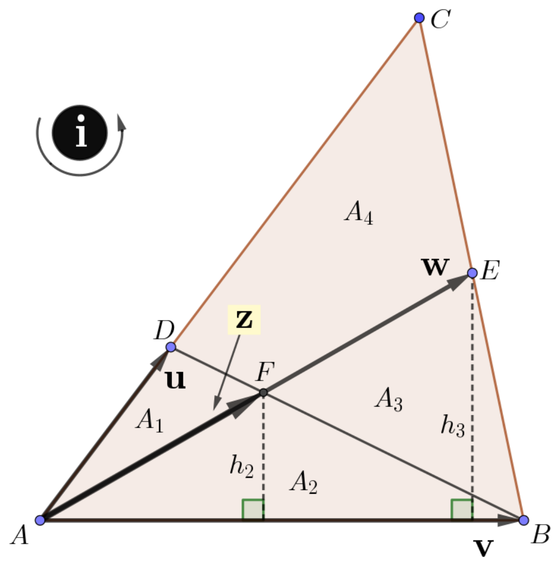
A Geometric Algebra solution to a "Divided Triangle" Problem

March 5, 2024

James Smith

Abstract

We show how to use properties of Geometric Algebra bivectors to solve the following problem: "A triangle is divided into three smaller triangles and a quadrilateral by two lines drawn from vertices to the opposite sides. Given only the areas of the three triangles, find the area of the quadrilateral."



Given the areas A_1, A_2, and A_3, find A_4.

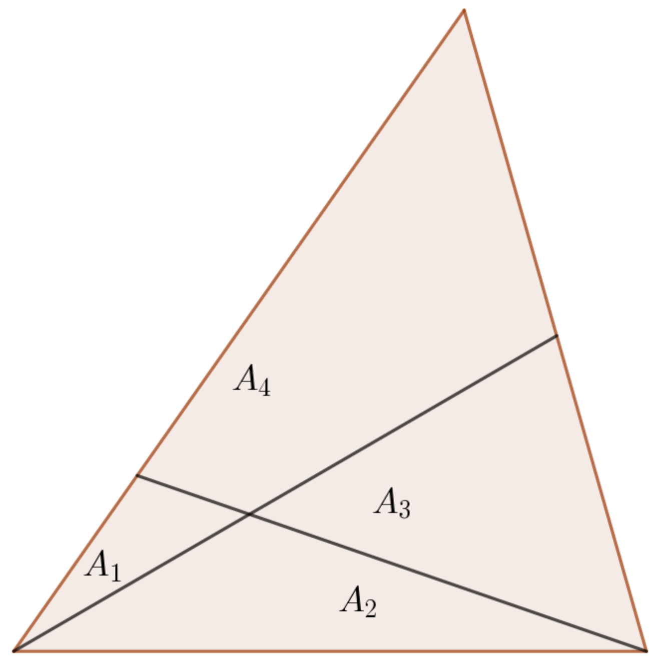


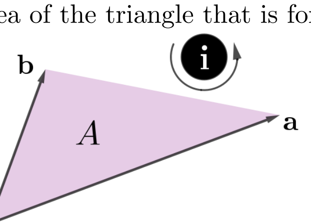
Figure 1: Given areas A_1, A_2, and A_3, find A_4.

1 Statement of the Problem

Given areas A_1, A_2, and A_3, find A_4 (Fig. 1).

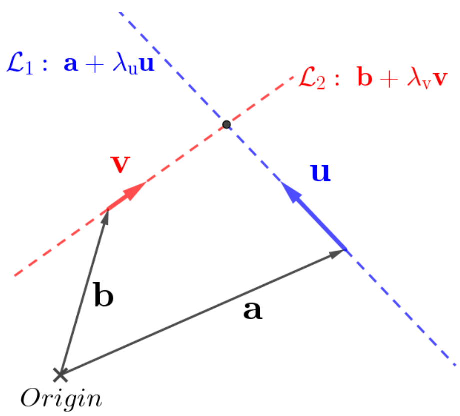
2 Ideas that We Will Use

1. The relationship between the outer product of two vectors and the oriented area of the triangle that is formed by them.

   $\mathbf{a} \wedge \mathbf{b} = 2A\mathbf{i}$; $\mathbf{b} \wedge \mathbf{a} = -2A\mathbf{i}$

2. How to find the intersection of two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ that are parameterized as $\mathcal{L}_1 = \mathbf{a} + \lambda_u \mathbf{u}$ and $\mathcal{L}_2 = \mathbf{b} + \lambda_v \mathbf{v}$, in which λ_u and λ_v are scalars (Fig. 2). We will see that in the present problem, we will not need to carry out the full procedure.



