Continued fraction approximation and bounds for the psi function

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ABSTRACT

In this paper, we provide a new continued fraction approximation for the psi function. Then we establish continued fraction bounds for the psi function.

Keywords: Euler connection, Continued fraction, Psi function

1. Introduction

Special functions and mathematical constants play an important role in several areas of mathematics and other branches of science such as number theory, analysis, probability theory, statistical physics and so on.

Especially, the classical Euler gamma function Γ defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0, \tag{1.1}$$

is one of the most important special functions and has a lot of applications in diverse areas. The logarithmic derivative $\psi(x)$ of the gamma function $\Gamma(x)$ given by

$$\psi(x) = \frac{\Gamma(x)'}{\Gamma(x)} \qquad \text{or} \qquad \ln \Gamma(x) = \int_1^x \psi(t) dt$$

is well-known as the psi(or digamma) function.

The successive derivatives of the psi function $\psi(x)$

$$\psi^{(n)}(x) = \frac{d^n}{dx^n}\{\psi(x)\} \quad (n \in \mathbb{N})$$

are called the polygamma functions. The following recurrence formula is well known for the psi function (see [1, p. 258]):

$$\psi(x+1) = \psi(x) + \frac{1}{x} \tag{1.2}$$

The psi function is connected to the Euler-Mascheroni constant and harmonic numbers through the well-known relation (see [1, p. 258, Eq. (6.3.2)]):

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$$\psi(n+1) = -\gamma + H_n, \qquad n \in \mathbb{N}, \tag{1.3}$$

where

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \; (n \in \mathbb{N})$$

is the n^{th} harmonic number and γ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n \to \infty} D_n = 0.577215664 \cdots,$$

where

$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n. \tag{1.4}$$

The constant γ, now universally known as gamma, is generally accepted to be the most significant of the 'constant' and as such is the important special constant of mathematics, after π and e. It is deeply related to the gamma function $\Gamma(x)$ thanks to the Weierstrass formula:

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{x}{k}\right)^{-1} e^{x/k} \right\} \tag{1.5}$$

As you can see, the gamma function, psi function and Euler-Mascheroni constant are related to each other.

In this field, the most important problem is to find more accurate approximation and bounds for them, so during the past several decades, many mathematicians and scientists have worked on this subject.

Up to now, many researchers made great efforts in this area of establishing more accurate approximations and inequalities for the gamma function, psi function and Euler-Mascheroni constant and had lots of inspiring results.

Recently, authors have focused on continued fractions in order to obtain new approximation and bounds.

For example, Lu([5]) provided faster sequence convergent to γ as follows.

$$r_{n,s} = H_n - \ln n - \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \ddots}}}}, \tag{1.6}$$

where

$$\begin{aligned} &a_1 = \frac{1}{2}, a_2 = \frac{1}{6}, a_4 = \frac{3}{5}, a_6 = \frac{79}{126}, a_8 = \frac{7230}{6241}, a_{10} = \frac{4146631}{3833346}, \cdots . \\ &a_{2k+1} = -a_{2k} \ (1 \le k \le 6) \end{aligned}$$

Moreover, he used continued fraction approximation to consider new classes of sequences for the Euler–Mascheroni constant as follows. ([6])

$$\begin{aligned} L_{r,n} &= H_{n-1} + \frac{1}{rn} - \ln n - \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \ddots}}}} \quad (r \neq 2) \\ L_{2,n} &= H_{n-1} + \frac{1}{2n} - \ln n - \frac{1}{n} \cfrac{b_1}{n + \cfrac{b_2}{n + \cfrac{b_3}{n + \cfrac{b_4}{n + \ddots}}}} \end{aligned}, \tag{1.7}$$

where

$$\begin{aligned} &a_1 = \frac{2-r}{2r}, a_2 = \frac{r}{6(2-r)}, a_3 = \frac{r}{6(r-2)}, a_4 = \frac{3(2-r)}{5r}, \cdots \\ &b_1 = -\frac{1}{12}, b_2 = \frac{1}{10}, b_3 = \frac{79}{210}, b_4 = \frac{1205}{1659}, \cdots \end{aligned} .$$

In [7], he introduced new classes of sequences.

$$L_{n,k}=H_n-\ln n-\frac{1}{k}\ln\left(1+\cfrac{a_1}{n+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\cfrac{a_4 n}{n+\ddots}}}}\right), \tag{1.8}$$

where

$$a_1=\frac{k}{2}, a_2=\frac{2-3k}{12}, a_3=\frac{3k^2+4}{12(3k-2)}, a_4=-\frac{15k^4-30k^3+60k^2-104k+96}{20(3k-2)(3k^2+4)},\cdots$$

In this paper, based on continued fractions, we provide a new continued fraction approximation for the psi function and continued fraction bounds for the psi function and polygamma function.

