Supportive intersection

Bin Wang (汪 镔)

February 25, 2024

Abstract

Let X be a differentiable manifold. Let $\mathscr{D}'(X)$ be the space of currents, and $S^{an}(X)$ the Abelian group freely generated by analytic cells, i.e. the pairs of a polyhedron Π and a real analytic map $\Pi \to X$, that can be extended to a real analytic embedding of neighborhood of Π. In this paper, we define a bilinear map

$$\begin{array}{ccc} S^{an}(X) \times S^{an}(X) & \rightarrow & \mathscr{D}'(X) \\ (\sigma_1, \sigma_2) & \rightarrow & [\sigma_1 \wedge \sigma_2] \end{array} \tag{0.1}$$

such that

1) the support of $[\sigma_1 \wedge \sigma_2]$ is contained in the set-intersection of the supports of σ_1, σ_2;
2) if σ_1, σ_2 are closed, $[\sigma_1 \wedge \sigma_2]$ is also closed and its cohomology class is the cup-product of the cohomology classes of σ_1, σ_2.

We call elements in S^{an} the chains, and the current $[\sigma_1 \wedge \sigma_2]$ the supportive intersection of the chains.

Contents

Key words: de Rham's regularization, support, intersection
2020 Mathematics subject classification Primary: 58A25, 58A12, 28A25, 28A33

1 Introduction

Let X be a differentiable manifold. Consider two types of intersections based on singular chains: 1) cup-product of the cohomology ring; 2) set-intersection of the supports of chains. While the cup-product is more structure-oriented, the set-intersection is entirely object-oriented and requires nothing more than a set. As the relation between these two extremes is rather obscured, we would like to raise a question:

To what extent, is the cup-product related to the set-intersection ?

To answer the question, in this paper we are going to set-up the tool, the supportive intersection. Explicitly, we are going to construct a bilinear map as the intersection,

$$\begin{array}{ccc} S^{an}(X)\times S^{an}(X) & \to & \mathscr{D}'(X) \\ (\sigma_1,\sigma_2) & \to & [\sigma_1\wedge\sigma_2] \end{array} \tag{1.1}$$

such that

Condition 1.1. *the support of $[\sigma_1 \wedge \sigma_2]$ is contained in the set-intersection of the supports of σ_1, σ_2;*

Condition 1.2. *if σ_1, σ_2 are closed, $[\sigma_1 \wedge \sigma_2]$ is also closed and its cohomology class is the cup-product of the cohomology classes of σ_1, σ_2.*

The idea of the construction is based on de Rham's work on currents. Originally in order to understand the homology of the complex of currents, de Rham constructed, for an arbitrary current T, the regularization $R_\epsilon T$ that is a family of C^∞ forms for a real number $\epsilon > 0$, weakly converging to T as $\epsilon \to 0$. Among many other properties, the regularization in particular satisfies that

1) the support of $R_\epsilon T$ is contained in any given neighborhood of the support of T provided ϵ is sufficiently small;
2) there exists another operator A_ϵ on currents such that

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT \tag{1.2}$$

where b is the *boundary* operator on currents.

So if we can define the intersection as the weak limit of the currents

$$\sigma_1 \wedge R_\epsilon(\sigma_2), \text{(see [1] for the notation } \wedge)$$

for $\epsilon \to 0$, Condition 1.1 and Condition 1.2 are immediate consequences of these two properties. This is our assertion of the main theorem.

Theorem 1.3. *(Main theorem) Let σ_1, σ_2 be two chains in $S^{an}(X)$. The following current,*

$$\sigma_1 \wedge R_\epsilon(\sigma_2) \tag{1.3}$$

as $\epsilon \to 0$, converges weakly to a current. Furthermore, the weak limit of (1.3) denoted by $[\sigma_1 \wedge \sigma_2]$ satisfies Condition 1.2 and Condition 1.3.

As the conditions 1.2, 1.3 follow easily from the properties of de Rham's regularization, the only remaining difficulty is the convergence of (1.3) which is the main focus of this paper. But the conventional convergence in any piecewise Euclidean structure will fail. So the central point of our technique is to interpret the convergence of (1.3) as the convergence of measures. Then by the portemanteau theorem about measure convergence (in probability theory), we obtain the limit in terms of measures (see Example 4.3).

The paper is organized as follows. In Section 2, we review the de Rham's regularization and give a further description of its kernel. In Section 3, we show the convergence of (1.3). In section 4, we verify that the convergence of (1.3) has the properties for the supportive intersection

2 De Rham's Regularization

• De Rham's construction

We start with de Rham's regularization in [1], but with our own interpretation. [*]

Definition 2.1. *Let X be a differentiable manifold. Let ϵ be a small positive number. Linear operators R_ϵ and A_ϵ:*

$$\mathscr{D}'(X) \to \mathscr{D}'(X)$$

are called the regulator and homotopy operator respectively if for $T \in \mathscr{D}'(X)$ they satisfy

(1) a homotopy formula

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT. \tag{2.1}$$

where b is the boundary operator.

(2) $supp(R_\epsilon T), supp(A_\epsilon T)$ are contained in any given neighborhood of $supp(T)$ provided ϵ is sufficiently small.

(3) $R_\epsilon T$ is C^∞;

(4) $A_\epsilon T$ is C^r, provided T is C^r;

(5) $R_\epsilon\phi, A_\epsilon\phi$ are bounded, provided that a smooth differential form ϕ varies in a bounded set and ϵ is bounded above;

(6)

$$\lim_{\epsilon\to 0} R_\epsilon T = T, \quad \lim_{\epsilon\to 0} A_\epsilon T = 0$$

in the weak topology of $\mathscr{D}'(X)$.

[*] All mistakes belong to us.

Theorem 2.2. *(G. de Rham) The operators R_ϵ, A_ϵ exist.*

Proof. In the following, we'll review the construction but omit the verification whose detail is given in §15, [1]. There are three steps in de Rham's original construction.

Step 1: Local construction. Use bump functions to construct an operator in $X = \mathbb{R}^m$ to regularize the current.

Step 2: Preparation for the extension to global. Apply step 1 to construct an operator that regularizes the current at the interior points of a bounded domain B in the chart, but remains to be the identity outside.

Step 3: From local to global. Assume X is covered by countably many such domains B_i that are locally finite. Then take the infinite composition to extend the local operators to the global operator,

$$R_\epsilon, A_\epsilon \tag{2.2}$$

Step 1: Let $X = \mathbb{R}^m$ be the Euclidean space of dimension m with the standard linear structure. Let $x_1, \cdots, x_m$ be its standard coordinates, and vectors and points in $\mathbb{R}^m$ will be denoted by the **bold** letters.

Let $f(\mathbf{x}) \in C_c^\infty(\mathbb{R}^m)$ satisfy

$$\int_{\mathbf{x}\in\mathbb{R}^m} f(\mathbf{x}) d\mu = 1, \tag{2.3}$$

where μ is the Lebesgue measure, $d\mu$ is the volume form

$$dx_1 \wedge \cdots \wedge dx_m.$$

Let

$$\vartheta_1(\mathbf{x}) = f(\mathbf{x})d\mu, \qquad \vartheta_\epsilon(\mathbf{x}) = \vartheta_1(\frac{\mathbf{x}}{\epsilon})$$

be the m-forms on $\mathbb{R}^m$.

