon]$ such that

$$\begin{tikzcd} v \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook, "i"'] & v \pm \varepsilon \arrow[dl, hook, "i'"] \\ \mathfrak{Cone}(v) & \end{tikzcd}$$

is a commutative diagram of homeomorphic inclusions. This is proven by letting $\mathfrak{T} \supset \mathfrak{Cone}(v)$ be a one-object overcategory of $\mathfrak{Cone}(v)$, and writing

$$\begin{tikzcd} \mathfrak{Cone}(v) \arrow[r, "E_1"'] \arrow[d, "E_2"] & End(\mathfrak{Cone}(v)) \arrow[dl, equal, "Exp"] \\ End(\mathfrak{Cone}(v)) & \end{tikzcd}$$

By prolonging the diagram, we have $\mathfrak{Cone}(v) \xrightarrow{E_n} End(\mathfrak{Cone}(v)) \xrightarrow{E_{n+2}} \mathfrak{Cone}(v)$, and our reduced normal cone becomes a distinguished triangle via after cancelling the lift $Exp : End(\mathfrak{Cone}(v)) = End(\mathfrak{Cone}(v))$ via the map $Exp \to \emptyset$, which cancels as $Id_\emptyset \in Pic^0(F)$, and so it becomes redundant. We also set $Exp = [v^x] \to \mathfrak{Cone}(v)\ \forall x$.

2 Main Results

The main results here allow us to construct spacetimes out of Lagrangian knots.

Theorem 2.1 (Main theorem)**.** *For every link, ℓ, there is an associated manifold $\mathcal{W}$ such that*

$$\mathcal{W} = \bigcup_\beta \Lambda_0^\beta$$

where Λ_0 is a Reeb chord and β is the highest dimension of any link.

Proof. This is essentially a form of the gluing condition for D-branes. Given a link ℓ, let $\Sigma\ell$ denote its suspension.

Let

$$\bigcup_\beta \Lambda_0^\beta = \Sigma \bigcup_\beta \Lambda_0^{\beta-1}$$

Then, we can take the cross product

$$\Lambda_0^{\beta-1} \times S_t$$

$$= fib^{n-1} \times \Lambda_0$$

with the Tate circle to construct arbitrary contactomorphisms $\zeta \to \eta$ between contact forms on Reeb chords. □

Further,

Theorem 2.2 (Construction)**.** *Every Lagrangian knot on an orientable manifold consists of chiral Reeb chords.*

Proof. Let $\mathfrak{P} \in \mathfrak{Posets}$ be a pre-ordered set under the preorder $\wp$. Then, one has

$$\wp(\Lambda_0) \ni v$$

for some v with nullparity. We have

$$v' = \begin{cases} v^+ \text{ if } \ v' > v \\ v^- \text{ if } \ v' < v \\ v \ \text{ if } f(v') = f(v) \end{cases}$$

so that $f(v') \in Hom(v, v')$ and $f^{-1}(v) \simeq f(v')$. This shows that we have a unique identity for each $v \in \wp(\Lambda_0)$. We then lift v to $\mathfrak{P}$ via the map $i' = i \circ \wp$.

$$\begin{tikzcd} \wp(\Lambda_0) \arrow[r, hook, "i"] & \mathfrak{P} \\ v \arrow[u, "\wp"] \arrow[ur, dashed, "i'"'] & \end{tikzcd}$$

where $v \in Pic^0(F)$ for a Reeb chord $F \simeq \wp(\Lambda_0)$. We then gain the pair of functors

$$F \underset{m}{\overset{n}{\leftrightarrows}} \wp(\Lambda_0)$$

and by applying Reidemiester moves via

$$\mathcal{R}_{ijk}(F)$$

we have shown that we can construct knots from Reeb chords. ◻

Proposition 2.1. *$\mathcal{W}$ is an Eilenberg-Maclane space.*

Proof. Since $min(\beta) = 0$, we have that a single contact form α is the smallest possible link in a given manifold, which specifically has a point as its only good cover. We have $\mathcal{W} = K(\beta, \ell)$ so that the βth homology group of the space yields the desired link, which is unique. ◻

2.1 Constructing Links

Let $T_x^*(X) \in \mathcal{W}$ be a cotangent space in a Weinstein manifold, with characteristic vectors.

Definition 2.1. *A θ-link is a link $\ell_\theta : A \to A^\vee$ between a topological space and its formal dual.*

Proposition 2.2. *θ-links are contactomorphisms.*

Proof. Since ℓ_θ is a geometric object to begin with, we have $|\ell_\theta| = \theta$, where θ is a contactomorphism whose number of points with contact structure is:

$$\#_C(A) = \#(C(\bar{\ell}, A \in A \cup A^\vee) \geq 1$$

If A is self-dual, then $\#_C(A)$ is equal to the one object set $\{\xi\}$, which contains only a two-sided ideal. □

Definition 2.2. *The class $[\ell_\theta]$ is henceforth known as the formal class of contact-dual vector spaces, and $[\theta]$ the class of dual contactomorphisms.*

We fix a parameter, called the "naive Legendrian" of a space S. The naive Legendrian is given by

$$\mathscr{L}_{Naive}(S) : \#\ell \cup \chi$$

where χ is the crossing number of links in the ambient space. Technically speaking, this number is given by

$$\int_{\mathbb{K}} \Lambda_0 \cup C(\ell, \ell')$$

where each positively polarized action on each Reeb chord is given δ weight, and each negatively polarized action is given $\delta - 2\varepsilon$ weight.

Example 2.1. *Let D^1 be the one-dimensional disc embedded in an ambient space $\mathcal{A}$. Let there be two links: one from $\partial D^1 \to \mathcal{A} \setminus D^1$, and the second from $\partial^- D^1$ to $\partial^+ D^1$.*

The first link is a 2-ε-small chain, while the second has a length of $\int \frac{(dy)^2}{\varepsilon+\delta}$ where y is the height of the link's right parity elements with respect to the hieght of v where y is set to v_*. We have

$$\#_C(S) = \frac{len(\ell)}{\delta}$$

which is equal to 2 for the first link, assuming $\varepsilon = \delta$, and is $\int \frac{(dy)^2}{4\delta^2}$ for the second, which is equal to

$$\int dy = \frac{d((\partial^- D^1), (\partial^+ D^1))}{dt}$$

Since we know D^1 (as a background) has a single link and region, we only need to count the number of crossings between $int(D^1)$ and its boundary link. This is zero when travelling along the fiber from $\partial^- D^1$ to $\partial^+ D^1$, but one when exiting D^1 into $\mathcal{A}$. So, the naive Legendrian is:

$$\mathscr{L}_{Naive}(S \supset D^1) = (1 + a) \cup 1 = \{3, 1\}$$

because we have added the two links a to the single Reeb chord of D^1.

3 References

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