Advanced continued fraction approximations and bounds for the Gamma Function and the generalized Wallis ratio

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ABSTRACT

In this paper, we provide a main method for construction of continued fraction based on a given power series using Euler connection. Then we establish very innovative results in continued fraction approximation for the Gamma function as applications of our method. Also new continued fraction bounds for the Gamma function are obtained. Finally new continued fraction approximations and bounds for Wallis ratio are established.

Keywords: Euler connection, Continued fraction, Gamma function, Bernoulli number, Wallis ratio

1. Introduction

The classical Euler Gamma function Γ defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt, \qquad x>0 \tag{1.1}$$

was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non-integer values.

Today the Stirling's formula

$$n!\approx\sqrt{2\pi\, n}\left(\frac{n}{e}\right)^n \tag{1.2}$$

is one of the most well-known formulas for approximation of the factorial function by being widely applied in number theory, combinatorics, statistical physics, probability theory and other branches of science.

Up to now, many researchers made great efforts in the area of establishing more accurate approximations for the factorial function and more precise inequalities, and had lots of inspiring results.

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The Stirling's series for the Gamma function is presented (see [1]) by

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x} \exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}}\right), \qquad x \to \infty \tag{1.3}$$

where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) denotes the Bernoulli numbers defined by the generating formula

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi,$$

then the first few terms of B_n are as follows:

$$B_{2n+1} = 0, n \geq 1,$$
$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, \cdots$$

It was proved in [2] by Alzer (see also [18]) that for given $n \in \mathbb{N}$, the function

$$F_n = \ln\Gamma(x+1) - \left(x + \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) - \sum_{i=1}^{n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

is strictly completely monotonic on $(0, \infty)$ of n is even, and so is $-F_n$ if n is odd.

It thus follows that double inequality

$$\exp\left(\sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right) < \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}} < \exp\left(\sum_{i=1}^{2n-1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right) \tag{1.4}$$

holds for all $x > 0$.

The Burnside's formula [4]

$$n! \approx \sqrt{2\pi}\left(\frac{n + \frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \tag{1.5}$$

is more precise than (1.2).

An asymptotic expansion of which for the Gamma function

$$\Gamma\left(x+\frac{1}{2}\right)\approx\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\exp\left(-\sum_{i=1}^{\infty}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}\right) \tag{1.6}$$

as $x\to\infty$ was given in [25]. In [39], Yang showed that the function

$$G_n=\begin{cases}\ln\dfrac{\Gamma(x+1/2)}{\sqrt{2\pi}(x/e)^{x}}+\displaystyle\sum_{i=1}^{n}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}} & \text{if } n\geq 1,\\ \ln\dfrac{\Gamma(x+1/2)}{\sqrt{2\pi}(x/e)^{x}} & \text{if } n=0\end{cases} \tag{1.7}$$

is completely monotonic on $(0,\infty)$ of n is odd, and so is $-G_n$ if n is even. These yield that for n ∈ ℕ, the double inequality

$$\exp\left(-\sum_{i=1}^{2n-1}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}\right)<\frac{\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}}<\exp\left(-\sum_{i=1}^{2n}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}\right) \tag{1.8}$$

holds for all $x>0$.

More asymptotic expansion developed by some closed approximation formulas for the Gamma function can be found in [3], [6], [7], [8], [9], [10], [11], [12], [15], [16], [21], [22], [24], [29], [30], [32], [33], [36], [41],[42] and the references cited therein.

Then two approximation formulas for the Gamma function in terms of hyperbolic functions have attracted the attention of scholars, the first one of which, Windschitl's approximation formula [47], is given by

$$\Gamma(x+1)\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\left(x\sinh\frac{1}{x}\right)^{\frac{x}{2}} \quad \text{as } x\to\infty, \tag{1.9}$$

and the second one is the Smith's approximation formula for the Gamma function

$$\Gamma\left(x+\frac{1}{2}\right)\approx\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\left(2x\tanh\frac{1}{2x}\right)^{\frac{x}{2}} \quad \text{as } x\to\infty, \tag{1.10}$$

which was introduced in [35] by Smith. These two formulas are based on the Stirling's formula and the Burnside's formula respectively.

In recent papers [43] and [44], Yang and Tian developed the Windschitl's approximation formula (1.9) to an asymptotic expansion as

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{\frac{x}{2}} \exp\left(\sum_{i=3}^{\infty} \frac{2i(2i-2)! - 2^{2i-1}}{2i(2i)!} \frac{B_{2i}}{x^{2i-1}}\right), \tag{1.11}$$

as $x \to \infty$, and the Smith's approximation formula (1.10) to an asymptotic expansion as

$$\Gamma\left(x+\frac{1}{2}\right) \approx \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x \tanh\frac{1}{2x}\right)^{\frac{x}{2}} \exp\left(-\sum_{i=3}^{\infty} \frac{(2i)! - (2i-1)2^{2i-1}}{2i(2i-1)(2i)!}(1-2^{1-2i})\frac{B_{2i}}{x^{2i-1}}\right), \tag{1.12}$$

as $x \to \infty$.

Especially the Wallis ratio defined by

$$W_n = \frac{(2n-1)!!}{(2n)!!} \frac{\Gamma(n+1/2)}{\Gamma(n+1)\Gamma(1/2)}$$

for n ∈ ℕ (see [17]) also has attracted the attention of many researchers(see [13, 19, 20, 26] and references therein).

Some properties involving the generalized Wallis ratio $\Gamma(x+1/2)/\Gamma(x+1)$ for $x > -1/2$, as a ratio of two Gamma functions, can be found in [5], [31], [37], [38], [40], [45], [46].

In particular, by (1.3) and (1.6), we immediately get that

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x} \exp\left(\sum_{i=1}^{\infty} \frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}}\right) \quad \text{as} \quad x \to \infty, \tag{1.13}$$

and for n ∈ ℕ, the double inequality

$$\sqrt{x} \exp\left(\sum_{i=1}^{2n} \frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}}\right) < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x} \exp\left(\sum_{i=1}^{2n-1} \frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}}\right) \tag{1.14}$$

holds for all $x > 0$ (see [14], [34], [39]).