The rest of this paper is arranged as follows.

In Sect. 2, preliminaries are given. In Sect. 3, we provide a main method to construct continued fraction based on a given power series. In Sect. 4, a new continued fraction approximation for the psi function are provided. In the last section, the conclusions are given.

2. Preliminaries

The basic approximation is given by the well-known asymptotic expansion of the psi function (see e.g. [1])

$$\psi(x+1)\sim \ln x+\frac{1}{2x}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2nx^{2n}}, \tag{2.1}$$

where B_n $(n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\})$ denotes the Bernoulli numbers defined by the generating formula

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!},\ |z|<2\pi,$$

then the first few terms of B_n are as follows.

$$B_{2n+1}=0, n\geq 1,$$
$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42}, B_8=-\frac{1}{30}, B_{10}=\frac{5}{66},\cdots.$$

The following two lemmas shows the double inequalities for the psi function and polygamma function.

Lemma 2.1(see [2]). For $x>0$,

$$\ln x+\sum_{i=1}^{2n}\frac{(1-2^{1-2i})B_{2i}}{2ix^{2i}}<\psi\left(x+\frac{1}{2}\right)<\ln x+\sum_{i=1}^{2n+1}\frac{(1-2^{1-2i})B_{2i}}{2ix^{2i}},\qquad n\in\mathbb{N}_0. \tag{2.2}$$

Lemma 2.2([3, Theorem 9]). Let $k\geq 1$ and $n\geq 0$ be integers. Then for all real num bers $x>0$:

$$S_k(2n;x)<(-1)^{k+1}\psi^{(k)}(x)<S_k(2n+1;x)\ , \tag{2.3}$$

where

$$S_k(p,x)=\frac{(k-1)!}{x^k}+\frac{k!}{2x^{k+1}}+\sum_{i=1}^{p}\left[B_{2i}\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}}.$$

3. A main method

In this section, we present a main method to construct continued fraction based on a given power series using Euler connection.

The Euler connection states the connection between series and continued fractions as follows.

Lemma 3.1.(The Euler connection [4, p.19, Eq. (1.7.1, 1.7.2)]) Let $\{c_k\}$ be a sequence in $\mathbb{C} \setminus \{0\}$ and

$$f_n = \sum_{k=0}^{n} c_k \, , \quad \mathrm{n} \in \mathbb{N}_0. \tag{3.1}$$

Since $f_0 \neq \infty, \ f_n \neq f_{n-1}$, $\mathrm{n} \in \mathbb{N}$, there exists a continued fraction $b_0 + K(a_m / b_m)$ with n[th] approximant f_n for all *n*. This continued fraction is given by

$$c_0 + \frac{c_1}{1} \, \frac{-c_2 / c_1}{+1 + c_2 / c_1} + \cdots + \frac{-c_m / c_{m-1}}{1 + c_m / c_{m-1}} + \cdots. \tag{3.2}$$

The following theorem states our main method.

Theorem 3.1. For every $x \neq 0$,

$$\sum_{i=1}^{n} \frac{c_{2i}}{x^{2i}} = \frac{1}{x^2} \mathop{K}_{i=1}^{n} \frac{a_i x^2}{x^2 + b_i} = \frac{1}{x^2} \frac{a_1 x^2}{x^2 + \mathop{K}_{i=2}^{n} \dfrac{a_i x^2}{x^2 - a_i}}, \qquad \mathrm{n} \in \mathbb{N}, \tag{3.3}$$

where

$$c_{2i} \neq 0, \quad i = 1, 2, \cdots, n$$

and

$$\begin{aligned} &a_1 = c_2, \ b_1 = 0, \\ &a_i = -\frac{c_{2i}}{c_{2(i-1)}}, \ b_i = -a_i, \ i = 2, 3, \cdots, n \end{aligned}.$$

Proof. Assume that

$$f_0(x) \neq \infty, \quad f_n(x) = \sum_{i=1}^{n} \frac{c_{2i}}{x^{2i}}, \quad \mathrm{n} \in \mathbb{N}, \quad x \neq 0, \tag{3.4}$$

where

$$c_{2i} \neq 0, \quad i = 1, 2, \cdots, n.$$

Since

$$f_0(x) \neq \infty, \quad f_n(x) \neq f_{n-1}(x), \quad n \in \mathbb{N},$$

the left-side of (3.3) is equal to $f_n(x)$ ($n \in \mathbb{N}$).

