Supportive intersection

Bin Wang (汪 镔)

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Abstract

Let X be a differentiable manifold. Let $\mathscr{D}'(X)$ be the space of currents, and $S^\infty(X)$ the Abelian group freely generated by C^∞ cells, i.e. the maps from polyhedrons to X can be extended defferentiablelly to a neighborhoods of the polyhedrons. In this paper, we define a bilinear map

$$\begin{array}{ccc} S^\infty(X)\times S^\infty(X) & \to & \mathscr{D}'(X) \\ (\sigma_1,\sigma_2) & \to & [\sigma_1\wedge\sigma_2] \end{array} \tag{0.1}$$

such that

1) the support of $[\sigma_1\wedge\sigma_2]$ is contained in the set-intersection of the supports of σ_1,σ_2;
2) if σ_1,σ_2 are closed, $[\sigma_1\wedge\sigma_2]$ is also closed and its cohomology class is the cup-product of the cohomology classes of σ_1,σ_2.

We call the current $[\sigma_1\wedge\sigma_2]$ the supportive intersection of σ_1,σ_2.

Contents

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1 Introduction

Let X be a differentiable manifold. Consider two types of intersections based on singular chains: 1) cup-product of the cohomology ring; 2) set-intersection of the supports of chains. While the cup-product is more structure-oriented, the set-intersection is entirely object-oriented and requires nothing more than a set. As the relation between these two extremes is rather obscured, we would like to raise a question:

To what extent, is the cup-product related to the set-intersection ?

To answer the question, in this paper we are going to set-up the tool, the supportive intersection. Specifically, we'll construct a bilinear map as the intersection,

$$\begin{array}{ccc} S^{\infty}(X) \times S^{\infty}(X) & \rightarrow & \mathscr{D}'(X) \\ (\sigma_1, \sigma_2) & \rightarrow & [\sigma_1 \wedge \sigma_2] \end{array} \tag{1.1}$$

such that

Condition 1.1. *the support of* $[\sigma_1 \wedge \sigma_2]$ *is contained in the set-intersection of the supports of* σ_1, σ_2*;*

Condition 1.2. *if* σ_1, σ_2 *are closed,* $[\sigma_1 \wedge \sigma_2]$ *is also closed and its cohomology class is the cup-product of the cohomology classes of* σ_1, σ_2*.*

The idea of the construction is based on de Rham's work on currents. Originally in order to understand the homology of the complex of currents, de Rham constructed, for an arbitrary current T, the regularization $R_\epsilon T$ that is a family of C^∞ forms for a real number $\epsilon > 0$, weakly converging to T as $\epsilon \to 0$. Among many other properties, the regularization in particular satisfies that

1) the support of $R_\epsilon T$ is contained in any given neighborhood of the support of T provided ϵ is sufficiently small;
2) there exists another operator A_ϵ on currents such that

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT \tag{1.2}$$

where b is the *boundary* operator on currents.

So if we can define the intersection as the weak limit of the currents

$$\sigma_1 \wedge R_\epsilon(\sigma_2)$$

for $\epsilon \to 0$, Condition 1.1 and Condition 1.2 are immediate consequences of these two properties. This is our assertion of the main theorem.

Theorem 1.3. *(Main theorem) Let* σ_1, σ_2 *be two singular chains in* $S^\infty(X)$*. The following currents,*

$$\sigma_1 \wedge R_\epsilon(\sigma_2), \tag{1.3}$$

as $\epsilon \to 0$*, converge weakly to a current. Furthermore, the weak limit of (1.3) denoted by* $[\sigma_1 \wedge \sigma_2]$ *satisfies Condition 1.2 and Condition 1.3.*

As the conditions 1.2, 1.3 follow easily from the properties of de Rham's regularization, only remaining difficulty is the convergence of (1.3) which is the main focus of this paper. But the conventional convergence in any piecewise Euclidean structure will fail. So the central point of our technique is to interpret the convergence of (1.3) as the convergence of Lebesgue measures. Then by using the standard theorem, the Portemanteau theorem, we obtain the convergence whose limit is interpreted by Lebesgue measures.

The paper is organized as follows. In Section 2, we review the de Rham's regularization and give a further description of its kernel. In Section 3, we show the convergence of (1.3). In section 4, we verify that the convergence of (1.3) has the properties for the supportive intersection

2 De Rham's Regularization

• De Rham's construction

We start with de Rham's regularization in [1], but with our own interpretation. *

Definition 2.1. *Let X be a differentiable manifold. Let ϵ be a small positive number. Linear operators R_ϵ and A_ϵ:*

$$\mathscr{D}'(X) \to \mathscr{D}'(X)$$

are called the regulator and homotopy operator respectively if for $T \in \mathscr{D}'(X)$ they satisfy

(1) a homotopy formula

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT. \tag{2.1}$$

where b is the boundary operator.