In [14] Chen and Paris showed that

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x \quad \text{as} \quad x \to \infty, \tag{1.15}$$

and in a recent paper [44], Yang and Tian developed this formula to an asymptotic expansion as

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x}\left(\cosh\frac{1}{2x}\right)^{x} \exp\left(\sum_{i=3}^{\infty}\frac{(2i)!-(2i-1)2^{2i-1}}{i(2i-1)(2i)!}(1-2^{-2i})\frac{B_{2i}}{x^{2i-1}}\right) \tag{1.16}$$

as $x \to \infty$.

In our study, we focus the continued fraction approximations.

Recently, some authors have focused on continued fractions in order to obtain new asymptotic formulas.

For example, on the one hand, Mortici [27] found Stieltjes' continued fraction

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x} exp\left(\cfrac{a_0}{x+\cfrac{a_1}{x+\cfrac{a_2}{x+\ddots}}}\right), \tag{1.17}$$

where

$$a_0 = \frac{1}{12}, \quad a_1 = \frac{1}{30}, \quad a_2 = \frac{53}{210}, \quad \cdots.$$

Also Mortici [28] provided a new continued fraction approximation starting from the Nemes' formula (1.7) as follows:

$$\Gamma(x+1) \approx \sqrt{2\pi x}e^{-x}\left(x+\cfrac{1}{12x-\cfrac{1}{10x+\cfrac{a}{x+\cfrac{b}{x+\cfrac{c}{x+\cfrac{d}{x+\ddots}}}}}}\right)^{x}, \tag{1.18}$$

where

$$a=-\frac{2369}{252}, b=\frac{2117009}{1193976}, c=\frac{3930321915\ 11}{1324011300\ 744}, d=\frac{3326589616\ 4277124002\ 451}{1427802410\ 4089641878\ 840}\cdots.$$

On the other hand, Lu [23] provided a new continued fraction approximation based on the Burnside's formula (1.5) as follows;

$$n!\approx\sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\left(1+\cfrac{a_1}{n^2+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\ddots}}}\right)^{\frac{n-\frac{1}{2}}{k}}, \tag{1.19}$$

where

$$a_1=-\frac{k}{24},\quad a_2=\frac{k}{48}-\frac{23}{120},\quad a_3=\frac{14}{5k-46},\quad \cdots.$$

Also Lu [48] found two asymptotic formulas

$$\Gamma(x+1)\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\left(1+\cfrac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}+\frac{1}{x^2}\cfrac{a_1}{x+\cfrac{a_2}{x+\cfrac{a_3}{x+\ddots}}}}\right)^{x^2+\frac{53}{210}}, \tag{1.20}$$

where

$$a_1=\frac{2117}{35280}, a_2=\frac{188098}{116435}, a_3=\frac{1681526854\ 38}{1993008347\ 33}, a_4=\frac{1513751180\ 9264254261\ 577}{4973625898\ 1948852651\ 52},\cdots,$$

and

$$\Gamma(x+1)\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}}\left(1+\frac{1}{x^6}\cfrac{b_1}{x+\cfrac{b_2}{x+\cfrac{b_3}{x+\ddots}}}\right), \tag{1.21}$$

where

$$b_1 = -\frac{2117}{5080320}, b_2 = \frac{1892069}{978054}, b_3 = \frac{4064269668\,5}{9288453088\,2}, b_4 = \frac{8499178650\,3793092231}{1599494758\,6610983120}, \cdots.$$

Until now many continued fraction approximations for the Gamma function were given, but it's very uncomfortable to determine the parameters of the continued fractions because of the limitation of method.

So we establish an effective method to construct continued fraction.

In this paper, using Euler connection, we provide a main method for construction of continued fraction based on a given power series and determine all parameters of the continued fractions simply. Then we establish several continued fraction approximations for the Gamma function as applications of our method. Also new continued fraction bounds for the Gamma function are obtained. Finally new continued fraction approximations and bounds for Wallis ratio are established.

2. A main method to construct continued fractions

In this section, we present a main method to construct continued fraction based on a given power series using Euler connection.

The Euler connection states the connection between series and continued fractions as follows;

Lemma 2.1. (The Euler connection [14]) Let $\{c_k\}$ be a sequence in $\mathbb{C} \setminus \{0\}$ and

$$f_n = \sum_{k=0}^{n} c_k, \quad \mathrm{n} \in \mathbb{N}_0. \tag{2.1}$$

Since $f_0 \neq \infty$, $f_n \neq f_{n-1}$, $\mathrm{n} \in \mathbb{N}$, there exists a continued fraction $b_0 + K(a_m / b_m)$ with n[th] approximant f_n for all *n*. This continued fraction is given by

$$c_0 + \frac{c_1}{1} \, \frac{-c_2/c_1}{+1+c_2/c_1} \, \cdots \, \frac{-c_m/c_{m-1}}{+1+c_m/c_{m-1}} \, \frac{}{+\cdots}. \tag{2.2}$$

The following theorem states our main method.

Theorem 2.1. For every $x \neq 0$,

$$\sum_{i=1}^{n} \frac{c_{2i}}{x^{2i-1}} = \mathop{K}_{i=1}^{n} \frac{a_i}{x + \frac{b_i}{x}} = \frac{a_1}{x + \mathop{K}\limits_{i=2}^{n} \frac{a_i}{x - \frac{a_i}{x}}}, \quad \mathrm{n} \in \mathbb{N}, \tag{2.3}$$

where $c_{2i} \neq 0$, $i = 1, 2, \cdots, n$ and

$$a_1 = c_2,\ b_1 = 0,$$
$$a_i = -\frac{c_{2i}}{c_{2(i-1)}},\ b_i = -a_i,\ \ i = 2, 3, \cdots, n.$$