The construction is based on the general form of a map $s_{\mathbf{y}}(\mathbf{x})$ as follows. Let

$$s_{\mathbf{y}}(\mathbf{x})$$

be C^∞ maps parametrized by $\mathbf{y} \in \mathbb{R}^m$,

$$\begin{array}{ccc} \mathbb{R}^m & \to & \mathbb{R}^m \\ \mathbf{x} & \to & s_{\mathbf{y}}(\mathbf{x}) \end{array}$$

such that all partial derivatives of the components with respect to the variables of $\mathbf{x}$ are continuous functions in $(\mathbf{x}, \mathbf{y})$. Let ϕ be a test form on $\mathbb{R}^m$.

Let T be a homogeneous current of degree p on $\mathbb{R}^m$. Then de Rham defined operators R_ϵ, A_ϵ of currents by the functional

$$\begin{cases} R_\epsilon T[\phi] = T\left[\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \phi\big(s_{\mathbf{y}}(\mathbf{x})\big)\right], \\ A_\epsilon T[\phi] = T\left[\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \int_{t=0}^{t=1} \phi\big(s_{t\mathbf{y}}(\mathbf{x})\big)\right] \end{cases} \tag{2.4}$$

where ϕ is a test form, and T is evaluated at the forms of $\mathbf{x}$ variables. We shall note that

(1) the continuity assumption about $s_{\mathbf{y}}(\mathbf{x})$ guarantees the existence of the first of (2.4),

(2) $\int_{t=0}^{t=1} \phi\big(s_{t\mathbf{y}}(\mathbf{x})\big)$ is the fibre integral along the t variable. So

$$\begin{cases} dim(R_\epsilon(T)) = dim(T), \\ dim(A_\epsilon(T)) = dim(T) - 1. \end{cases}$$

If furthermore the map

$$\begin{array}{ccc} \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^m \times \mathbb{R}^m \\ (\mathbf{x}, \mathbf{y}) & \to & (\mathbf{x}, s_{\mathbf{y}}(\mathbf{x})) \end{array}$$

is a diffeomorphism, we denote the inverse map by

$$(\mathbf{y}, \mathbf{x}) \to \big(\mathbf{y}, g(\mathbf{x}, \mathbf{y})\big)$$

(we switch the letters $\mathbf{x}$ and $\mathbf{y}$) to obtain a C^∞ form

$$R_\epsilon T = T_{\mathbf{y}}\left[\vartheta_\epsilon\big(g(\mathbf{x}, \mathbf{y})\big)\right] \tag{2.5}$$

where $T_{\mathbf{y}}$ is the evaluation of T at the form in $\mathbf{y}$ variables.

Next we use the specific map

$$s_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{y}, \tag{2.6}$$

where the $+$ is from the standard linear structure of $\mathbb{R}^m$. Then

$$R_\epsilon T = T_{\mathbf{y}}\left[\vartheta_\epsilon\big(\mathbf{x} - \mathbf{y}\big)\right] \tag{2.7}$$

Then all properties in Definition 2.1 are satisfied. We refer the proof to [1].

Note: The local construction in this step is well known (see [2]). Next we'll see the global extension which is the main focus of this paper.

Step 2: Choose the unit ball $B \subset \mathbb{R}^m$ diffeomorphic to $\mathbb{R}^m$. Let h be the specific diffeomorphism

$$\mathbb{R}^m \quad \to \quad B,$$

defined on p66, [1]. Then we define the new C^∞ map

$$s_{\mathbf{y}}(\mathbf{x}) = \begin{cases} h s^B_{\mathbf{y}} h^{-1}(\mathbf{x}) & \text{for} \quad \mathbf{x} \in B \\ \mathbf{x} & \text{for} \quad \mathbf{x} \notin B \end{cases} \tag{2.8}$$

where $s^B_{\mathbf{y}}$ is the specific map (2.6) as in Step 1 for $B \simeq \mathbb{R}^m$. We would like to point out that $s_{\mathbf{y}}(\mathbf{x})$ satisfies assumption. Then we can define the operators $R^B_\epsilon, A^B_\epsilon$ depending on B in the same way (with a test form ϕ):

$$\begin{cases} R^B_\epsilon T[\phi] = T\left[\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \phi\big(s_{\mathbf{y}}(\mathbf{x})\big)\right], \\ A^B_\epsilon T[\phi] = T\left[\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \int_{t=0}^{t=1} \phi\big(s_{t\mathbf{y}}(\mathbf{x})\big)\right]. \end{cases} \tag{2.9}$$

Then the operators $R^B_\epsilon, A^B_\epsilon$ will satisfy
(a) properties (1), (4), (5) and (6) in Definition 2.1.
(b) $R^B_\epsilon(T)$ is C^∞ in B, $R^B_\epsilon(T) = T$ in the complement of $\overline{B}$;
We refer the verification to [1].

Step 3: Cover the X with countably many, locally finite open sets B_i. Now we regard each B^i as a subset of B in step 2. Let a neighborhood U_i of B_i. Let h_i be a diffeomorphism

$$\begin{array}{ccc} U_i & \rightarrow & \mathbb{R}^m \\ \cup & & \cup \\ B_i & \rightarrow & B. \end{array}$$

Let $g_i \geq 0$ be a function on X, which is 1 on B_i and supported in U_i. Let $T' = g_i T$ and $T'' = T - T'$. Then we let

$$R^i_\epsilon T = h_i^{-1} \circ R^B_\epsilon \circ h_i(T') + T''$$
$$A^i_\epsilon T = h_i^{-1} \circ A^B_\epsilon \circ h_i(T').$$

Finally we extend it from local to global by taking the composition,

$$\begin{aligned} R^{(N)}_\epsilon &= R^1_\epsilon \circ \cdots \circ R^N_\epsilon, \\ A^{(N)}_\epsilon &= R^1_\epsilon \circ \cdots \circ R^{N-1}_\epsilon \circ A^N_\epsilon. \end{aligned} \tag{2.10}$$

Then the limits

$$R_\epsilon := \lim_{N\to\infty} R^{(N)}_\epsilon$$
$$A_\epsilon := \sum_{N=1}^{\infty} A^{(N)}_\epsilon$$

exist and satisfy all properties in Definition 2.1. We refer the verification to [1]. †

□

†In [1], for each open set U_i there is a different positive ϵ_i. We used the same number ϵ for all U_i.

Definition 2.3. *(de Rham's regularization)*

(a) *We call R_ϵ in Theorem 2.2 the de Rham's regulator, the regularization the de Rham's regularization.*

(b) *We define the de Rham data to be all items in the construction of de Rham's regularization.*

Remark According to the definition de Rham data always exist. However, they can't be chosen canonically. Hence the supportive intersection that will be defined later is not canonical.

• C^∞ **Kernel of the de Rham's regulator**

G. de Rham further showed in chapter III, §17, [1],

Corollary 2.4. *De Rham's operator R_ϵ constructed in Theorem 2.2 is a regularizing operator which is defined to be an operator expressed by a C^∞ form $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ on $X \times X$, called the C^∞ kernel of R_ϵ, as*

$$R_\epsilon T = T_{\mathbf{y}} \left[\varrho_\epsilon(\mathbf{x}, \mathbf{y}) \right]$$

for any current T, where the current's evaluation $T_{\mathbf{y}}$ of T on $\mathbf{y}$-form is defined as in Theorem 9, [1] through a double form.