At the intersection point,

$$\mathbf{a} + \lambda_u \mathbf{u} = \mathbf{b} + \lambda_v \mathbf{v}$$

$$\therefore \; (\mathbf{a} + \lambda_u \mathbf{u}) \wedge \mathbf{v} = (\mathbf{b} + \lambda_u \mathbf{v}) \wedge \mathbf{v}$$

$$\lambda_u \mathbf{u} \wedge \mathbf{v} = (\mathbf{b} - \mathbf{a}) \wedge \mathbf{v}$$

$$\textit{and} \;\; \lambda_u = [(\mathbf{b} - \mathbf{a}) \wedge \mathbf{v}](\mathbf{u} \wedge \mathbf{v})^{-1}.$$

Figure 2: Method for finding the intersection point of two lines that are parameterized as $\mathcal{L}_1 = \mathbf{a} + \lambda_u \mathbf{u}$ and $\mathcal{L}_2 = \mathbf{b} + \lambda_v \mathbf{v}$, in which λ_u and λ_v are scalars. Shown is the detailed procedure for finding the value of λ_u of the intersection point: the value of λ_v for that point can be found by taking the outer product of both sides of $\mathcal{L}_1 = \mathbf{a} + \lambda_u \mathbf{u} = \mathcal{L}_2 = \mathbf{b} + \lambda_v \mathbf{v}$ with $\mathbf{u}$.

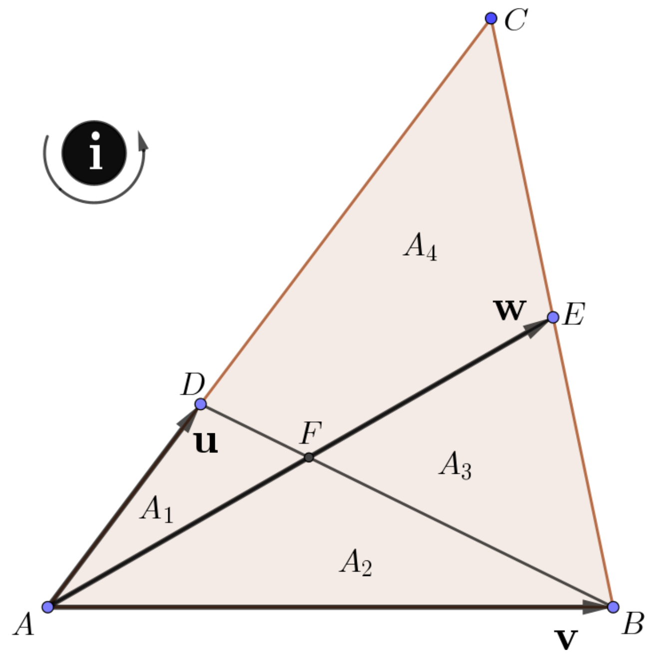


Figure 3: Formulation of the problem in terms of vectors.

3 Solution Strategy

We will express each of the three given areas, and the total area of $\triangle ABC$, in terms of outer products, then find A_4 as $Total\ -\ A_1 - A_2 - A_3$.

4 Formulation in Terms of Vectors

We formulate the problem as shown in Fig. 3.

5 Solution

To identify the area of the triangle $\triangle ABC$, we will need to express the vector $\mathbf{c}$ from A to C in terms of the given areas (Fig. 4). To do so, we begin by expressing point C as the intersection of lines:

$$\begin{aligned} \mathbf{c} &= \lambda_u \mathbf{u} \text{ and} \\ \mathbf{c} &= \mathbf{v} + \lambda_{wv} (\mathbf{w} - \mathbf{v})\,; \\ \therefore \lambda_u \mathbf{u} &= \mathbf{v} + \lambda_{wv} (\mathbf{w} - \mathbf{v})\,. \end{aligned}$$

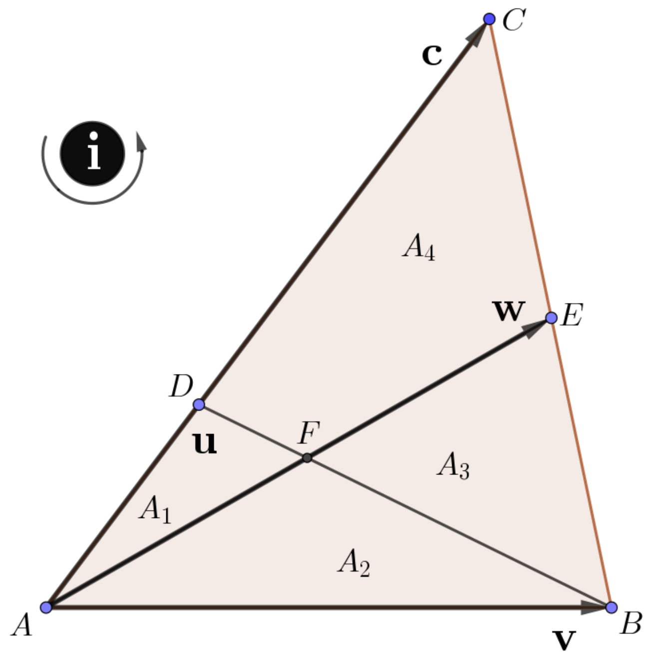


Figure 4: Showing the vector **c** that we will use to express the area of $\triangle ABC$, via $\mathbf{v}\wedge\mathbf{c} = 2\left[Area\ of\ \triangle ABC\right]\mathbf{i}$.