Proof. Assume that

$$f_0(x) \neq \infty, \quad f_n(x) = \sum_{i=1}^{n} \frac{c_{2i}}{x^{2i-1}}, \quad \mathrm{n} \in \mathbb{N}, \quad x \neq 0, \tag{2.4}$$

where $c_{2i} \neq 0$, $i = 1, 2, \cdots, n$. Since

$$f_0(x) \neq \infty, \quad f_n(x) \neq f_{n-1}(x), \quad \mathrm{n} \in \mathbb{N},$$

the left-side of (2.3) is equal to $f_n(x)$ (n ∈ ℕ). Using Lemma 2.1,

$$f_n(x) = \sum_{i=1}^{n} \frac{c_{2i}}{x^{2i-1}} = \frac{\frac{c_2}{x}}{1} + \frac{-\frac{c_4}{c_2 x^2}}{1 + \frac{c_4}{c_2 x^2}} + \frac{-\frac{c_6}{c_4 x^2}}{1 + \frac{c_6}{c_4 x^2}} + \cdots + \frac{-\frac{c_{2i}}{c_{2(i-1)} x^2}}{1 + \frac{c_{2i}}{c_{2(i-1)} x^2}} + \cdots + \frac{-\frac{c_{2n}}{c_{2(n-1)} x^2}}{1 + \frac{c_{2n}}{c_{2(n-1)} x^2}}$$

$$= \frac{c_2}{x} + \frac{-\frac{c_4}{c_2 x}}{1 + \frac{c_4}{c_2 x^2}} + \frac{-\frac{c_6}{c_4 x^2}}{1 + \frac{c_6}{c_4 x^2}} + \cdots + \frac{-\frac{c_{2i}}{c_{2(i-1)} x^2}}{1 + \frac{c_{2i}}{c_{2(i-1)} x^2}} + \cdots + \frac{-\frac{c_{2n}}{c_{2(n-1)} x^2}}{1 + \frac{c_{2n}}{c_{2(n-1)} x^2}}$$

$$= \frac{c_2}{x} + \frac{-\frac{c_4}{c_2}}{x + \frac{c_4}{c_2 x}} + \frac{-\frac{c_6}{c_4 x}}{1 + \frac{c_6}{c_4 x^2}} + \cdots + \frac{-\frac{c_{2i}}{c_{2(i-1)} x^2}}{1 + \frac{c_{2i}}{c_{2(i-1)} x^2}} + \cdots + \frac{-\frac{c_{2n}}{c_{2(n-1)} x^2}}{1 + \frac{c_{2n}}{c_{2(n-1)} x^2}}$$

$$= \cdots \ \cdots \ \cdots$$

$$= \frac{c_2}{x} + \frac{-\frac{c_4}{c_2}}{x + \frac{c_4}{c_2 x}} + \frac{-\frac{c_6}{c_4}}{x + \frac{c_6}{c_4 x}} + \cdots + \frac{-\frac{c_{2i}}{c_{2(i-1)}}}{x + \frac{c_{2i}}{c_{2(i-1)} x}} + \cdots + \frac{-\frac{c_{2n}}{c_{2(n-1)}}}{x + \frac{c_{2n}}{c_{2(n-1)} x}}$$

$$= \frac{c_2}{x + \underset{i=2}{\overset{n}{K}} \frac{-\frac{c_{2i}}{c_{2(i-1)}}}{x + \frac{c_{2i}}{c_{2(i-1)} x}}} = \frac{c_2}{x + \frac{0}{x} + \underset{i=2}{\overset{n}{K}} \frac{-\frac{c_{2i}}{c_{2(i-1)}}}{x + \frac{c_{2i}}{c_{2(i-1)} x}}}. \tag{2.5}$$

The right-side of (2.3) is equal to

$$\mathop{K}_{i=1}^{n} \frac{a_i}{x+\frac{b_i}{x}} = \frac{a_1}{x+\frac{b_1}{x}+\mathop{K}_{i=2}^{n} \frac{a_i}{x+\frac{b_i}{x}}}, \qquad x \neq 0. \tag{2.6}$$

Thus,

$$a_1 = c_2,\ b_1 = 0,$$
$$a_i = -\frac{c_{2i}}{c_{2(i-1)}}, b_i = \frac{c_{2i}}{c_{2(i-1)}} = -a_i, \quad i = 2, 3, \cdots, n.$$

The proof of Theorem 2.1 is complete.

Remark 2.1. As you can see, Theorem 2.1 is simply proved by Euler connection. This theorem is very useful for construction of continued fraction approximations and comfortable to determine the parameters of the continued fractions.

3. Continued fraction approximations and bounds for the Gamma function

In this section, we establish continued fraction approximations and continued fraction bounds for the Gamma function as applications of our method.

Theorem 3.1. As $x \to \infty$, we have the continued fraction approximation of $\Gamma(x+1)$,

$$\begin{aligned} \Gamma(x+1) &\approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\mathop{K}_{i=1}^{\infty} \frac{a_i}{x+\frac{b_i}{x}}\right) \\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\cfrac{a_1}{x+\frac{b_1}{x}+\cfrac{a_2}{x+\frac{b_2}{x}+\cfrac{a_3}{x+\frac{b_3}{x}+\ddots}}}\right), \end{aligned} \tag{3.1}$$

where

$$a_1 = \frac{B_2}{2},\ b_1 = 0,$$
$$a_i = -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}}, b_i = -a_i, \quad i = 2, 3, \cdots.$$

Proof. Assume that

$$c_{2i} = \frac{B_{2i}}{2i(2i-1)}, \qquad i = 1, 2, 3, \cdots. \tag{3.2}$$

From (3.2) and Theorem 2.1, as $x \to \infty$,

$$\sum_{i=1}^{\infty} \frac{c_{2i}}{x^{2i-1}} = \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} = \mathop{K}_{i=1}^{\infty} \frac{a_i}{x + \frac{b_i}{x}} \Leftrightarrow \exp\left(\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \right) = \exp\left(\mathop{K}_{i=1}^{\infty} \frac{a_i}{x + \frac{b_i}{x}} \right), \tag{3.3}$$

where

$$a_1 = c_2 = \frac{B_2}{2},\ b_1 = 0,$$
$$a_i = -\frac{c_{2i}}{c_{2(i-1)}} = -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}}, b_i = \frac{c_{2i}}{c_{2(i-1)}} = -a_i,\ i = 2, 3, \cdots.$$

According to the Stirling's series,

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \right) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\mathop{K}_{i=1}^{\infty} \frac{a_i}{x + \frac{b_i}{x}} \right). \tag{3.4}$$

Thus, our new continued fraction approximation can be obtained.