Remark The general definition of kernels is attached in the Appendix in which de Rham shows a general operator from currents to forms has a smooth kernel. In particular, the de Rham's regularization has a smooth kernel. In the following, we'll go further to show this kernel has a particular type of local property that allows the convergence of (1.3).

Definition 2.5. *(local blow-up family of forms)*

Let ω_ϵ for $\epsilon > 0$ be a family of smooth forms of degree p (any whole number) on an Euclidean space $\mathbb{R}^n$. If there are a decomposition $\mathbb{R}^n \simeq \mathbb{R}^p \times \mathbb{R}^{n-p}$ with Euclidean coordinates $(\mathbf{u}, \mathbf{v})$ and a smooth form $\omega_1(\mathbf{u}, \mathbf{v})$ on $\mathbb{R}^n$ with a compact support such that

$$\omega_\epsilon(\mathbf{u}, \mathbf{v}) = \omega_1(\frac{\mathbf{u}}{\epsilon}, \mathbf{v}), \tag{2.11}$$

then ω_ϵ is called a blow-up family of ω_1 along $\mathbb{R}^p$.

Remark The blow-up family is well-known in a special case where $p = n$ (for instance, see [1] or [2]).

Theorem 2.6. *Let X be a differentiable manifold of degree m. Let ϱ_ϵ be the C^∞ kernel of the de Rham's regulator R_ϵ with de Rham data. Then around each point, there is a chart U such that $\varrho_\epsilon|_U$ is a local blow-up.*

Proof. We'll show the blow-up structure comes from a fibre integral. Let $q \in X$, and U_q be a small neighborhood of q. Also we may assume q does not belong to the boundary of each ball B_i in the de Rham data (because the collection of those points has Lebesgue measure 0). Consider the kernel ϱ_ϵ of the de Rham's regulator

$$R_\epsilon = R_\epsilon^1 \circ \cdots \circ R_\epsilon^N \tag{2.12}$$

restricted to $U_q \times U_q$, where N is finite. Each operator $R_\epsilon^i, i = 1, \cdots, N$ regularizes inside an Euclidean ball B_i. Since R_ϵ^i remains to be the identity outside of B_i, we may only consider the regularization inside of B_i. We denote those balls by $B_1, B_2, \cdots, B_n$. Let's denote the coordinates for each B_i by the letter $\mathbf{x}_i$, and the second copy of B_i by $\mathbf{y}_i$ (as in (2.7)). Then according to de Rham's construction the smooth kernel of each R_ϵ^i is the pullback form

$$\vartheta_1^i(\frac{\mathbf{x}_i}{\epsilon} \overset{i}{-} \frac{\mathbf{y}_i}{\epsilon}). \tag{2.13}$$

This is equivalent to have the operator on a current T

$$R_\epsilon^i T = \int_{\mathbf{y}_i \in T} \vartheta_\epsilon^i(\mathbf{x}_i \overset{i}{-} \mathbf{y}_i)$$

where the subtraction $\overset{i}{-}$, also addiction $\overset{i}{+}$ and the scalar multiplication $\frac{\mathbf{x}_i}{\epsilon}$, are the particular linear structures of $U_i \supset B_i$ in the de Rham data. Notice that the local expression for the composition of R_ϵ^i is just the fibre integral. Precisely, the kernel ϱ_ϵ of R_ϵ inside $B_1 \cap \cdots \cap B_n$ is the degree m form that can be calculated by the fibre integral

$$\varrho_\epsilon = \int_{(\mathbf{x}_2, \cdots, \mathbf{x}_n) \in \prod_{n-1} \mathbb{R}^m} \vartheta_\epsilon^1(\mathbf{x}_1 \overset{1}{-} \mathbf{x}_2) \wedge \vartheta_\epsilon^2(\mathbf{x}_2 \overset{2}{-} \mathbf{x}_3) \wedge \cdots$$
$$\wedge \vartheta_\epsilon^{n-1}(\mathbf{x}_{n-1} \overset{n-1}{-} \mathbf{x}_n) \wedge \vartheta_\epsilon^n(\mathbf{x}_n \overset{n}{-} \mathbf{y}_n) \tag{2.14}$$

where $\vartheta_\epsilon^i(\mathbf{x}_i \overset{i}{-} \mathbf{x}_{i+1})$ is regarded as the pullback of the kernel of each R_ϵ^i (see 2.13) to the product space $\prod_{n+1} \mathbb{R}^m$. So the kernel ϱ_ϵ is a m-form on the product $\mathbb{R}^m \times \mathbb{R}^m$ where $\mathbf{x}_1, \mathbf{y}_n$ are the coordinates for the first and second factor respectively. We express (2.14) in the new coordinates

$$\mathbf{w}_i = \mathbf{x}_i \overset{i}{-} \mathbf{x}_{i+1}, \tag{2.15}$$

where $i = 2, \cdots, n, \mathbf{x}_{n+1} = \mathbf{y}_n$. Also

$$\mathbf{x}_1 \overset{1}{-} \mathbf{x}_2 = \mathbf{x}_1 \overset{1}{-} \left(\mathbf{y}_n \overset{n}{+} (\mathbf{w}_n \overset{n-1}{+} \cdots \overset{3}{+} \mathbf{w}_3 \overset{2}{+} \mathbf{w}_2)\right). \tag{2.16}$$

Next we notice that for each fixed $\mathbf{x} \in \mathbb{R}^m$, the map

$$\begin{array}{ccc} \mathbb{R}^m & \to & \mathbb{R}^m \\ y & \to & x+y \end{array} \tag{2.17}$$

is a diffeomorphism. Hence there is a C^∞ map

$$\begin{array}{ccc} k : \prod_{n-1} \mathbb{R}^m & \to & \mathbb{R}^m \\ (\mathbf{w}_n, \cdots, \mathbf{w}_2) & \to & \mathbf{w}_n \overset{n-1}{+} \cdots \overset{3}{+} \mathbf{w}_3 \overset{2}{+} \mathbf{w}_2 \end{array} \tag{2.18}$$

such that

$$\mathbf{x}_1 \overset{1}{-} \mathbf{x}_2 = \mathbf{x}_1 \overset{1}{-} \mathbf{y}_n \overset{1}{-} k(\mathbf{w}_n, \cdots, \mathbf{w}_2). \tag{2.19}$$

Then

$$\varrho_\epsilon = \int_{(\mathbf{w}_2, \cdots, \mathbf{w}_n) \in \prod_{n-1} \mathbb{R}^m} \vartheta_\epsilon^1 \left(\mathbf{x}_1 \overset{1}{-} \mathbf{y}_n \overset{1}{-} k(\mathbf{w}_n, \cdots, \mathbf{w}_2) \right) \wedge \vartheta_\epsilon^2(\mathbf{w}_2) \wedge \cdots \wedge \vartheta_\epsilon^{n-1}(\mathbf{w}_n)) \tag{2.20}$$

Let's see the fibre integral (2.17) is a pullback form from $\mathbb{R}^m$. Recall $\mathbf{x}$ denotes the points of $\mathbb{R}^m$. If we let

$$F_\epsilon(\mathbf{x}) = \int_{(\mathbf{w}_2, \cdots, \mathbf{w}_n) \in \prod_{n-1} \mathbb{R}^m} \vartheta_\epsilon^1 \left(\mathbf{x} \overset{1}{-} k(\mathbf{w}_n, \cdots, \mathbf{w}_2) \right) \wedge \vartheta_\epsilon^2(\mathbf{w}_2) \wedge \cdots \wedge \vartheta_\epsilon^{n-1}(\mathbf{w}_n) \tag{2.21}$$

be the m form on $\mathbb{R}^m$, then

$$\varrho_\epsilon(\mathbf{x}_1, \mathbf{y}_n) = \kappa^*(F_\epsilon) \tag{2.22}$$

where κ is the map: $(\mathbf{x}_1, \mathbf{y}_n) \to \mathbf{x}_1 \overset{1}{-} \mathbf{y}_n$. So ϱ_ϵ is a blow-up of a compactly supported form $F_1(\mathbf{x})$.