Using Lemma 3.1,

$$\begin{aligned}
f_n(x) &= \sum_{i=1}^{n} \frac{c_{2i}}{x^{2i}} \\
&= \frac{\frac{c_2}{x^2}}{1} + \frac{-\frac{c_4}{c_2x^2}}{1+\frac{c_4}{c_2x^2}} + \frac{-\frac{c_6}{c_4x^2}}{1+\frac{c_6}{c_4x^2}} + \frac{-\frac{c_8}{c_6x^2}}{1+\frac{c_8}{c_6x^2}} + \cdots + \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} + \cdots + \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}} \\
&= \frac{1}{x^2}\frac{c_2}{1} + \frac{-\frac{c_4}{c_2x^2}}{1+\frac{c_4}{c_2x^2}} + \frac{-\frac{c_6}{c_4x^2}}{1+\frac{c_6}{c_4x^2}} + \frac{-\frac{c_8}{c_6x^2}}{1+\frac{c_8}{c_6x^2}} + \cdots + \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} + \cdots + \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}} \\
&= \frac{1}{x^2}\frac{c_2x^2}{x^2} + \frac{-\frac{c_4}{c_2}}{1+\frac{c_4}{c_2x^2}} + \frac{-\frac{c_6}{c_4x^2}}{1+\frac{c_6}{c_4x^2}} + \frac{-\frac{c_8}{c_6x^2}}{1+\frac{c_8}{c_6x^2}} + \cdots + \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} + \cdots + \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}} \\
&= \frac{1}{x^2}\frac{c_2x^2}{x^2} + \frac{-\frac{c_4}{c_2}x^2}{x^2+\frac{c_4}{c_2}} + \frac{-\frac{c_6}{c_4}}{1+\frac{c_6}{c_4x^2}} + \frac{-\frac{c_8}{c_6x^2}}{1+\frac{c_8}{c_6x^2}} + \cdots + \frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}} + \cdots + \frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}}
\end{aligned}$$

$$=\frac{1}{x^2}\frac{c_2x^2}{x^2}\,\underset{+}{}\,\frac{-\frac{c_4}{c_2}x^2}{x^2+\frac{c_4}{c_2}}+\frac{-\frac{c_6}{c_4}x^2}{x^2+\frac{c_6}{c_4}}+\frac{-\frac{c_8}{c_6}}{1+\frac{c_8}{c_6x^2}}+\cdots+\frac{-\frac{c_{2i}}{c_{2(i-1)}x^2}}{1+\frac{c_{2i}}{c_{2(i-1)}x^2}}+\cdots+\frac{-\frac{c_{2n}}{c_{2(n-1)}x^2}}{1+\frac{c_{2n}}{c_{2(n-1)}x^2}}$$

$$=\cdots\ \cdots\ \cdots$$

$$=\frac{1}{x^2}\frac{c_2x^2}{x^2}\,\underset{+}{}\,\frac{-\frac{c_4}{c_2}x^2}{x^2+\frac{c_4}{c_2}}+\frac{-\frac{c_6}{c_4}x^2}{x^2+\frac{c_6}{c_4}}+\frac{-\frac{c_8}{c_6}x^2}{x^2+\frac{c_8}{c_6}}+\cdots+\frac{-\frac{c_{2i}}{c_{2(i-1)}}x^2}{x^2+\frac{c_{2i}}{c_{2(i-1)}}}+\cdots+\frac{-\frac{c_{2n}}{c_{2(n-1)}}x^2}{x^2+\frac{c_{2n}}{c_{2(n-1)}}}\,. \quad (3.5)$$

$$=\frac{1}{x^2}\frac{c_2x^2}{x^2+\overset{n}{\underset{i=2}{K}}\frac{-\frac{c_{2i}}{c_{2(i-1)}}x^2}{x^2+\frac{c_{2i}}{c_{2(i-1)}}}}=\frac{1}{x^2}\frac{c_2x^2}{x^2+0+\overset{n}{\underset{i=2}{K}}\frac{-\frac{c_{2i}}{c_{2(i-1)}}x^2}{x^2+\frac{c_{2i}}{c_{2(i-1)}}}}$$

The right-side of (3.3) is equal to

$$\frac{1}{x^2}\overset{n}{\underset{i=1}{K}}\frac{a_ix^2}{x^2+b_i}=\frac{1}{x^2}\frac{a_1x^2}{x^2+b_1+\overset{n}{\underset{i=2}{K}}\frac{a_ix^2}{x^2+b_i}}, \qquad x\neq 0\,. \quad (3.6)$$

Thus,

$$a_1=c_2,\ b_1=0,$$
$$a_i=-\frac{c_{2i}}{c_{2(i-1)}},b_i=\frac{c_{2i}}{c_{2(i-1)}}=-a_i,\quad i=2,3,\cdots,n.$$

Then, it is obviously true that

$$\frac{1}{x^2}\overset{n}{\underset{i=1}{K}}\frac{a_ix^2}{x^2+b_i}=\frac{1}{x^2}\frac{a_1x^2}{x^2+\overset{n}{\underset{i=2}{K}}\frac{a_ix^2}{x^2-a_i}}, \qquad x\neq 0\,. \quad (3.7)$$

The proof of Theorem 3.1 is complete.