Remark 3.1. As you can see, our new continued fraction approximation for the Gamma function is equal to the Stirling's series.
From Theorem 2.1, we have another expression of (3.4) as follows;

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{a_1}{x + \mathop{K}_{i=2}^{\infty} \frac{a_i}{x - \frac{a_i}{x}}} \right) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\cfrac{a_1}{x + \cfrac{a_2}{x - \frac{a_2}{x} + \cfrac{a_3}{x - \frac{a_3}{x} + \ddots}}} \right), \tag{3.5}$$

Theorem 3.2. For every $x > 0$, we have continued fraction bounds for the Gamma function:

$$\exp\left(\mathop{K}_{i=1}^{2n}\frac{a_i}{x+\frac{b_i}{x}}\right)<\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x}<\exp\left(\mathop{K}_{i=1}^{2n-1}\frac{a_i}{x+\frac{b_i}{x}}\right),\qquad n\in\mathbb{N},\tag{3.6}$$

where

$$\begin{aligned} a_1 &= \frac{B_2}{2},\ b_1=0,\\ a_i &= -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}},b_i=-a_i,\quad i=2,3,\cdots,2n. \end{aligned}.$$

Proof. Using (1.4) and the same method from (3.2) to (3.3), for $x>0$,

$$\sum_{i=1}^{2n}\frac{B_{2i}}{2i(2i-1)x^{2i-1}}=\mathop{K}_{i=1}^{2n}\frac{a_i}{x+\frac{b_i}{x}}\Leftrightarrow$$

$$\exp\left(\sum_{i=1}^{2n}\frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right)=\exp\left(\mathop{K}_{i=1}^{2n}\frac{a_i}{x+\frac{b_i}{x}}\right)<\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x},\tag{3.7}$$

and

$$\sum_{i=1}^{2n-1}\frac{B_{2i}}{2i(2i-1)x^{2i-1}}=\mathop{K}_{i=1}^{2n-1}\frac{a_i}{x+\frac{b_i}{x}}\Leftrightarrow$$

$$\exp\left(\sum_{i=1}^{2n-1}\frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right)=\exp\left(\mathop{K}_{i=1}^{2n-1}\frac{a_i}{x+\frac{b_i}{x}}\right)>\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x},\tag{3.8}$$

where

$$\begin{aligned} a_1 &= \frac{B_2}{2},\ b_1=0,\\ a_i &= -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}},b_i=-a_i,\quad i=2,3,\cdots,2n \end{aligned}.$$

Thus, our new continued fraction bounds for the Gamma function are obtained.

Theorem 3.3. As $x\to\infty$, we have the continued fraction approximation of $\Gamma(x+\frac{1}{2})$,

$$\Gamma\left(x+\frac{1}{2}\right) \approx \sqrt{2\pi}\left(\frac{x}{e}\right)^x \exp\left(-\mathop{K}_{i=1}^{\infty}\frac{p_i}{x+\frac{q_i}{x}}\right)$$

$$= \sqrt{2\pi}\left(\frac{x}{e}\right)^x \exp\left(-\cfrac{p_1}{x+\cfrac{q_1}{x}+\cfrac{p_2}{x+\cfrac{q_2}{x}+\cfrac{p_3}{x+\cfrac{q_3}{x}+\ddots}}}\right), \tag{3.9}$$

where

$$p_1 = \frac{B_2}{4},\ q_1 = 0,$$
$$p_i = -\frac{(i-1)(2i-3)(4^i-2)B_{2i}}{i(2i-1)(4^i-8)B_{2(i-1)}}, q_i = -p_i,\ \ i = 2, 3, \cdots.$$

Proof. Assume that

$$c_{2i} = \frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)}, \qquad i = 1, 2, 3, \cdots. \tag{3.10}$$

From (3.10) and Theorem 2.1, as $x \to \infty$,

$$\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i-1}} = \sum_{i=1}^{\infty}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}} = \mathop{K}_{i=1}^{\infty}\frac{p_i}{x+\frac{q_i}{x}} \Leftrightarrow$$

$$\exp\left(-\sum_{i=1}^{\infty}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}\right) = \exp\left(-\mathop{K}_{i=1}^{\infty}\frac{p_i}{x+\frac{q_i}{x}}\right), \tag{3.11}$$

where

$$p_1 = c_2 = \frac{B_2}{4},\ q_1 = 0,$$
$$p_i = -\frac{c_{2i}}{c_{2(i-1)}} = -\frac{(i-1)(2i-3)(4^i-2)B_{2i}}{i(2i-1)(4^i-8)B_{2(i-1)}}, q_i = \frac{c_{2i}}{c_{2(i-1)}} = -p_i,\ \ i = 2, 3, \cdots.$$

According to (1.6),

$$\Gamma\left(x+\frac{1}{2}\right)\approx\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\exp\left(-\sum_{i=1}^{\infty}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}\right)=\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\exp\left(-\mathop{K}_{i=1}^{\infty}\frac{p_i}{x+\frac{q_i}{x}}\right). \quad (3.12)$$

Thus, our new continued fraction approximation can be obtained.

Remark 3.2. As you can see, our new continued fraction approximation for the Gamma function is also equal to (1.6).
From Theorem 2.1, we have another expression of (3.12) as follows;

$$\Gamma\left(x+\frac{1}{2}\right)\approx\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\exp\left(-\frac{p_1}{x+\mathop{K}_{i=2}^{\infty}\frac{p_i}{x-\frac{p_i}{x}}}\right)=\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\exp\left(-\frac{p_1}{x+\frac{p_2}{x-\frac{p_2}{x}+\frac{p_3}{x-\frac{p_3}{x}+\ddots}}}\right) \quad (3.13)$$

Theorem 3.4. For every $x>0$, we have continued fraction bounds for the Gamma function

$$\exp\left(-\mathop{K}_{i=1}^{2n-1}\frac{p_i}{x+\frac{q_i}{x}}\right)<\frac{\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}}<\exp\left(-\mathop{K}_{i=1}^{2n}\frac{p_i}{x+\frac{q_i}{x}}\right),\qquad n\in\mathbb{N}, \quad (3.14)$$

where

$$p_1=\frac{B_2}{4},\; q_1=0,$$
$$p_i=-\frac{(i-1)(2i-3)(4^i-2)B_{2i}}{i(2i-1)(4^i-8)B_{2(i-1)}},\, q_i=-p_i,\;\; i=2,3,\cdots,2n.$$

