Solution of the Klein-Gordon equation in a moving well and quantum state of complex momentum

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Abstract

One-dimensional infinite well is an important model in quantum mechanics, and the solutions of Schrodinger equation and Klein-Gordon equation in this case have been studied extensively. In this paper, we discuss the solution of the Klein-Gordon equation in a moving one-dimensional infinite well, we find that the momentum of the particle should be complex numbers in a particular case.

Keywords: Klein-Gordon equation, complex momentum, one-dimensional infinite well

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1 Introduction

One-dimensional infinite potential well is a common model in quantum mechanics. The solution of Schrodinger equation in one-dimensional infinite potential well is discussed in many quantum mechanics textbooks [1-2], and the solution of Klein-Gordon equation in one-dimensional infinite potential well is discussed in some study. [3]Usually, we discuss the case which the well wall is stationary, but the solution of the Klein-Gordon equation is discussed by Koehn.M [4] when one side of the well wall is moving, that is, when the width of the well changes. It will be widely used in the study of Fermi acceleration mechanism.The research in reference [5] further shows that if we decrease the width of the potential well , the particles will follow the change of the potential wall quickly and return to the potential well. It is rarely discussed that the two sides of the well wall move together at the same speed, such as the two sides of the well wall move to the right at the speed v ,According to the conclusion of reference [5], we can guess that the particles near the left potential well wall will follow the changes of the potential well wall quickly and return to the potential well wall, while the right potential well wall may follow the adiabatic evolution. This can also be seen as the whole well moving without changing the width of the well. A simple idea is to change the frame of reference, but a special cases will be discussed in this article, and we will get some interesting results such as particles with complex momentum .

2 The solution and discussion of Klein-Gordon equation

We can discuss those questions in the inertial frame K first, consider the potential well as

$$V(x)=\begin{cases}0 & vt<x<a+vt\\ +\infty & x\le vt, x\ge a+vt\end{cases} \tag{1}$$

Where v is the velocity of the coordinate change of the potential well wall in the inertial frame K, t is time, and a is a constant.
Considering only one-dimensional waves moving along the x-direction, the Klein Gordon equation has the following form:

$$\hbar^2\frac{\partial^2\varphi(x,t)}{\partial t^2}-\hbar^2c^2\frac{\partial^2\varphi(x,t)}{\partial x^2}+m^2c^4\varphi(x,t)=0 \tag{2}$$

Where $\hbar$ is the reduced Planck constant, m is the particle mass, and c is the speed of light. The wave function of a particle that satisfies the Klein Gordon equation should be 0 at the potential well wall [4]. Therefore,So the boundary conditions are: $\varphi(vt,t)=\varphi(a+vt,t)=0$.

Use the following variables for substitution

$$\eta=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}},\xi=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3}$$

Where $b=\frac{a}{\sqrt{1-\frac{v^2}{c^2}}}$, equation (2) becomes

$$\hbar^2\frac{\partial^2\varphi(\eta,\xi)}{\partial\xi^2}-\hbar^2c^2\frac{\partial^2\varphi(\eta,\xi)}{\partial\eta^2}+m^2c^4\varphi(\eta,\xi)=0 \tag{4}$$

The boundary conditions become

$$\varphi(0,\xi)=\varphi(b,\xi)=0 \tag{5}$$

The general solution to equation (4) can be simply found

$$\varphi(\eta,\xi)=\sum_n\sin(\frac{n\pi}{b}\eta)(c_{1n}e^{-\frac{iE_n\xi}{\hbar}}+c_{2n}e^{\frac{iE_n\xi}{\hbar}}) \tag{6}$$

Where c_{1n} and c_{2n} are constants, $E_n=\sqrt{\frac{\hbar^2n^2\pi^2(c^2-v^2)}{a^2}+m^2c^4}$, $n\in\mathbb{Z}$.

Consider another inertial frame K' moving at velocity v relative to the reference frame K. If

$v < c$, then obviously E_n is the energy level of the particle in the inertial frame K' , and the expression in the sum sign of equation (6) is the eigenstate of the particle which the energy eigenvalue is E_n in the inertial frame K' .

In equation (1), v is the velocity of the coordinate change of the potential well wall in the inertial frame K , this velocity can be the velocity of the potential well wall movement, But this speed can also be the velocity which the potential well wall establishes and collapses, The velocity which the potential well wall establishes and collapses is not limited by the speed of light, Therefore, it is possible for $v > c$,In this case, it is obvious that formula (3) cannot be seen as a change of the reference frame, And this condition indicates that there can be particles with complex momentum in the potential well.