□

Example 2.7. *Let $X = \mathbb{R}^m$ have the standard coordinates $\mathbf{x}$. Let μ be the Lebesgue measure of $\mathbb{R}^m$. Let $f(\mathbf{x})$ be a C^∞ function of $\mathbb{R}^m$ with a compact support in a ball of the origin such that*

$$\int_{\mathbb{R}^m} f(\mathbf{x}) d\mu = 1.$$

So $(\mathbb{R}^m, f)$ is the de Rham data of X. For a positive number ϵ, the kernel ϱ_ϵ of the de Rham's regulator is then the blow-up family

$$s^* \left(\frac{1}{\epsilon^m} f(\frac{\mathbf{x}}{\epsilon}) d\mu \right).$$

where $d\mu$ is the volume form and s is the map sending $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^m$ to $\mathbf{x} - \mathbf{y} \in \mathbb{R}^m$.

3 Convergence of the regularization

Now we step back to focus on the particular types of currents: the chains in S^{an}. Notice that the convergence (1.3) only concerns the local Euclidean space. So we focus on an Euclidean space.

In general, we denote the Lebesgue measure on an Euclidean space $\mathbb{R}^l$ by $\mu_{\mathbf{w}}$ where $\mathbf{w}$ is the standard coordinate or a point. We abuse the notation to denote the volume form with the maximal degree in the coordinates and the volume element in the Lebesgue integral by the same expression $d\mu_{\mathbf{w}}$. In the context, the current of the integration over a set σ is also denoted by σ.

Lemma 3.1. *Let Π_{m+r} be an $m+r$ dimensional polyhedron in $\mathbb{R}^{2m}$. Let ω_ϵ be a blow-up family of forms of degree $p \neq 0$ in $\mathbb{R}^{2m}$. Then the current*

$$\Pi_{m+r} \wedge \omega_\epsilon \tag{3.1}$$

converges weakly to a functional as $\epsilon \to 0$.

Proof. Let ϕ be a test form on $\mathbb{R}^{2m}$. Then it suffices to show the convergence of the numbers

$$\int_{\Pi_{m+r}} \omega_\epsilon \wedge \phi \tag{3.2}$$

We assume $\omega_1 = w_1(\mathbf{u}_1, \mathbf{u}_2) d\mu_{(\mathbf{u}_1,\mathbf{u}_2)}$ where w_1 is a smooth function with a compact support in $\mathbb{R}^{2m} = \mathbb{R}^p \times \mathbb{R}^{2m-p}$, and $\mathbf{u}_1, \mathbf{u}_2$ are the coordinates of $\mathbb{R}^p, \mathbb{R}^{2m-p}$. Let $D_{\frac{1}{\epsilon}}$ be the scalar multiplication on the $\mathbf{u}_1$ component by the number $\frac{1}{\epsilon}$. Next we make a change of variables

$$\begin{aligned} \frac{\mathbf{u}_1}{\epsilon} &\Rightarrow \mathbf{u}_1 \\ \mathbf{u}_2 &\Rightarrow \mathbf{u}_2 \end{aligned} \tag{3.3}$$

where the left column represents the old variables and right column represents the new variables. Then (3.2) is equal to

$$\int_{D_{\frac{1}{\epsilon}}(\Pi_{m+r})} \omega_1(\mathbf{u}_1, \mathbf{u}_2) \wedge \phi(\epsilon \mathbf{u}_1, \mathbf{u}_2) \tag{3.4}$$

Since $\omega_1(\mathbf{u}_1, \mathbf{u}_2)$ is bounded, the variable $\mathbf{u}_1$ in the integral in (3.4) is also bounded. Then since $\omega_1(\mathbf{u}_1, \mathbf{u}_2)$ is bounded, due to the uniform continuity of ϕ, the limit of (3.4) as $\epsilon \to 0$ is the same as

$$\int_{D_{\frac{1}{\epsilon}}(\Pi_{m+r})} \omega_1(\mathbf{u}_1, \mathbf{u}_2) \wedge \phi(\mathbf{0}, \mathbf{u}_2). \tag{3.5}$$

Notice the integrand gives a rise to a smooth form independent of ϵ. Then the convergence of it as $\epsilon \to 0$ will be implied by the weak convergence of the measures obtained as the restricted Lebesgue measures to the set $D_{\epsilon^{-1}}(\Pi_{m+r})$. Let's work with measures. Let $\mathcal{R}$ be a ray starting in the space. Since Π_{m+r} is a convex set, the intersection

$$\mathcal{R} \cap \Pi_{m+r}$$

is an interval on the ray. Hence

$$D_{\epsilon^{-1}}(\mathcal{R} \cap \Pi_{m+r}) \subset D_{(\epsilon')^{-1}}(\mathcal{R} \cap \Pi_{m+r}), \quad \text{for } \epsilon' < \epsilon$$

Now taking the union of all rays, we obtain

$$D_{\epsilon^{-1}}(\Pi_{m+r}) \subset D_{(\epsilon')^{-1}}(\Pi_{m+r}), \quad \text{for } \epsilon' < \epsilon. \tag{3.6}$$

Taking the union $\cup_{\epsilon \in (0,1]} \Big(D_{\epsilon^{-1}}(\Pi_{m+r}) \Big)$, we obtain the measurable set

$$\mathcal{D}_0 := \lim_{\epsilon \to 0} D_{\epsilon^{-1}}(\Pi_{m+r}).$$

We denote the Lebesgue measure restricted to the set $\Big(D_{\epsilon^{-1}}(\Pi_{m+r}) \Big)$ by μ_ϵ, to $\mathcal{D}_0$ by μ_0. Then (3.6) implies the measure μ_ϵ converges to the measure μ_0 setwisely. By the portmanteau theorem (Theorem 13.16, [3]), the setwise convergence implies the weak convergence of measures. Hence (3.4) converges.

□

The convergence in Lemma 3.1 only holds for polyhedrons. Next proposition goes further to address a cell, i.e. a cell with non-linear characteristic map.

Lemma 3.2. *Let $c \in S^{an}(\mathbb{R}^{2m})$. Assume that around any point of $\bar{c}$, there is a particular type of Taylor expansion*

$$c(\mathbf{u}) = c(\mathbf{0}) + B(\mathbf{u}) + \mathcal{O}(2) \tag{3.7}$$

where $\mathbf{0}$ is translated to the center of the expansion and B is a non-zero Jacobian matrix such that if $B(\mathbf{u}) = 0$, then $c(\mathbf{u}) = 0$. Let ω_ϵ be a blow-up of degree m as in Lemma 3.1. Then current, $c \wedge \omega_\epsilon$ converges weakly to a functional as $\epsilon \to 0$.