We find λ_u as follows:

$$\begin{aligned}\lambda_u \mathbf{u}\wedge(\mathbf{w}-\mathbf{v}) &= \mathbf{v}\wedge(\mathbf{w}-\mathbf{v}) + \lambda_{wv}(\mathbf{w}-\mathbf{v})\wedge(\mathbf{w}-\mathbf{v})\\ \lambda_u\mathbf{u}\wedge\mathbf{w} - \lambda_u\mathbf{u}\wedge\mathbf{v} &= \mathbf{v}\wedge\mathbf{w}. \end{aligned} \tag{5.1}$$

Now, we note that

$$\begin{aligned}\mathbf{u}\wedge\mathbf{v} &= -2\left(A_1 + A_1\right)\mathbf{i}, \text{and}\\ \mathbf{v}\wedge\mathbf{w} &= 2\left(A_2 + A_3\right)\mathbf{i}\end{aligned}$$

As explained in Fig. 5, $\mathbf{u}\wedge\mathbf{w} = -2\left[A_1\left(A_2 + A_3\right)/A_2\right]\mathbf{i}$. Thus, Eq. (5.1) becomes

$$\lambda_u\left\{-2\left[\frac{A_2 + A_3}{A_2}\right]A_1\right\}\mathbf{i} - \lambda_u\left[-2\left(A_1 + A_2\right)\right]\mathbf{i} = 2\left(A_2 + A_3\right)\mathbf{i},$$

from which

$$\lambda_u = \frac{A_2\left(A_2 + A_3\right)}{A_2^2 - A_1 A_3}.$$

Note that we have been able to identify λ_u without having to use the full procedure that is shown in Fig. 2.

Thus,

$$\begin{aligned}\mathbf{v}\wedge\mathbf{c} &= \mathbf{v}\wedge\left\{\left[\frac{A_2\left(A_2 + A_3\right)}{A_2^2 - A_1 A_3}\right]\mathbf{u}\right\}\\ 2\left(A_1 + A_2 + A_3 + A_4\right)\mathbf{i} &= \left[\frac{A_2\left(A_2 + A_3\right)}{A_2^2 - A_1 A_3}\right]\mathbf{v}\wedge\mathbf{u}\\ &= \left[\frac{A_2\left(A_2 + A_3\right)}{A_2^2 - A_1 A_3}\right]\left[2\left(A_1 + A_2\right)\right]\mathbf{i}.\end{aligned}$$

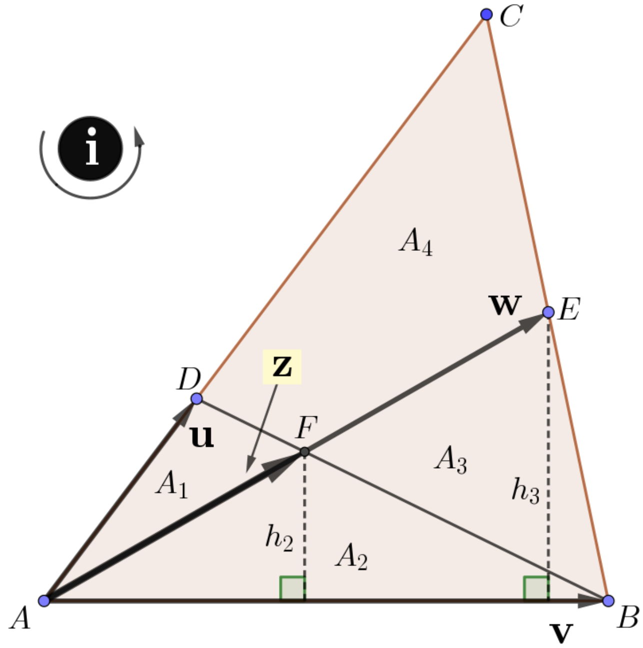


Figure 5: Obtaining an expression for $\mathbf{u} \wedge \mathbf{w}$. Vector $\mathbf{w}$ is $\frac{h_3}{h_2}\mathbf{z}$. Because $\triangle ABF$ and $\triangle ABE$ have the same base ($\overline{AB}$), $\frac{h_3}{h_2} = \frac{A_3 + A_2}{A_2}$. In addition, $\mathbf{z} \wedge \mathbf{u} = 2A_1\mathbf{i}$. Therefore, $\mathbf{u} \wedge \mathbf{w} = -2\left[\frac{A_2 + A_3}{A_2}\right] A_1$.

Solving for A_4,

$$A_4 = \frac{A_1 A_3 \left(A_1 + 2A_2 + A_3\right)}{A_2^2 - A_1 A_3}. \tag{5.2}$$