(2) $supp(R_\epsilon T), supp(A_\epsilon T)$ are contained in any given neighborhood of $supp(T)$ provided ϵ is sufficiently small.

(3) $R_\epsilon T$ is C^∞;

(4) $A_\epsilon T$ is C^r, provided T is C^r;

(5) $R_\epsilon\phi, A_\epsilon\phi$ are bounded, provided that a smooth differential form ϕ varies in a bounded set and ϵ is bounded above;

(6)

$$\lim_{\epsilon\to 0} R_\epsilon T = T, \quad \lim_{\epsilon\to 0} A_\epsilon T = 0$$

in the weak topology of $\mathscr{D}'(X)$.

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*All mistakes belong to us.

Theorem 2.2. *(G. de Rham) The operators R_ϵ, A_ϵ exist.*

Proof. In the following, we'll review the construction but omit the verification whose detail is given in §15, [1]. There are three steps in de Rham's original construction.

Step 1: Local construction. Use bump functions to construct an operator in $X = \mathbb{R}^m$ to regularize the current.

Step 2: Preparation for the extension to global. Apply step 1 to construct an operator that regularizes the current at the interior points of a bounded domain B in the chart, but remains to be the identity outside.

Step 3: From local to global. Assume X is covered by countable many such domains B_i that are locally finite. Then take the infinite composition to extend the local operators to the global operator,

$$R_\epsilon, A_\epsilon \tag{2.2}$$

Step 1: Let $X = \mathbb{R}^m$ be the Euclidean space of dimension m with the standard linear structure. Let $x_1, \cdots, x_m$ be its standard coordinates, and vectors and points in $\mathbb{R}^m$ will be denoted by the **bold** letters.

Let $f(\mathbf{x}) \in C_c^\infty(\mathbb{R}^m)$ satisfy

$$\int_{\mathbf{x}\in\mathbb{R}^m} f(\mathbf{x}) d\mu = 1, \tag{2.3}$$

where μ is the Lebesgue measure, $d\mu$ is the volume form

$$dx_1 \wedge \cdots \wedge dx_m.$$

Let

$$\vartheta_1(\mathbf{x}) = f(\mathbf{x}) d\mu, \qquad \vartheta_\epsilon(\mathbf{x}) = \vartheta_1(\frac{\mathbf{x}}{\epsilon})$$

be the m-forms on $\mathbb{R}^m$.

The construction is based on the general form of a map $s_{\mathbf{y}}(\mathbf{x})$ as follows. Let

$$s_{\mathbf{y}}(\mathbf{x})$$

be C^∞ maps parametrized by $\mathbf{y} \in \mathbb{R}^m$,

$$\begin{array}{ccc} \mathbb{R}^m & \rightarrow & \mathbb{R}^m \\ \mathbf{x} & \rightarrow & s_{\mathbf{y}}(\mathbf{x}) \end{array}$$

such that all partial derivatives of the components with respect to the variables of $\mathbf{x}$ are continuous functions in $(\mathbf{x}, \mathbf{y})$. Let ϕ be a test form on $\mathbb{R}^m$.

Let T be a homogeneous current of degree p on $\mathbb{R}^m$. Then de Rham defined operators R_ϵ, A_ϵ of currents by the functional

$$\begin{cases} R_\epsilon T[\phi] = T\left[\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \phi\big(s_{\mathbf{y}}(\mathbf{x})\big)\right], \\ A_\epsilon T[\phi] = T\left[\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \int_{t=0}^{t=1} \phi\big(s_{t\mathbf{y}}(\mathbf{x})\big)\right] \end{cases} \tag{2.4}$$

where ϕ is a test form, and T is evaluated at the forms of $\mathbf{x}$ variables. We shall note that

(1) the continuity assumption about $s_{\mathbf{y}}(\mathbf{x})$ guarantees the existence of the first of (2.4),
(2) $\int_{t=0}^{t=1} \phi\big(s_{t\mathbf{y}}(\mathbf{x})\big)$ is the fibre integral along the t variable. So

$$\begin{cases} dim(R_\epsilon(T)) = dim(T), \\ dim(A_\epsilon(T)) = dim(T) - 1. \end{cases}$$