Proof. Using (1.8) and the same method from (3.10) to (3.11), for $x>0$,

$$\sum_{i=1}^{2n-1}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}=\mathop{K}_{i=1}^{2n-1}\frac{p_i}{x+\frac{q_i}{x}}\Leftrightarrow$$

$$\exp\left(-\sum_{i=1}^{2n-1}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}\right)=\exp\left(-\mathop{K}_{i=1}^{2n-1}\frac{p_i}{x+\frac{q_i}{x}}\right)<\frac{\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}}, \quad (3.15)$$

and

$$\sum_{i=1}^{2n}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}=\mathop{K}_{i=1}^{2n}\frac{p_i}{x+\frac{q_i}{x}}\Leftrightarrow$$

$$\exp\left(-\sum_{i=1}^{2n}\frac{(1-2^{1-2i})B_{2i}}{2i(2i-1)x^{2i-1}}\right)=\exp\left(-\mathop{K}_{i=1}^{2n}\frac{p_i}{x+\frac{q_i}{x}}\right)>\frac{\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^x},\tag{3.16}$$

where

$$p_1=\frac{B_2}{4},\ q_1=0,$$
$$p_i=-\frac{(i-1)(2i-3)(4^i-2)B_{2i}}{i(2i-1)(4^i-8)B_{2(i-1)}},q_i=-p_i,\quad i=2,3,\cdots,2n.$$

Thus, our new continued fraction bounds for the Gamma function are obtained.

Theorem 3.5. As $x\to\infty$, we have the continued fraction approximation of $\Gamma(x+1)$,

$$\Gamma(x+1)\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{\frac{x}{2}}\exp\left(\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{m_i}{x+\frac{n_i}{x}}\right)$$
$$==\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{\frac{x}{2}}\exp\left(\frac{1}{x^4}\cfrac{m_1}{x+\cfrac{n_1}{x}+\cfrac{m_2}{x+\cfrac{n_2}{x}+\cfrac{m_3}{x+\cfrac{n_3}{x}+\ddots}}}\right),\tag{3.17}$$

where

$$m_1=\frac{7}{270}B_6,\ n_1=0,$$
$$m_i=-\frac{(i+1)}{2(i+2)^2(2i+3)}\frac{[(i+2)(2i+2)!-4^{i+1}]}{[(i+1)(2i)!-4^i]}\frac{B_{2(i+2)}}{B_{2(i+1)}},n_i=-m_i,\quad i=2,3,\cdots.$$

Proof. In (1.11), it's easy to see that

$$\exp\left(\sum_{i=3}^{\infty}\frac{2i(2i-2)!-2^{2i-1}}{2i(2i)!}\frac{B_{2i}}{x^{2i-1}}\right)=\exp\left(\sum_{i=1}^{\infty}\frac{(2i+4)(2i+2)!-2^{2i+3}}{(2i+4)(2i+4)!}\frac{B_{2(i+2)}}{x^{2i+3}}\right). \tag{3.18}$$

Assume that

$$c_{2i}=\frac{(2i+4)(2i+2)!-2^{2i+3}}{(2i+4)(2i+4)!}B_{2(i+2)}\,,\qquad i=1,2,3,\cdots. \tag{3.19}$$

From (3.19) and Theorem 2.1, as $x\to\infty$,

$$\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i+3}}=\frac{1}{x^4}\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i-1}}=\frac{1}{x^4}\sum_{i=1}^{\infty}\frac{(2i+4)(2i+2)!-2^{2i+3}}{(2i+4)(2i+4)!}\frac{B_{2(i+2)}}{x^{2i-1}}=\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{m_i}{x+\frac{n_i}{x}}, \tag{3.20}$$

from (3.18),

$$\exp\left(\sum_{i=3}^{\infty}\frac{2i(2i-2)!-2^{2i-1}}{2i(2i)!}\frac{B_{2i}}{x^{2i-1}}\right)=\exp\left(\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{m_i}{x+\frac{n_i}{x}}\right), \tag{3.21}$$

where

$$\begin{aligned}&m_1=c_2=\frac{7}{270}B_6,\ n_1=0,\\ &m_i=-\frac{c_{2i}}{c_{2(i-1)}}=-\frac{(i+1)}{2(i+2)^2(2i+3)}\frac{[(i+2)(2i+2)!-4^{i+1}]}{[(i+1)(2i)!-4^{i}]}\frac{B_{2(i+2)}}{B_{2(i+1)}},\ n_i=\frac{c_{2i}}{c_{2(i-1)}}=-m_i,\quad i=2,3,\cdots.\end{aligned}$$

According to (1.11),

$$\begin{aligned}\Gamma(x+1)&\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{\frac{x}{2}}\exp\left(\sum_{i=3}^{\infty}\frac{2i(2i-2)!-2^{2i-1}}{2i(2i)!}\frac{B_{2i}}{x^{2i-1}}\right)\\ &=\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{\frac{x}{2}}\exp\left(\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{m_i}{x+\frac{n_i}{x}}\right).\end{aligned} \tag{3.22}$$

Thus, our new continued fraction approximation can be obtained.

Remark 3.3. As you can see, our new continued fraction approximation for the Gamma function is equal to (1.11).