Proof. By the linearlity of c, it suffices to deal with the case when c is a single cell. So assume

$$c : \Pi_{m+r} \to \mathbb{R}^{2m} \tag{3.8}$$

can be extended a neighborhood of Π_{m+r}, whose the affine variable is denoted by $\mathbf{u}$. Let $c\wedge\omega_\epsilon$ be the functional

$$\begin{array}{c} \phi \\ \downarrow \\ \int_{\Pi_{m+r}} c^*(\omega_\epsilon\wedge\phi) \end{array} \tag{3.9}$$

where ϕ is a test form of the degree r. As in Lemma 3.1, we may assume the smooth form ω_1 in $\mathbb{R}^m\subset\mathbb{R}^{2m}$ is written as $\omega_1 = w_1(\mathbf{x})d\mu_{\mathbf{x}}$ for some volume form in coordinate $\mathbf{x}$ of $\mathbb{R}^m$ of degree m. Let $\mathbf{y}$ be the coordinate of the orthogonal subspace $(\mathbb{R}^m)^\perp$ of $\mathbb{R}^m\subset\mathbb{R}^{2m}$. Also we may assume the form ϕ is simple, i.e. it can be expressed as $\psi(\mathbf{x},\mathbf{y})d\mu_{\mathbf{z}}$ where $\mathbf{z}$ is the coordinate for a subspace of $(\mathbb{R}^m)^\perp$. Then we can write

$$\int_{\Pi_{m+r}} c^*(\omega_\epsilon)\wedge\phi = \int_{\mathbf{u}\in\Pi_{m+r}} \epsilon^{-m} w_1\Big(\epsilon^{-1}\mathbf{x}(\mathbf{u})\Big)\psi\Big(\mathbf{x}(\mathbf{u}),\mathbf{y}(\mathbf{u})\Big)d\mu_{\mathbf{x}(\mathbf{u})}d\mu_{\mathbf{z}(\mathbf{u})} \tag{3.10}$$

where the expression of $c:\mathbf{u}\to(\mathbf{x}(\mathbf{u}),\mathbf{y}(\mathbf{u}))$. Next we expand the differential map

$$\begin{array}{ccc} c:\Pi_{m+r} & \to & \mathbb{R}^{2m} \\ \cap & & \\ \mathbb{R}^{m+r}. & & \end{array} \tag{3.11}$$

Since we can always translate the center to the origin, so we may assume the center of the expansion is the origin. We'll make an appropriate linear transformation of $\mathbf{u}$ such that the first order of $\mathbf{x}(\mathbf{u})$ is $\mathbf{u}_1$ and $\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2$. We note length of the vector $\mathbf{u}_1$ is not zero. With this setting we write down the Taylor expansion

$$\mathbf{x}(\mathbf{u}) = \mathbf{a}_0 + (\mathbf{u}_1,\mathbf{0}) + \sum_{i=1}^{l} u_1^i\mathbf{k}_1(\mathbf{u}_1,\mathbf{u}_2) + \mathbf{k}_2(\mathbf{u}_1,\mathbf{u}_2) \tag{3.12}$$

where $(u_1^1,\cdots,u_1^l)=\mathbf{u}_1$ and $\mathbf{k}_2(\mathbf{0},\mathbf{u}_2)\neq 0$. By the assumption, $\mathbf{k}_2(\mathbf{u}_1,\mathbf{u}_2)=0$. We obtain the differential

$$d\mathbf{x}(\mathbf{u}) = (d\mathbf{u}_1,\mathbf{0}) + \sum_{i=1}^{l} du_1^i\mathbf{k}_1(\mathbf{u}_1,\mathbf{u}_2) + \sum_{i=1}^{l} u_1^i d\mathbf{k}_1(\mathbf{u}_1,\mathbf{u}_2). \tag{3.13}$$

Then the exterior product is

$$d\mu_{\mathbf{x}(\mathbf{u})} = \sum_{i_1,j,i_2} u_{(1,m-j,i_3)}\xi_{(i_1,j,i_2)}(\mathbf{u})d\mu_{\mathbf{u}_1,i_1,j}\wedge d\mu_{\mathbf{u}_2,i_2,m-j} \tag{3.14}$$

where $u_{(1,m-j,i_3)}$ is the product of $m-j$ many variables from $\mathbf{u}_1$, $d\mu_{\mathbf{u}_1,i_1,j}$ is a volume form of degree j in Euclidean variables from $\mathbf{u}_1$, $d\mu_{\mathbf{u}_2,i_2,m-j}$ is a volume form of degree $m-j$ in variables from $\mathbf{u}_2$, $\xi_{(i_1,j,i_2)}(\mathbf{u})$ is a real analytic function that does not vanish at $\mathbf{u} = 0$. Now we can write down (3.11) to have

$$\begin{gathered} \int_{\Pi_{m+r}} c^*(\omega_\epsilon) \wedge \phi \\ \| \\ \sum_{i_1, i_2 i_3, j} \int_{\mathbf{u} \in \Pi_{m+r}} \epsilon^{-m} u_{(1,m-j,i_3)} w_1 \Big(\epsilon^{-1} \mathbf{x}(\mathbf{u}) \Big) \psi \Big(\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{u}) \Big) \xi_{(i_1,j,i_2)}(\mathbf{u}) \\ d\mu_{\mathbf{u}_1, i_1, j} \wedge d\mu_{\mathbf{u}_2, i_2, m-j} \wedge d\mu_{\mathbf{z}(\mathbf{u})} \end{gathered} \tag{3.15}$$

Next we make a change of variables

$$\begin{gathered} \frac{\mathbf{u}_1}{\epsilon} \Rightarrow \mathbf{u}_1 \\ \mathbf{u}_2 \Rightarrow \mathbf{u}_2 \end{gathered} \tag{3.16}$$

where the left column represents the old variables and right column represents the new variables. Then

$$\begin{gathered} (3.15) \\ \| \\ \sum_{i_1, i_2, i_3, j} \int_{\mathbf{u} \in D_{\frac{1}{\epsilon}}(\Pi_{m+r})} u_{(1,m-j,i_3)} w_1 \left(\frac{\mathbf{a}_0}{\epsilon} + (\mathbf{u}_1, \mathbf{0}) + \sum_{i=1}^{l} u_1^i \mathbf{k}_1(\epsilon \mathbf{u}_1, \mathbf{u}_2) \right) \\ \psi \Big(\mathbf{x}(\epsilon \mathbf{u}_1, \mathbf{u}_2), \mathbf{y}(\epsilon \mathbf{u}_1, \mathbf{u}_2) \Big) \xi_{(i_1,j,i_2)}(\epsilon \mathbf{u}_1, \mathbf{u}_2) \\ d\mu_{\mathbf{u}_1, i_1, j} \wedge d\mu_{\mathbf{u}_2, i_2, m-j} \wedge d\mu_{\mathbf{z}(\mathbf{u})} \end{gathered} \tag{3.17}$$

where $D_{\frac{1}{\epsilon}}$ is the scalar multiplication on the $\mathbf{u}_1$ component by the number $\frac{1}{\epsilon}$. If $\mathbf{a}_0 \neq 0$, then because w_1 has a bounded support,