If furthermore the map

$$\begin{array}{ccc} \mathbb{R}^m \times \mathbb{R}^m & \rightarrow & \mathbb{R}^m \times \mathbb{R}^m \\ (\mathbf{x}, \mathbf{y}) & \rightarrow & (\mathbf{x}, s_{\mathbf{y}}(\mathbf{x})) \end{array}$$

is a diffeomorphism, we denote the inverse map by

$$(\mathbf{y}, \mathbf{x}) \rightarrow \big(\mathbf{y}, g(\mathbf{x}, \mathbf{y})\big)$$

(we switch the letters $\mathbf{x}$ and $\mathbf{y}$) to obtain a C^∞ form

$$R_\epsilon T = T_{\mathbf{y}}\left[\vartheta_\epsilon\big(g(\mathbf{x}, \mathbf{y})\big)\right] \tag{2.5}$$

where $T_{\mathbf{y}}$ is the evaluation of T at the form in $\mathbf{y}$ variables.

Next we use the specific map

$$s_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{y}, \tag{2.6}$$

where the $+$ is from the standard linear structure of $\mathbb{R}^m$. Then

$$R_\epsilon T = T_{\mathbf{y}}\left[\vartheta_\epsilon\big(\mathbf{x} - \mathbf{y}\big)\right] \tag{2.7}$$

Then all properties in Definition 2.1 are satisfied. We refer the proof to [1].

Note: The local construction in this step is well known (see [2]). Next we'll see the global extension which is the main focus of this paper.

Step 2: Choose the unit ball $B \subset \mathbb{R}^m$ diffeomorphic to $\mathbb{R}^m$. Let h be the specific diffeomorphism

$$\mathbb{R}^m \quad \rightarrow \quad B,$$

defined on p66, [1]. Then we define the new C^∞ map

$$s_{\mathbf{y}}(\mathbf{x}) = \begin{cases} hs^B_{\mathbf{y}}h^{-1}(\mathbf{x}) & \text{for} \quad \mathbf{x} \in B \\ \mathbf{x} & \text{for} \quad \mathbf{x} \notin B \end{cases} \tag{2.8}$$

where $s^B_{\mathbf{y}}$ is the specific map (2.6) as in Step 1 for $B \simeq \mathbb{R}^m$. We would like to point out that $s_{\mathbf{y}}(\mathbf{x})$ satisfies assumption. Then we can define the operators $R^B_\epsilon, A^B_\epsilon$ depending on B in the same way (with a test form ϕ):

$$\begin{cases} R^B_\epsilon T[\phi] = T\left[\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \phi\big(s_{\mathbf{y}}(\mathbf{x})\big)\right], \\ A^B_\epsilon T[\phi] = T\left[\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \int_{t=0}^{t=1} \phi\big(s_{t\mathbf{y}}(\mathbf{x})\big)\right]. \end{cases} \tag{2.9}$$

Then the operators $R^B_\epsilon, A^B_\epsilon$ will satisfy
(a) properties (1), (4), (5) and (6) in Definition 2.1.
(b) $R^B_\epsilon(T)$ is C^∞ in B, $R^B_\epsilon(T) = T$ in the complement of $\overline{B}$;
We refer the verification to [1].

Step 3: Cover the X with countable many, locally finite open sets B_i. Now we regard each B^i as a subset of B in step 2. Let a neighborhood U_i of B_i. Let h_i be a diffeomorphism

$$\begin{array}{ccc} U_i & \rightarrow & \mathbb{R}^m \\ \cup & & \cup \\ B_i & \rightarrow & B. \end{array}$$

Let $g_i \geq 0$ be a function on X, which is 1 on B_i and supported in U_i. Let $T' = g_i T$ and $T'' = T - T'$. Then we let

$$\begin{array}{c} R^i_\epsilon T = h_i^{-1} \circ R^B_\epsilon \circ h_i(T') + T'' \\ A^i_\epsilon T = h_i^{-1} \circ A^B_\epsilon \circ h_i(T'). \end{array}$$

Finally we extend it from local to global by taking the composition,

$$\begin{array}{c} R^{(N)}_\epsilon = R^1_\epsilon \circ \cdots \circ R^N_\epsilon, \\ A^{(N)}_\epsilon = R^1_\epsilon \circ \cdots \circ R^{N-1}_\epsilon \circ A^N_\epsilon. \end{array} \tag{2.10}$$

Then the limits

$$R_\epsilon := \lim_{N\to\infty} R^{(N)}_\epsilon$$
$$A_\epsilon := \sum_{N=1}^{\infty} A^{(N)}_\epsilon$$

exist and satisfy all properties in Definition 2.1. We refer the verification to [1]. †

□

†In [1], for each open set U_i there is a different positive ϵ_i. We used the same number ϵ for all U_i.