From Theorem 2.1, we have another expression of (3.22) as follows;

$$\Gamma(x+1)\approx\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{\frac{x}{2}}\exp\left(\frac{1}{x^4}\frac{m_1}{x+\mathop{K}_{i=2}^{\infty}\frac{m_i}{x-\frac{m_i}{x}}}\right)$$

$$= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{\frac{x}{2}} \exp\left(\frac{1}{x^4}\cfrac{m_1}{x+\cfrac{m_2}{x-\frac{m_2}{x}+\cfrac{m_3}{x-\frac{m_3}{x}+\ddots}}}\right). \tag{3.23}$$

Theorem 3.6. As $x \to \infty$, we have the continued fraction approximation of $\Gamma(x+\frac{1}{2})$,

$$\Gamma\left(x+\frac{1}{2}\right) \approx \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{\frac{x}{2}} \exp\left(-\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{u_i}{x+\frac{v_i}{x}}\right)$$

$$= \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{\frac{x}{2}} \exp\left(-\frac{1}{x^4}\cfrac{u_1}{x+\frac{v_1}{x}+\cfrac{u_2}{x+\frac{v_2}{x}+\cfrac{u_3}{x+\frac{v_3}{x}+\ddots}}}\right), \tag{3.24}$$

where

$$u_1 = \frac{217}{8640}B_6,\ v_1 = 0,$$

$$u_i = -\frac{(i+1)}{2(i+2)^2(2i+3)}\frac{[(i+2)(2i+2)!-4^{i+1}]}{[(i+1)(2i)!-4^i]}\frac{4^{i+2}-2}{4^{i+2}-8}\frac{B_{2(i+2)}}{B_{2(i+1)}}, v_i = -u_i,\ \ i = 2,3,\cdots.$$

Proof. In (1.12), it's easy to see that

$$\exp\left(-\sum_{i=3}^{\infty}\frac{(2i)!-(2i-1)2^{2i-1}}{2i(2i-1)(2i)!}(1-2^{1-2i})\frac{B_{2i}}{x^{2i-1}}\right) = \exp\left(-\sum_{i=1}^{\infty}\frac{(2i+4)!-(2i+3)2^{2i+3}}{(2i+3)(2i+4)(2i+4)!}(1-2^{-3-2i})\frac{B_{2(i+2)}}{x^{2i+3}}\right)$$

.(3.25)

Assume that

$$c_{2i} = \frac{(2i+4)!-(2i+3)2^{2i+3}}{(2i+3)(2i+4)(2i+4)!}(1-2^{-3-2i})B_{2(i+2)}, \quad i = 1,2,3,\cdots. \tag{3.26}$$

From (3.26) and Theorem 2.1, as $x \to \infty$,

$$\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i+3}}=\frac{1}{x^4}\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i-1}}=\frac{1}{x^4}\sum_{i=1}^{\infty}\frac{(2i+4)!-(2i+3)2^{2i+3}}{(2i+3)(2i+4)(2i+4)!}(1-2^{-3-2i})\frac{B_{2(i+2)}}{x^{2i-1}}=\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{u_i}{x+\frac{v_i}{x}},\quad(3.27)$$

from (3.25),

$$\exp\left(-\sum_{i=3}^{\infty}\frac{(2i)!-(2i-1)2^{2i-1}}{2i(2i-1)(2i)!}(1-2^{1-2i})\frac{B_{2i}}{x^{2i-1}}\right)=\exp\left(-\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{u_i}{x+\frac{v_i}{x}}\right),\tag{3.28}$$

where

$$u_1=c_2=\frac{217}{8640}B_6,\ v_1=0,$$
$$u_i=-\frac{c_{2i}}{c_{2(i-1)}}=-\frac{(i+1)}{2(i+2)^2(2i+3)}\frac{[(i+2)(2i+2)!-4^{i+1}]}{[(i+1)(2i)!-4^i]}\frac{4^{i+2}-2}{4^{i+2}-8}\frac{B_{2(i+2)}}{B_{2(i+1)}},\ \cdot$$
$$v_i=\frac{c_{2i}}{c_{2(i-1)}}=-u_i,\ \ i=2,3,\cdots.$$

According to (1.12),

$$\Gamma\left(x+\frac{1}{2}\right)\approx\sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{\frac{x}{2}}\exp\left(-\sum_{i=3}^{\infty}\frac{(2i)!-(2i-1)2^{2i-1}}{2i(2i-1)(2i)!}(1-2^{1-2i})\frac{B_{2i}}{x^{2i-1}}\right)$$
$$=\sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{\frac{x}{2}}\exp\left(-\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{u_i}{x+\frac{v_i}{x}}\right).\tag{3.29}$$

Thus, our new continued fraction approximation can be obtained.

Remark 3.4. As you can see, our new continued fraction approximation for the Gamma function is also equal to (1.12).
From Theorem 2.1, we have another expression of (3.29) as follows;

$$\Gamma\left(x+\frac{1}{2}\right)\approx\sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{\frac{x}{2}}\exp\left(-\frac{1}{x^4}\frac{u_1}{x+\mathop{K}_{i=2}^{\infty}\frac{u_i}{x-\frac{u_i}{x}}}\right)$$

$$= \sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\left(2x\tanh\frac{1}{2x}\right)^{\frac{x}{2}}\exp\left(-\frac{1}{x^{4}}\cfrac{u_1}{x+\cfrac{u_2}{x-\cfrac{u_2}{x}+\cfrac{u_3}{x-\cfrac{u_3}{x}+\ddots}}}\right) \tag{3.30}$$

4. Continued fraction approximations and bounds for the generalized Wallis ratio

In this section, we present continued fraction approximations and continued fraction bounds for the Wallis ratio as applications of our method.

Theorem 4.1. As $x \to \infty$, we have the continued fraction approximation for the generalized Wallis ratio:

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x}\exp\left(\mathop{K}_{i=1}^{\infty}\frac{s_i}{x+\frac{t_i}{x}}\right) = \sqrt{x}\exp\left(\cfrac{s_1}{x+\cfrac{t_1}{x}+\cfrac{s_2}{x+\cfrac{t_2}{x}+\cfrac{s_3}{x+\cfrac{t_3}{x}+\ddots}}}\right), \tag{4.1}$$

where

$$s_1 = \frac{3B_2}{4},\ t_1 = 0,$$
$$s_i = -\frac{(i-1)(2i-3)}{i(2i-1)}\frac{4^i-1}{4^i-4}\frac{B_{2i}}{B_{2(i-1)}}, t_i = -s_i,\quad i = 2, 3, \cdots.$$

Proof. Assume that

$$c_{2i} = \frac{(1-2^{-2i})B_{2i}}{i(2i-1)},\qquad i = 1, 2, 3, \cdots. \tag{4.2}$$