$$w_1 \left(\frac{\mathbf{a}_0}{\epsilon} + (\mathbf{u}_1, \mathbf{0}) + \sum_{i=1}^{l} u_1^i \mathbf{k}_1(\epsilon \mathbf{u}_1, \mathbf{u}_2) \right)$$

is 0 for a sufficiently small ϵ. Hence (3.15) is 0 for all sufficiently small ϵ. If $\mathbf{a} = 0$, we observe the expansion of

$$\mathbf{y}(\epsilon \mathbf{u}_1, \mathbf{u}_2).$$

Since

$$\mathbf{u} \to (\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{u}))$$

is an embedding and ψ has a compact support, $\mathbf{u}_2$ in the integral (3.17) is bounded. Similarly, since w_1 has a compact support, $\mathbf{u}_1$ in the integral (3.17) is also bounded. Hence the variable $\mathbf{u}$ in the integral (3.17) is bounded. Then since the integrand in (3.17) is bounded, the limit of (3.17) is equal to

$$\begin{gathered} \lim_{\epsilon \to 0} \int_{\mathbf{u} \in D_{\frac{1}{\epsilon}}(\Pi_{m+r})} \sum_{i_1, i_2, i_3, j} u_{(1,m-j,i_3)} w_1 \left((\mathbf{u}_1, \mathbf{0}) + \sum_{i=1}^{l} u_1^i \mathbf{k}_1(\mathbf{0}, \mathbf{u}_2) \right) \\ \psi \Big(\mathbf{x}(\mathbf{0}, \mathbf{u}_2), \mathbf{y}(\mathbf{0}, \mathbf{u}_2) \Big) \xi_{(i_1,j,i_2)}(\mathbf{0}, \mathbf{u}_2) \\ d\mu_{\mathbf{u}_1, i_1, j} \wedge d\mu_{\mathbf{u}_2, i_2, m-j} \wedge d\mu_{\mathbf{z}(\mathbf{u})} \end{gathered} \tag{3.18}$$

Denote the form

$$\sum_{i_1,i_2,i_3,j} u_{(1,m-j,i_3)} w_1\Bigg((\mathbf{u}_1,\mathbf{0})+\sum_{i=1}^{l} u_1^i \mathbf{k}_1(\mathbf{0},\mathbf{u}_2)\Bigg)$$
$$\psi\Bigg(\mathbf{x}(\mathbf{0},\mathbf{u}_2),\mathbf{y}(\mathbf{0},\mathbf{u}_2)\Bigg)\xi_{(i_1,j,i_2)}(\mathbf{0},\mathbf{u}_2)$$
$$d\mu_{\mathbf{u}_1,i_1,j}\wedge d\mu_{\mathbf{u}_2,i_2,m-j}\wedge d\mu_{\mathbf{z}(\mathbf{u})}$$

by $\alpha(\mathbf{u}_1,\mathbf{u}_2)$. Then (3.18) is equal to

$$\Pi_{m+r}\wedge\alpha(\epsilon^{-1}\mathbf{u}_1,\mathbf{u}_2) \tag{3.19}$$

Notice $\alpha(\epsilon^{-1}\mathbf{u}_1,\mathbf{u}_2)$ is a family of blow-up forms. By Lemma 3.1, we complete the proof.

□

The following example is a well-known local regularization in cohomology theory (see [3]).

Example 3.3. *Let c have the dimension n and contain the origin of $\mathbb{R}^n$, and blow-up forms ω_ϵ have the top degree n along $\mathbb{R}^n$. Then as $\epsilon\to 0$, $c\wedge\omega_\epsilon$ converges weakly to a constant multiple of the delta function at the origin.*

The following proposition proves the first part of Main theorem 1.3.

Proposition 3.4. *Let X be a manifold of dimension m. For chains σ_1,σ_2 in $S^{an}(X)$ such that $dim(\sigma_1)+dim(\sigma_2)\geq m$, the exterior product*

$$\sigma_1\wedge R_\epsilon(\sigma_2)$$

weakly converges to a current as $\epsilon\to 0$, for the fixed de Rham data.

Proof. (1) Let ϕ be a test form. Then

$$\big(\sigma_1\wedge R_\epsilon(\sigma_2)\big)[\phi]=\int_{\sigma_1} R_\epsilon\sigma_2\wedge\phi=\int_{\sigma_1\times\sigma_2}\varrho_\epsilon(\mathbf{x},\mathbf{y})\wedge\phi(\mathbf{y}), \tag{3.20}$$

where $\mathbf{x},\mathbf{y}$ are the local coordinates for the first and second X in $X\times X$. By Theorem 2.6, the kernel $\varrho_\epsilon(\mathbf{x},\mathbf{y})$ of R_ϵ is a local blow-up whose local structure consists of locally finite open covering subsets U of X and a subspace

$$V\simeq\mathbb{R}^m\subset U\times U$$

such that

$$\varrho_\epsilon(\mathbf{x},\mathbf{y})|_{U\times U}=\varrho_1(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{y}}{\epsilon})|_{U\times U}=\pi^*(\theta(\frac{\mathbf{v}}{\epsilon})), \tag{3.21}$$

where $\pi : U \times U \to V$ is a C^∞ map to $V \simeq \mathbb{R}^m$, θ is a C^∞ m-form on V and $\mathbf{v}$ is the point in V. By a partition of unity it suffices to focus on one open set U. Precisely it suffices to show the convergence of the real numbers

$$\int_{\sigma_1|_U \times \sigma_2|_U} \pi^*(\theta(\frac{\mathbf{v}}{\epsilon})) \wedge \phi \tag{3.22}$$

where $\sigma_1|_U \times \sigma_2|_U$ is the restriction to $U \times U \simeq \mathbb{R}^{2m}$. Notice that $\pi^*(\theta(\frac{\mathbf{v}}{\epsilon}))$ is a blow-up and $\sigma_1|_U \times \sigma_2|_U$ is a C^∞ chain of dimension $m + r$. Then according to Lemma 3.2, the convergence follows. We complete the proof.

(2) Let ϕ be an element of a subset of $\mathscr{D}(X)$ bounded to any orders. Applying a partition of unity, we may address it on the sufficiently small local chart U only. By observing the local expression (3.18) where the bound of the function ψ is a multiple of $||\phi||_\infty$. Thus, we obtain that

$$\int_{\sigma_1} R_\epsilon(\sigma_2) \wedge \phi$$

is bounded by a multiple of $||\phi||_\infty$. Hence globally,

$$\lim_{\epsilon \to 0} \int_{\sigma_1} R_\epsilon \sigma_2 \wedge \phi$$

is bounded for ϕ in the bounded set. This shows the weak limit

$$\lim_{\epsilon \to 0} \big(\sigma_1 \wedge R_\epsilon(\sigma_2)\big)$$

is a continuous functional, thus a current.

□

4 The supportive intersection

Definition 4.1. *Let σ_1, σ_2 be two chains in S^{an} where X is a differentiable manifold equipped with de Rham data. We define*

$$[\sigma_1 \wedge \sigma_2]$$

to be the weak limit

$$\lim_{\epsilon \to 0} \big(\sigma_1 \wedge R_\epsilon \sigma_2\big)$$

where $R_\epsilon \sigma_2$ is the de Rham's regularization associated to the given de Rham data.