For simplicity, taking $c_{2n} = 0$ in (6) as an example, choose the particle state as

$$\varphi(x,t) = A\sin[\frac{n\pi}{a}(x - vt)]e^{-\frac{i}{\hbar}\sqrt{\frac{\hbar^2 n^2 \pi^2 (c^2 - v^2)}{a^2} + m^2 c^4}\frac{(t - \frac{vx}{c^2})}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad (7)$$

In the formula, A is the normalization constant.

If $c^2 < v^2 < c^2 + \frac{m^2 c^2 a^2}{\hbar^2 n^2 \pi^2}$, E_n will be a real number, We discuss the mathematical expectation of particle momentum at time $t = 0$ in this case.

$$< \hat{p} >= -i\hbar\int_0^a \varphi^*(x,0)\frac{\partial}{\partial x}\varphi(x,0)dx = (\frac{v}{\sqrt{c^2 - v^2}})E_n - i\hbar\,\frac{n^2\pi^2 A^* A}{4a^2}(\frac{1 + e^{\frac{2iEva}{\hbar\sqrt{c^2 - v^2}}}}{\frac{n^2\pi^2}{a^2} + \frac{E_n{}^2 v^2}{v^2 - c^2}}) \qquad (8)$$

It can be seen that both terms in the formula are pure imaginary, meaning that in this case, the average momentum of particles is imaginary, it can be seen that in this situation, there are particles with complex momentum in the potential well. If we consider the expression for the momentum of particles with velocity u in special relativity $p = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}}$, It can be observed that the velocity of the particles with imaginary momentum will be faster than the speed of light,in other words, it is possible to produce particles that exceed the speed of light in this way.

Note that the limit of $\varphi(x,t)$ does not exist when $v = c$, so further in-depth research is needed on the state of particles when $v = c$.

3 conclusion

This article discusses the solution of the Klein Gordon equation in a moving one-dimensional infinite depth potential well. When the velocity of the potential well $v > c$, it can be simply understood by the change of the reference frame, but the velocity which the potential well wall establishes and collapses is not limited by the speed of light, so the momentum of particles could be complex numbers.It may mean that there are particles in the potential well with velocities greater than the speed of light under these conditions. I believe that this study can encourage more scholars to explore this field more deeply.

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运动势阱中 Klein-Gordon 方程的解与复数动量态

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摘 要

一维无限深势阱是一类被广泛讨论的模型，不论薛定谔方程还是 Klein-Gordon 方程在这种情况下的解都被大量研究，本文讨论了一类两侧势阱壁会一同运动的一维无限深势阱中 Klein-Gordon 方程的解，并证明了在一类特殊情况下，这类势阱中的粒子可以具有复数的动量或者说可以超过光速这一有趣性质。

关键词： Klein-Gordon 方程；复数动量；一维无限深势阱

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1 引 言

一维无限深势阱是量子力学中一种常见的模型，众多教材[1-2]都对薛定谔方程在一维无限深势阱中的解进行了讨论，文献[3]对一维无限深势阱中 Klein-Gordon 方程的解进行了讨论，并给出了相应的能级。通常我们讨论的是势阱壁静止不动的情况，不过文献[4]讨论了一侧势阱壁运动时，也就是势阱宽度发生变化时 Klein-Gordon 方程的解，这可以广泛应用于费米加速机制的研究。文献[5]的研究进一步表明，势阱宽度变小时，粒子会迅速跟随势阱壁变化而回到允许的势阱内。两侧势阱壁以相同速度一同运动的情况很少有人讨论，如两侧势阱壁均以速度 v 向右运动，根据文献[5]的结论我们可以猜测左侧势阱壁附近会出现粒子迅速跟随势阱壁变化而回到势阱内而右侧势阱壁附近可能会接近绝热演化，这种情况也可以看做整个势阱在运动而势阱宽度不变，一种简单的思路是通过参考系变换求解，不过本文中会讨论一类特殊情况，并得到诸如复数动量粒子等有趣结果。

2 对 Klein-Gordon 方程的求解与讨论

讨论在惯性系 K 下形如下式的一维无限深势阱

$$V(x)=\begin{cases}0 & vt<x<a+vt\\ \\ +\infty & x\le vt, x\ge a+vt\end{cases} \tag{1}$$

其中 v 为惯性系 K 下势阱壁坐标变化的速度，t 为时间，a 为常数。
仅考虑沿着 x 方向一维运动的波，Klein-Gordon 方程具有如下形式：

$$\hbar^2\frac{\partial^2\varphi(x,t)}{\partial t^2}-\hbar^2c^2\frac{\partial^2\varphi(x,t)}{\partial x^2}+m^2c^4\varphi(x,t)=0 \tag{2}$$

式中 $\hbar$ 为约化普朗克常量，m 为粒子质量，c 为光速，满足 Klein-Gordon 方程的粒子的波函数在势阱壁处应为 0[4]，因此边界条件为 $\varphi(vt,t)=\varphi(a+vt,t)=0$.