4. A new continued fraction approximation for the psi function

In this section, we present a new continued fraction approximation for the psi function using our main method and one remark.

Theorem 4.1. We have a new continued fraction approximation for the psi function:

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \frac{1}{x^2} \operatorname*{K}_{i=1}^{\infty} \frac{a_i x^2}{x^2 + b_i}$$

$$= \ln x + \frac{1}{2x} - \frac{1}{x^2} \cfrac{a_1 x^2}{x^2 + b_1 + \cfrac{a_2 x^2}{x^2 + b_2 + \cfrac{a_3 x^2}{x^2 + b_3 + \ddots}}} \tag{4.1}$$

where

$$\begin{aligned} & a_1 = \frac{B_2}{2},\ b_1 = 0, \\ & a_i = -\frac{(i-1)B_{2i}}{iB_{2(i-1)}},\ b_i = -a_i,\ \ i = 2, 3, \cdots \end{aligned}.$$

Proof. Let

$$c_{2i} = \frac{B_{2i}}{2i}, \qquad i = 1, 2, 3, \cdots. \tag{4.2}$$

From (4.2) and Theorem 3.1,

$$\sum_{i=1}^{\infty} \frac{c_{2i}}{x^{2i}} = \sum_{i=1}^{\infty} \frac{B_{2i}}{2ix^{2i}} = \frac{1}{x^2} \operatorname*{K}_{i=1}^{\infty} \frac{a_i x^2}{x^2 + b_i}, \tag{4.3}$$

where

$$\begin{aligned} & a_1 = c_2 = \frac{B_2}{2},\ b_1 = 0, \\ & a_i = -\frac{c_{2i}}{c_{2(i-1)}} = -\frac{(i-1)B_{2i}}{iB_{2(i-1)}}, b_i = \frac{c_{2i}}{c_{2(i-1)}} = -a_i,\ \ i = 2, 3, \cdots \end{aligned}.$$

According to (2.1) and (4.3),

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \sum_{i=1}^{\infty} \frac{B_{2i}}{2ix^{2i}} = \ln x + \frac{1}{2x} - \frac{1}{x^2} \mathop{K}_{i=1}^{\infty} \frac{a_i x^2}{x^2 + b_i}. \tag{4.4}$$

Thus, our new continued fraction approximation can be obtained.

Remark 4.1. As you can see, our new continued fraction approximation for the psi function is equal to (2.1) but the expression is totally different.

From (3.7), we have another expression of (4.4) as follows:

$$\psi\left(x+1\right) \approx \ln x + \frac{1}{2x} - \frac{1}{x^2} \frac{a_1 x^2}{x^2 + \mathop{K}_{i=2}^{n} \frac{a_i x^2}{x^2 - a_i}}$$

$$= \ln x + \frac{1}{2x} - \frac{1}{x^2} \cfrac{a_1 x^2}{x^2 + \cfrac{a_2 x^2}{x^2 - a_2 + \cfrac{a_3 x^2}{x^2 - a_3 + \ddots}}}, \tag{4.5}$$

where

$$a_1 = \frac{1}{12}, a_2 = \frac{1}{10}, a_3 = \frac{10}{21}, a_4 = \frac{21}{20}, a_5 = \frac{20}{11}, a_6 = \frac{7601}{2730}, a_7 = \frac{2730}{691}, \cdots.$$

For the convenience of readers, we rewrite.

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \frac{1}{x^2} \cfrac{\frac{1}{12}x^2}{x^2 + \cfrac{\frac{1}{10}x^2}{x^2 - \frac{1}{10} + \cfrac{\frac{10}{21}x^2}{x^2 - \frac{10}{21} + \cfrac{\frac{21}{20}x^2}{x^2 - \frac{21}{20} + \cfrac{\frac{20}{11}x^2}{x^2 - \frac{20}{11} + \ddots}}}}}. \tag{4.6}$$

5. Conclusion

As mentioned above, in our investigation, we have provided a generally applicable and useful method to construct continued fraction and have successfully found its applications.

Two approximations for the gamma function are simply represented by continued fractions and then all parameters of continued fractions are clearly determined by Bernoulli numbers.

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