Remark The notation does not specify the de Rham data which plays an important role in the determination of the supportive intersection. Because of this role, the supportive intersection is not invariant of any structures, but its existence is a C^∞ invariant.

Property 4.2.
Let X a differentiable manifold of dimension m equipped with de Rham data. For chains σ_1, σ_2 in S^{an}, the intersection $[\sigma_1 \wedge \sigma_2]$ satisfies:

(1) (Supportivity)

$$supp([\sigma_1 \wedge \sigma_2]) \subset supp(\sigma_1) \cap supp(\sigma_2). \tag{4.1}$$

(2) (Closedness) The intersection current $[\sigma_1 \wedge \sigma_2]$ is closed if σ_1, σ_2 are.

(3) (Cohomologicity) According to the de Rham's theory in [1], the homology of the complex of currents coincides with the singular cohomology with real coefficients. Hence we use $\langle \sigma \rangle$ to denote the singular cohomology class represented by a closed current σ, and $\smile$ the cup-product. If σ_1, σ_2 are closed, then in singular cohomology

$$\langle [\sigma_1 \wedge \sigma_2] \rangle = \langle \sigma_1 \rangle \smile \langle \sigma_2 \rangle \tag{4.2}$$

(4) (Leibniz rule) If $deg(\sigma_1) = p$, then the differential map of chains follows Leibniz rule,

$$d[\sigma_1 \wedge \sigma_2] = [d\sigma_1 \wedge \sigma_2] + (-1)^p [\sigma_1 \wedge d\sigma_2], \tag{4.3}$$

where the differential map d is the operator $(-1)^{p+1} b$ for the boundary operator b acting on chains of the codimension p.

Proof. (1) Suppose

$$\mathbf{a} \notin supp(\sigma_1) \cap supp(\sigma_2).$$

Then $\mathbf{a}$ must be either outside of $supp(\sigma_1)$ or outside of $supp(\sigma_2)$. Let's assume first it is not in $supp(\sigma_2)$. Since the support of a currents is closed, we choose a small neighborhood $U_{\mathbf{a}}$ of $\mathbf{a}$ in X, but disjoint from $supp(\sigma_2)$. Let ϕ be a C^∞-form of X with a compact support in $U_{\mathbf{a}}$. According to Definition 2.1, when ϵ is small enough $R_\epsilon(\sigma_2)$ is zero in $U_{\mathbf{a}}$. Hence

$$[\sigma_1 \wedge \sigma_2][\phi] = 0. \tag{4.4}$$

Hence $\mathbf{a} \notin supp([\sigma_1 \wedge \sigma_2])$. If $\mathbf{a} \notin supp(\sigma_1)$, $U_{\mathbf{a}}$ can be chosen disjoint with $supp(\sigma_1)$. Then since $\phi \in \mathscr{D}(U_{\mathbf{a}})$ is a C^∞-form of X with a compact support in $U_{\mathbf{a}}$ disjoint with $supp(\sigma_1)$, the restriction of ϕ to σ_1 is zero. Hence

$$[\sigma_1 \wedge \sigma_2][\phi] = 0.$$

Then $\mathbf{a} \notin supp([\sigma_1 \wedge \sigma_2])$. Thus

$$\mathbf{a} \notin supp(\sigma_1) \cap supp(\sigma_2)$$

will always imply

$$\mathbf{a} \notin supp([\sigma_1 \wedge \sigma_2]).$$

This completes the proof.

(2) Let ϕ be a test form. By the definition

$$\begin{aligned} & b[\sigma_1 \wedge \sigma_2][\phi] \\ & = \lim_{\epsilon \to 0} \sigma_1[R_\epsilon \sigma_2 \wedge d\phi] \\ & = \pm \lim_{\epsilon \to 0} \sigma_1[dR_\epsilon \sigma_2 \wedge \phi] \\ & = \pm \lim_{\epsilon \to 0} \sigma_1[bR_\epsilon \sigma_2 \wedge \phi] \end{aligned} \tag{4.5}$$

According to the homotopy formula (2.1)

$$bR_\epsilon \sigma_2 - b\sigma_2 = bbA_\epsilon \sigma_2 - bA_\epsilon b\sigma_2 \tag{4.6}$$

Because σ_2 is closed,

$$bR_\epsilon \sigma_2 = 0.$$

So $[\sigma_1 \wedge \sigma_2]$ is closed.

(3) Let ϕ be a closed C^∞ form of degree $deg(\sigma_1)+deg(\sigma_2)$, and has a compact support. Denote the cohomology class by $\langle \cdot \rangle$. The intersection number,

$$deg\Big(\big\langle [\sigma_1 \wedge \sigma_2] \big\rangle \smile \langle \phi \rangle\Big) \tag{4.7}$$

is a well-defined real number that is equal to

$$\lim_{\epsilon \to 0} \sigma_1[R_\epsilon(\sigma_2) \wedge \phi]. \tag{4.8}$$

By the definition in §20, [1], (4.7) is the de Rham's symbol

$$\Big(\sigma_1 \wedge (\sigma_2 \wedge \phi)\Big)[1].$$

which by de Rham is the intersection number

$$deg\Big((\langle \sigma_1 \rangle \smile \langle \sigma_2 \rangle) \smile \langle \phi \rangle \Big). \tag{4.9}$$

The formulas (4.7) and (4.9) yield

$$\big\langle [\sigma_1 \wedge \sigma_2] \big\rangle = \langle \sigma_1 \rangle \smile \langle \sigma_2 \rangle.$$

(4) (Leibniz Rule) Let $\phi \in \mathscr{D}(\mathcal{X})$ be a test form. Let

$$deg(T_1) = p, deg(T_2) = q.$$

Then

$$\begin{aligned}
& b[\sigma_1 \wedge \sigma_2](\phi) \\
& = \lim_{\epsilon \to 0} \int_{\sigma_1} R_\epsilon \sigma_2 \wedge d\phi \\
& (\text{ Leibniz Rule for } C^\infty \text{ forms }) \\
& = \lim_{\epsilon \to 0} \int_{\sigma_1} (-1)^q d(R_\epsilon \sigma_2 \wedge \phi) + (-1)^{q+1} dR_\epsilon \sigma_2 \wedge \phi \\
& = \lim_{\epsilon \to 0} (-1)^q \int_{b\sigma_1} R_\epsilon \sigma_2 \wedge \phi + \lim_{\epsilon \to 0} (-1)^{q+1} \int_{\sigma_1} dR_\epsilon \sigma_2 \wedge \phi \\
& \lim_{\epsilon \to 0} (-1)^q \int_{b\sigma_1} R_\epsilon \sigma_2 \wedge \phi + \lim_{\epsilon \to 0} (-1)^{q+1} \int_{\sigma_1} R_\epsilon (d\sigma_2) \wedge \phi \\
& = (-1)^q [b\sigma_1 \wedge \sigma_2][\phi] + (-1)^{q+1} [\sigma_1 \wedge d\sigma_2][\phi]
\end{aligned}$$

Hence

$$b[\sigma_1 \wedge \sigma_2] = (-1)^q [b\sigma_1 \wedge \sigma_2] + (-1)^{q+1} [\sigma_1 \wedge d\sigma_2]. \tag{4.10}$$

After change the sign, we found (4.10) is the same as (4.3).

□

The properties (1) and (3) together with Proposition 3.4 complete the proof of Main theorem 1.3.