做出如下换元

$$\eta=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}},\xi=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3}$$

并令 $b=\frac{a}{\sqrt{1-\frac{v^2}{c^2}}}$，方程（2）变为

$$\hbar^2\frac{\partial^2\varphi(\eta,\xi)}{\partial\xi^2}-\hbar^2c^2\frac{\partial^2\varphi(\eta,\xi)}{\partial\eta^2}+m^2c^4\varphi(\eta,\xi)=0 \tag{4}$$

边界条件变为

$$\varphi(0,\xi)=\varphi(b,\xi)=0 \tag{5}$$

可以简单求出其通解为

$$\varphi(\eta,\xi)=\sum_n\sin(\frac{n\pi}{b}\eta)(c_{1n}e^{-\frac{iE_n\xi}{\hbar}}+c_{2n}e^{\frac{iE_n\xi}{\hbar}}) \tag{6}$$

式中 c_{1n} 和 c_{2n} 为常数，$E_n=\sqrt{\frac{\hbar^2n^2\pi^2(c^2-v^2)}{a^2}+m^2c^4}$，$n\in\mathbb{Z}$。

考虑相对惯性系 K 以速度 v 运动的另一惯性系 K'，若 $v<c$ 则显然 E_n 就是惯性系 K' 下粒子的能级，而式（6）求和号内的表达式则为惯性系 K' 下粒子能量本征值为 E_n 的本征态。

式（1）中 v 为势阱壁坐标变化的速度，但这一速度可以是势阱壁（墙）运动的速度，也可以是势阱壁（墙）建立与倒塌的速度，而势阱壁建立与倒塌的速度并不会受到光速的限制，因此是 $v>c$ 允许的，这种情况下显然不能再将换元（3）看做是参考系变换，而这类情况表示势阱中可以存在动量为复数的粒子。

简单起见以（6）中 $c_{2n}=0$ 为例，选择粒子态为

$$\varphi(x,t)=A\sin[\frac{n\pi}{a}(x-vt)]e^{-\frac{i}{\hbar}\sqrt{\frac{\hbar^2n^2\pi^2(c^2-v^2)}{a^2}+m^2c^4}\frac{(t-\frac{vx}{c^2})}{\sqrt{1-\frac{v^2}{c^2}}}} \tag{7}$$

式中 A 为归一化常数。

若 v 满足 $c^2<v^2<c^2+\frac{m^2c^2a^2}{\hbar^2n^2\pi^2}$，则 E_n 为实数，讨论这一情况下 $t=0$ 时刻粒子动量的期望。

$$< \hat{p} > = -i\hbar\int_0^a \varphi^*(x,0)\frac{\partial}{\partial x}\varphi(x,0)dx = (\frac{v}{\sqrt{c^2-v^2}})E_n - i\hbar\frac{n^2\pi^2 A^* A}{4a^2}(\frac{1+e^{\frac{2iEva}{\hbar\sqrt{c^2-v^2}}}}{\frac{n^2\pi^2}{a^2}+\frac{E_n^2 v^2}{v^2-c^2}}) \quad (8)$$

可以看出两项均为纯虚数，即这种情况下粒子动量的平均值为虚数，可见这种情况下势阱内存在动量为复数的粒子。如果考虑狭义相对论中速度为u的粒子动量的表达式 $p=\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}$，可以发现动量为虚数意味着粒子的速度大于光速，换言之通过这种方式制备速度大于光速的粒子是可能的。

注意到$v=c$时的$\varphi(x,t)$极限不存在，因此势阱壁建立和倒塌速度为光速的势阱中粒子的状态还需要进一步深入研究。

5 结　论

本文讨论了运动的一维无限深势阱下 Klein-Gordon 方程的解，当势阱运动速度$v<c$时可以简单地通过参考系变换来理解，而势阱建立和倒塌的速度不会受到光速的影响，因此当势阱运动速度$v>c$时，势阱内粒子会出现复数动量的有趣性质，这或许意味着此时势阱内存在着速度大于光速的粒子，相信这项研究能促使更多学者们展开更深入的探索。

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