From (4.2) and Theorem 2.1, as $x \to \infty$,

$$\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i-1}} = \sum_{i=1}^{\infty}\frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}} = \mathop{K}_{i=1}^{\infty}\frac{s_i}{x+\frac{t_i}{x}} \Leftrightarrow \exp\left(\sum_{i=1}^{\infty}\frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}}\right) = \exp\left(\mathop{K}_{i=1}^{\infty}\frac{s_i}{x+\frac{t_i}{x}}\right), \tag{4.3}$$

where

$$s_1 = c_2 = \frac{3B_2}{4},\ t_1 = 0,$$
$$s_i = -\frac{c_{2i}}{c_{2(i-1)}} = -\frac{(i-1)(2i-3)}{i(2i-1)}\frac{4^i-1}{4^i-4}\frac{B_{2i}}{B_{2(i-1)}}, t_i = \frac{c_{2i}}{c_{2(i-1)}} = -s_i,\quad i = 2,3,\cdots.$$

According to (1.13),

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x}\exp\left(\sum_{i=1}^{\infty}\frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}}\right) = \sqrt{x}\exp\left(\underset{i=1}{\overset{\infty}{K}}\frac{s_i}{x+\frac{t_i}{x}}\right). \tag{4.4}$$

Thus, our new continued fraction approximation can be obtained.

Remark 4.1. As you can see, our new continued fraction approximation for the Gamma function is equal to (1.13).
From Theorem 2.1, we have another expression of (4.4) as follows;

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x}\exp\left(\frac{s_1}{x+\underset{i=2}{\overset{\infty}{K}}\frac{s_i}{x-\frac{s_i}{x}}}\right) = \sqrt{x}\exp\left(\frac{s_1}{x+\cfrac{s_2}{x-\frac{s_2}{x}+\cfrac{s_3}{x-\frac{s_3}{x}+\ddots}}}\right). \tag{4.5}$$

Theorem 4.2. For every $x > 0$, we have continued fraction bounds for the Wallis ratio:

$$\sqrt{x}\exp\left(\underset{i=1}{\overset{2n}{K}}\frac{s_i}{x+\frac{t_i}{x}}\right) < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\exp\left(\underset{i=1}{\overset{2n-1}{K}}\frac{s_i}{x+\frac{t_i}{x}}\right),\quad n \in \mathbb{N}, \tag{4.6}$$

where

$$s_1 = \frac{3B_2}{4},\ t_1 = 0,$$
$$s_i = -\frac{(i-1)(2i-3)}{i(2i-1)}\frac{4^i-1}{4^i-4}\frac{B_{2i}}{B_{2(i-1)}}, t_i = -s_i,\quad i = 2,3,\cdots,2n.$$

Proof. Using (1.14) and the same method from (4.2) to (4.3), for $x > 0$,

$$\sum_{i=1}^{2n}\frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}} = \underset{i=1}{\overset{2n}{K}}\frac{s_i}{x+\frac{t_i}{x}} \Leftrightarrow$$

$$\sqrt{x}\exp\left(\sum_{i=1}^{2n}\frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}}\right)=\sqrt{x}\exp\left(\mathop{K}_{i=1}^{2n}\frac{s_i}{x+\frac{t_i}{x}}\right)<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}, \tag{4.7}$$

and

$$\sum_{i=1}^{2n-1}\frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}}=\mathop{K}_{i=1}^{2n-1}\frac{s_i}{x+\frac{t_i}{x}}\Leftrightarrow$$

$$\sqrt{x}\exp\left(\sum_{i=1}^{2n-1}\frac{(1-2^{-2i})B_{2i}}{i(2i-1)x^{2i-1}}\right)=\sqrt{x}\exp\left(\mathop{K}_{i=1}^{2n-1}\frac{s_i}{x+\frac{t_i}{x}}\right)>\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}, \tag{4.8}$$

where

$$s_1=\frac{3B_2}{4},\ t_1=0,$$

$$s_i=-\frac{(i-1)(2i-3)}{i(2i-1)}\frac{4^i-1}{4^i-4}\frac{B_{2i}}{B_{2(i-1)}},t_i=-s_i,\quad i=2,3,\cdots,2n.$$

Thus, our new continued fraction bounds are obtained.

Theorem 4.3. As $x\to\infty$, we have the continued fraction approximation of the Wallis ratio:

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\approx\sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\exp\left(\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{w_i}{x+\frac{r_i}{x}}\right)$$

$$=\sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\exp\left(\frac{1}{x^4}\cfrac{w_1}{x+\cfrac{r_1}{x}+\cfrac{w_2}{x+\cfrac{r_2}{x}+\cfrac{w_3}{x+\cfrac{r_3}{x}+\ddots}}}\right), \tag{4.9}$$

where

$$w_1=\frac{49}{960}B_6,\ r_1=0,$$
$$w_i=-\frac{(i+1)}{2(i+2)^2(2i+3)}\frac{[(i+2)(2i+2)!-4^{i+1}]}{[(i+1)(2i)!-4^i]}\frac{4^{i+2}-1}{4^{i+2}-4}\frac{B_{2(i+2)}}{B_{2(i+1)}},r_i=-w_i,\ \ i=2,3,\cdots.$$

Proof. In (1.16), it's easy to see that
$$\exp\left(\sum_{i=3}^{\infty}\frac{(2i)!-(2i-1)2^{2i-1}}{i(2i-1)(2i)!}(1-2^{-2i})\frac{B_{2i}}{x^{2i-1}}\right)=\exp\left(\sum_{i=1}^{\infty}\frac{(2i+4)!-(2i+3)2^{2i+3}}{(i+2)(2i+3)(2i+4)!}(1-2^{-4-2i})\frac{B_{2(i+2)}}{x^{2i+3}}\right).\tag{4.10}$$
Assume that
$$c_{2i}=\frac{(2i+4)!-(2i+3)2^{2i+3}}{(i+2)(2i+3)(2i+4)!}(1-2^{-4-2i})B_{2(i+2)}\ ,\qquad i=1,2,3,\cdots.\tag{4.11}$$