Example 4.3. *Let $\mathcal{X} = \mathbb{R}^2$, and $\mathcal{U}$ the de Rham data consisting of single chart $\mathbb{R}^2$ with the bump function function f satisfying*

$$\int_{\mathbb{R}^2} f(x_1, x_2) dx_1 \wedge dx_2 = 1 \tag{4.11}$$

where x_1, x_2 are coordinates of $\mathbb{R}^2$. Let σ be a piece of the parabola

$$x_1 = x_2^2 \tag{4.12}$$

containing the origin $\mathbf{0}$. Since σ has dimension 1, $[\sigma \wedge \sigma]$ exists as a 0-dimensional current. Let $\phi(x)$ be a test function. Denote the coordinates for the second copy *of $\mathbb{R}^2$ by y_1, y_2. Then the regularization is the fibre integral (integration along the y_1, y_2)*

$$R_\epsilon(\sigma) = \frac{1}{\epsilon^2}\int_{(y_1,y_2)\in\sigma} f(\frac{x_1-y_1}{\epsilon}, \frac{x_2-y_2}{\epsilon})(dx_1-dy_1)\wedge(dx_2-dy_2) \tag{4.13}$$

which is a C^∞ 1-form in variables x_1, x_2. Then we calculate

$$\begin{array}{c} \int_{[\sigma\wedge\sigma]}\phi \\ \| \\ \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_{(x_1,x_2)\in\sigma}\int_{(y_1,y_2)\in\sigma} f(\frac{x_1-y_1}{\epsilon}, \frac{x_2-y_2}{\epsilon})\phi(x_1,x_2)(dx_1-dy_1)\wedge(dx_2-dy_2)\end{array} \tag{4.14}$$

Then the functional

$$\begin{array}{c} \phi \\ \downarrow \\ \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_{(x_2,y_2)\in I\times I} f(\frac{x_2^2-y_2^2}{\epsilon}, \frac{x_2-y_2}{\epsilon})\phi(x_2^2,x_2)(2y_2-2x_2)dx_2\wedge dy_2\end{array} \tag{4.15}$$

is the current $[\sigma\wedge\sigma]$ of degree 0, where ϕ is a test function on $\mathbb{R}^2$ and I is the given interval. According to Lemma 3.2, the limit in (4.15) exists and is equal to a Lebesgue integral over a measurable set.

Remark This example also shows the intersection $[\bullet\wedge\bullet]$ depends on de Rham data $\mathcal{U}$.

Example 4.4. *We give multiple cases where the supportive intersections are independent of de Rham data. All of them are known as Kronecker index in [1]. Let $\mathcal{X} = \mathbb{R}^2$. Let $\mathcal{U}$ be the de Rham data consisting of the single chart $\mathbb{R}^2$ with a bump function f satisfying*

$$\int_{\mathbb{R}^2} f(x_1,x_2)dx_1\wedge dx_2 = 1 \tag{4.16}$$

where x_1, x_2 are the coordinates of $\mathbb{R}^2$.

Case 1: Let σ_1 be a line through the origin $\mathbf{0}$ and σ_2 is another line segment through the origin. Then

$$[\sigma_1\wedge\sigma_2] = \delta_{\mathbf{0}}$$

if the order of "$\wedge$" matches to the orientation of $\mathbb{R}^2$, where $\delta_{\mathbf{0}}$ is the delta-function at the point $\mathbf{0}$.

Case 2: Continuing from the setting in case 1, let σ_2 be the line $x_1 = 0$. Let σ_1 be a piece of parabola

$$x_1 = x_2^2, x_2\in(-1,1). \tag{4.17}$$

Denote the second copy of $\mathbb{R}^2$ for the de Rham's regularization by (y_1, y_2). The regularization is the fibre integral along y_2,

$$R_\epsilon(\sigma_2) = \frac{1}{\epsilon^2}\int_{y_2\in\mathbb{R}} f(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} - \frac{y_2}{\epsilon})dy_2 \wedge dx_1. \tag{4.18}$$

To calculate $[\sigma_1 \wedge \sigma_2]$, let $\phi(x)$ be a test function supported in a neighborhood of the origin. Then

$$\begin{aligned} &\int_{[\sigma_1\wedge\sigma_2]} \phi \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_{x_1\in\sigma_1}\int_{y_2\in\mathbb{R}} f(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} - \frac{y_2}{\epsilon})\phi(x_1, x_2)dy_2 \wedge dx_1. \end{aligned} \tag{4.19}$$

Let

$$f_1(x_1) = \int_{y_2\in\mathbb{R}} f(x_1, -y_2)dy_2.$$

Now we continue to have

$$\begin{array}{c} \int_{[\sigma_1\wedge\sigma_2]} \phi \\ \| \\ \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{(x_1,x_2)\in T_1} f_1(\frac{x_1}{\epsilon})\phi(x_1, x_2)dx_1 \\ \| \\ \phi(\mathbf{0})\left(\int_{+\infty}^{0} f_1(x_1)dx + \int_0^{+\infty} f_1(x_1)dx_1\right) = 0, \end{array} \tag{4.20}$$

So

$$[\sigma_1 \wedge \sigma_2] = 0.$$

Case 3: Continuing from the setting in case 2, let σ_2 be the line $x_1 = 0$. Let σ_1 be a piece of the cubic curve

$$x_1 = x_2^3, x_2 \in (-1, 1). \tag{4.21}$$

The same calculation in case 2 shows if order of σ_1, σ_2 is concordant with the orientation of $\mathbb{R}^2$, then

$$[\sigma_1 \wedge \sigma_2] = \delta_{\mathbf{0}} \tag{4.22}$$

where $\delta_{\mathbf{0}}$ is the δ-function at the origin.

Appendix

A Kernel

In [1] G. de Rham defined the notion of "regularizing operator" which includes the de Rham's regulator R_ϵ. Let X, Y be two manifolds. Let $\mathscr{T} \in \mathscr{D}'(X \times Y)$. Then $\mathscr{T}$ derives an operator

$$\Lambda ; \mathscr{D}(X) \quad \rightarrow \quad \mathscr{D}'(Y) \tag{A.1}$$

We call $\mathscr{T}$ the kernel of Λ. Conversely given a homomorphism Λ, there is a kernel $\mathscr{T}$ on $X \times Y$. Notice

$$\begin{array}{cc} \mathscr{D}(X), & \mathscr{E}(Y) \\ \cap & \cap \\ \mathscr{E}'(X), & \mathscr{D}'(Y) \end{array} \tag{A.2}$$

where $\mathscr{E}(\bullet)$ is the set of C^∞ forms, and $'$ is the topological dual.

Definition A.1. *If operator Λ has an extension*

$$\Lambda : \mathscr{E}'(X) \quad \rightarrow \quad \mathscr{E}(Y) \tag{A.3}$$

we say Λ is regularizing.

Theorem A.2. *(G. de Rham)*
Λ is regularizing if and only if the kernel $\mathscr{T}$ is a C^∞ form on $X \times Y$. In particular R_ϵ is regularizing.

References

[1] G. de Rham, *Differential manifold*, English translation of "Variétés différentiables", Springer-Verlag (1984)

[2] P. Griffiths, J. Harris, *Principle of algebraic geometry*, John Wiley & Sons (1978)

[3] A. Klenke, *Probability Theory*, Springer-Verlag (2006)