A BOUND FOR THE ISOTROPIC CONSTANT

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ABSTRACT. We obtain a dimension independent bound for the isotropic constants for the convex bodies.

CONTENTS

1. INTRODUCTION

The isotropic conjecture or the Bourgain's slicing problem asks the existence of a following universal constant c.

Theorem 1.1. *There exists an affine hyperplane H and an universal constant c such that*

$$m_{n-1}(H \cap K) > c,$$

for convex bodies K of unit volume.

A classic reference for these kind of questions is [9]. More recently the claim is already proved up to a polylog with very modern methods [6].Those methods very introduced in the groundbreaking work by Chen [5]. The entries of the covariance matrix of a convex body K are defined as

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|} - \frac{\int_K x_i}{|K|}\frac{\int_K x_j}{|K|}.$$

We define the isotropic constant of any convex body K in scaling invariant way using

$$L_K^{2n} := \frac{\mathrm{Det}(CovK)}{|K|^2}.$$

The isotropic position is a position, when the covariance matrix is diagonal and all the diagonal entries are the same. Moreover, it is assumed that the volume is unit. This kind of position exists [9]. An another position that always exists is the John's position. It is the position of a convex body, where the minimal circumscribed ellipsoid is the unit ball. We prove the Bourgain's slicing conjecture by proving an universal upper bound for the isotropic constant.

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2. Previously known results

For any measurable set A we let $|A|$ be the n-dimensional Lebesque measure.The inner volume ratio for a convex body K is defined as

$$ivr(K) := \min_T (|K|/|T(B_n)|)^{1/n},$$

where T is an affine map, B_n the standard unit ball and $T(B_n) \subset K$. The outer volume ratio for a convex body K is defined as

$$ovr(K) := \min_T (|T(B_n)|/|K|)^{1/n}, \tag{2.1}$$

where T is an affine map, B_n the standard unit ball and $K \subset T(B_n)$. Ball [2] and Barthe [4] proved using the Braschamb-lieb [8] and reversed Braschamb-Lieb [4] inequalities, respectively, that in the non-symmetric case $ivr(K)$ and $ovr(K)$ are maximized when the convex body K is the standard simplex S_n. Moreover, in the symmetric case $ivr(K)$ is maximized when K is the cube C_n and $ovr(K)$ is maximized when K is the crosspolytope CP_n. The extended Khinchine inequality says that for any convex bodies

$$(\frac{1}{|K|}\int_K |x_i|^2 dx)^{1/2} \leq C\frac{1}{|K|}\int_K |x_i| dx. \tag{2.2}$$

A proof can be found in [7].

3. The proof

First we show a key fact.

Theorem 3.1. *Let K be a convex body of unit diameter in John's position. Then*

$$|K|^{1/n} \geq c'(n!)^{1/n} > cn^{-1}.$$

Proof. For K' in John's position we have that $K' \subset B(0,1)$. So for the diameter d we have that

$$1 \leq d \leq 2.$$

Moreover, via (2.1) we have that

$$\frac{|B(0,1)|}{|S_n|} \geq \frac{|B(0,1)|}{|K'|}.$$

So

$$\frac{1}{|S_n|} \geq \frac{1}{|K'|}.$$

Thus,

$$|S_n| \leq |K'|.$$

Now, the diameter of K was the unit. So we have

$$|S_n| \leq 2^n|K|.$$

Thus,

$$|S_n|^{1/n} \leq 2|K|^{1/n}. \tag{3.1}$$

Now, we just need to calculate the volume of the standard simplex S_n in John's position. We have that

$$|S|^{1/n} > Cn^{-1}, \tag{3.2}$$

where C is an universal constant. So combining (3.1) and (3.2) gives us the claim.

□

We will also need the lemma showing the essential monotonicity of the means.

Lemma 3.2. *Let K be a convex body. If $||x||_2 \leq a$ then*

$$\int_K \frac{\sum_{i=1}^n |x_i| dx}{n|K|} \leq C \int_{B(0,a)} \frac{|x_i| dx}{|B(0,a)|}.$$

Proof. We have

$$\int_K \frac{\sum_{i=1}^n |x_i| dx}{n|K|} \leq \int_K \frac{\sqrt{n}||x||_2 dx}{n|K|} \leq \frac{a}{\sqrt{n}}.$$

On the other hand we have

$$\int_{B(0,a)} \frac{|x_i|_2 dx}{|B(0,a)|} = \int_{B(0,a)} \frac{||x||_2 dx}{\sqrt{n}|B(0,a)|} = \frac{an}{(n+2)\sqrt{n}}.$$

□

The following theorem is the key theorem.

Theorem 3.3. *Let K be a convex body in a scaled John's position such that*

$$\int_K ||x||_1 dx = |K|. \tag{3.3}$$

Then it holds in a scaled John's position that

$$\int_K \frac{\frac{1}{n}\sum_{i=1}^n |x_i| dx}{|K|^{1+1/n}} \leq C. \tag{3.4}$$

Proof. We notice that the diameter of K' must be greater than a constant. Assuming that $||x||_2 \leq a$ we have from the essential monotonicity of the means (3.2), Jensen and from the parallelogram law that

$$\frac{1}{n^2} = (\frac{\int_{K'} \sum_{i=1} |x_i| dx}{n|K|})^2 \leq \frac{C\int_{B(0,a)} |x_i|^2 dx}{n|B(0,a)|} = \frac{Ca^2}{(n+2)n}.$$

Then little algebra gives us

$$a > c.$$

Remark 3.4. It's clear that the position (3.3) exists because the average can be the unit.

So we have from theorem 3.1 that

$$|K|^{1/n} \geq c' n^{-1}. \tag{3.5}$$

Thus, we have

$$\begin{aligned} &\int_K \frac{\frac{1}{n}\sum_{i=1}^n |x_i|}{|K|^{1+1/n}} dx \\ &\leq cn \int_K \frac{\frac{1}{n}\sum_{i=1}^n |x_i|}{|K|} dx \\ &= c, \end{aligned}$$

where we used the inequality (3.5) and the asumption (3.3). □

We can assume that the covariance matrix is diagonal, because it is real and symmetric. So it can be diagonalized by an orthogonal matrix. Because K is centralized, we have

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|}.$$

Moreover, we assume K is in a John's position. We have

$$\begin{aligned}
L_K &= (\prod_{i=1}^{n} \int_K \frac{x_i x_i dx}{|K|^{n+2}})^{1/(2n)} \\
&= \prod_{i=1}^{n} (\int_K \frac{|x_i|^2 dx}{|K|^{n+2}})^{1/(2n)} \\
&\leq \prod_{i=1}^{n} (C \int_K \frac{|x_i| dx}{|K|^{n+1}})^{1/n} \\
&\leq \frac{C}{n} \sum_{i=1}^{n} \int_K \frac{|x_i| dx}{|K|^{n+1}} \\
&\leq C,
\end{aligned}$$

where we firs used the extended Klinchine's inequality (2.2), then we used the GM-AM inequality and lastly we used the theorem 3.3, respectively. It's clear that the inequality (3.4) is scaling invariant. This ends the proof of the theorem 1.1.

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