From (4.11) and Theorem 2.1, as $x\to\infty$,
$$\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i+3}}=\frac{1}{x^4}\sum_{i=1}^{\infty}\frac{c_{2i}}{x^{2i-1}}=\frac{1}{x^4}\sum_{i=1}^{\infty}\frac{(2i+4)!-(2i+3)2^{2i+3}}{(i+2)(2i+3)(2i+4)!}(1-2^{-4-2i})\frac{B_{2(i+2)}}{x^{2i-1}}=\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{w_i}{x+\frac{r_i}{x}},\tag{4.12}$$
from (4.10),
$$\exp\left(\sum_{i=3}^{\infty}\frac{(2i)!-(2i-1)2^{2i-1}}{i(2i-1)(2i)!}(1-2^{-2i})\frac{B_{2i}}{x^{2i-1}}\right)=\exp\left(\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{w_i}{x+\frac{r_i}{x}}\right),\tag{4.13}$$

where

$$w_1=c_2=\frac{49}{960}B_6,\ r_1=0,$$
$$w_i=-\frac{c_{2i}}{c_{2(i-1)}}=-\frac{(i+1)}{2(i+2)^2(2i+3)}\frac{[(i+2)(2i+2)!-4^{i+1}]}{[(i+1)(2i)!-4^i]}\frac{4^{i+2}-1}{4^{i+2}-4}\frac{B_{2(i+2)}}{B_{2(i+1)}},$$
$$r_i=\frac{c_{2i}}{c_{2(i-1)}}=-w_i,\ \ i=2,3,\cdots.$$

According to (1.16),
$$\begin{aligned}\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}&\approx\sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\exp\left(\sum_{i=3}^{\infty}\frac{(2i)!-(2i-1)2^{2i-1}}{i(2i-1)(2i)!}(1-2^{-2i})\frac{B_{2i}}{x^{2i-1}}\right)\\&==\sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\exp\left(\frac{1}{x^4}\mathop{K}_{i=1}^{\infty}\frac{w_i}{x+\frac{r_i}{x}}\right).\end{aligned}\tag{4.14}$$

Thus, our new continued fraction approximation can be obtained.

Remark 4.2. As you can see, our new continued fraction approximation for the Gamma function is also equal to (1.16).
From Theorem 2.1, we have another expression of (4.14) as follows:

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x}\left(\cosh\frac{1}{2x}\right)^{x}\exp\left(\frac{1}{x^4}\frac{w_1}{x+\underset{i=2}{\overset{\infty}{K}}\dfrac{w_i}{x-\dfrac{w_i}{x}}}\right)$$
$$== \sqrt{x}\left(\cosh\frac{1}{2x}\right)^{x}\exp\left(\frac{1}{x^4}\frac{w_1}{x+\dfrac{w_2}{x-\dfrac{w_2}{x}+\dfrac{w_3}{x-\dfrac{w_3}{x}+\ddots}}}\right), \tag{4.15}$$

5. Concluding Remarks

In Remark 3.1 and Remark 3.2, we can see that

$$p_i = \begin{cases} \dfrac{4^i-2}{4^i}a_i, & i=1, \\ \dfrac{4^i-2}{4^i-8}a_i, & i=2,3,\cdots, \end{cases} \tag{5.1}$$

and in Remark 3.3 and Remark 3.4,

$$u_i = \begin{cases} \dfrac{4^{i+2}-2}{4^{i+2}}m_i, & i=1, \\ \dfrac{4^{i+2}-2}{4^{i+2}-8}m_i, & i=2,3,\cdots. \end{cases} \tag{5.2}$$

Also in Remark 3.1 and Remark 4.1, we can see that

$$s_i = \begin{cases} \dfrac{2(4^i-1)}{4^i}a_i, & i=1, \\ \dfrac{4^i-1}{4^i-4}a_i, & i=2,3,\cdots, \end{cases} \tag{5.3}$$

and in Remark 3.3 and Remark 4.2,

$$w_i = \begin{cases} \dfrac{2(4^{i+2}-1)}{4^{i+2}} m_i, & i = 1, \\ \dfrac{4^{i+2}-1}{4^{i+2}-4} m_i, & i = 2, 3, \cdots. \end{cases} \tag{5.4}$$

From Theorem 3.1, Theorem 3.5 and the relations of (5.1), (5.2), (5.3) and (5.4), we get

$$a_1 = \frac{1}{12}, a_2 = \frac{1}{30}, a_3 = \frac{2}{7}, a_4 = \frac{3}{4}, a_5 = \frac{140}{99}, a_6 = \frac{2073}{910}, a_7 = \frac{2310}{691}, \cdots,$$

$$p_1 = \frac{1}{24}, p_2 = \frac{7}{120}, p_3 = \frac{31}{98}, p_4 = \frac{381}{496}, p_5 = \frac{17885}{12573}, p_6 = \frac{4243431}{1860040}, p_7 = \frac{9460605}{2828954}, \cdots,$$

$$s_1 = \frac{1}{8}, s_2 = \frac{1}{24}, s_3 = \frac{3}{10}, s_4 = \frac{85}{112}, s_5 = \frac{217}{153}, s_6 = \frac{6219}{2728}, s_7 = \frac{60071}{17966}, \cdots,$$

and

$$m_1 = \frac{1}{1620}, m_2 = \frac{33}{35}, m_3 = \frac{13}{9}, m_4 = \frac{2260261}{990990},$$

$$m_5 = \frac{98232630}{29383393}, m_6 = \frac{2264216681}{491163150}, m_7 = \frac{132677262368890}{21824784588159}, \cdots,$$

$$u_1 = \frac{31}{51840}, u_2 = \frac{4191}{4340}, u_3 = \frac{6643}{4572}, u_4 = \frac{4626754267}{2025583560},$$

$$u_5 = \frac{402311736165}{120295610942}, u_6 = \frac{74191587986327}{16092469446600}, u_7 = \frac{8695070727976390595}{1430265433200411906}, \cdots,$$

$$w_1 = \frac{7}{5760}, w_2 = \frac{187}{196}, w_3 = \frac{4433}{3060}, w_4 = \frac{2260261}{990264},$$

$$w_5 = \frac{17881613081}{5347777526}, \ w_6 = \frac{581903687017}{126223151160}, \ w_7 = \frac{968544015292897}{159319104203286}, \cdots.$$

As mentioned above, in our investigation, we provide a generally applicable and very useful method to construct continued fraction and have successfully found its applications.

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