Definition 2.3. *(de Rham's regularization, uniformed de Rham data)*

(a) We call R_ϵ in Theorem 2.2 the de Rham's regulator, the regularization the de Rham's regularization.

(b) We define the de Rham data to be all items in the construction of de Rham's regularization.

(c) The de Rham data is uniformed if the linear structure determined by the coordinates of B_{i+1} is the same as that determined by the coordinates of B_i for all i.

Remark According to the definition, uniformed and non-uniformed de Rham data always exist. However, they can't be chosen canonically. Hence the supportive intersection that will be defined later is not canonical.

• C^∞ **Kernel of the de Rham's regulator**

G. de Rham further showed in chapter III, §17, [1],

Corollary 2.4. *De Rham's operator R_ϵ constructed in Theorem 2.2 is a regularizing operator that has an associated C^∞ form $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ on $X \times X$, called the C^∞ kernel of R_ϵ, i.e for any current T,*

$$R_\epsilon T = T_{\mathbf{y}}\big[\varrho_\epsilon(\mathbf{x}, \mathbf{y})\big].$$

where the current's evaluation $T_{\mathbf{y}}$ of T on $\mathbf{y}$-form is defined as in Theorem 9, [1] through a double form.

Remark The general definition of kernels is attached in the Appendix in which de Rham shows a general operator from currents to forms has a smooth kernel. In particular, the de Rham's regularization has a smooth kernel. In the following, we'll go further to show this kernel has a particular type of local property that allows the convergence of (1.3).

Definition 2.5. *(local blow-up family of forms)*

Let ω_ϵ for $\epsilon > 0$ be smooth forms of degree p on an Euclidean space $\mathbb{R}^n$. If there are a decomposition $\mathbb{R}^n \simeq \mathbb{R}^p \times \mathbb{R}^{n-p}$ with the orthogonal projection $\pi : \mathbb{R}^n \to \mathbb{R}^p$, and a smooth form $\omega_1(\mathbf{x})$ on $\mathbb{R}^p$ with a compact support such that $\omega_\epsilon = \pi^*(\omega_1(\frac{\mathbf{x}}{\epsilon}))$ where $\mathbf{x}$ is the coordinate of $\mathbb{R}^p$, then ω_ϵ is called a blow-up family of $\omega_1(\mathbf{x})$.

Remark The blow-up family is well-known in a special case where $p = n$ (for instance, see [2]).

Theorem 2.6. *Let X be a differentiable manifold of degree m. Let ϱ_ϵ be the C^∞ kernel of the de Rham's regulator R_ϵ with an uniformed de Rham data. Then around each point, there is a chart U such that $\varrho_\epsilon|_U$ is a local blow-up.*

Proof. We'll show the blow-up structure comes from a fibre integral. Let $q \in X$, and U_q be a small neighborhood of q. Also we may assume q does not belong to the boundary of each ball B_i in the de Rham data (because the collection of those points has Lebesgue measure 0). Consider the kernel ϱ_ϵ of the de Rham's regulator

$$R_\epsilon = R_\epsilon^1 \circ \cdots \circ R_\epsilon^N \tag{2.11}$$

restricted to $U_q \times U_q$, where N is finite. Each operator $R_\epsilon^i, i = 1, \cdots, N$ regularizes inside an Euclidean ball B_i. Since R_ϵ^i remains to be the identity outside of B_i, we may only consider the regularization inside of B_i. We denote those balls by $B_1, B_2, \cdots, B_n$. Let's denote the coordinates for each B_i by the letter $\mathbf{x}_i$, and the second copy of B_i by $\mathbf{y}_i$ (as in (2.7)). Then according to de Rham's construction the smooth kernel of each R_ϵ^i is the pullback form

$$\vartheta_1^i(\frac{\mathbf{x}_i}{\epsilon} - \frac{\mathbf{y}_i}{\epsilon}). \tag{2.12}$$

Since the de Rham data is uniformed, the linear structures on all B_i are the same. Then local expression for the composition of R_ϵ^i is just the fibre integral. Precisely, the kernel ϱ_ϵ of R_ϵ inside $B_1 \cap \cdots \cap B_n$ is the m-form that can be calculated by the fibre integral

$$\begin{aligned}
\varrho_\epsilon(\mathbf{x}_1, \mathbf{y}_n) = \\
&\int_{(\mathbf{x}_2, \cdots, \mathbf{x}_n) \in \prod_{n-1} \mathbb{R}^m} \vartheta_1^1(\frac{\mathbf{x}_1}{\epsilon} - \frac{\mathbf{x}_2}{\epsilon}) \wedge \vartheta_1^2 \frac{\mathbf{x}_2}{\epsilon} - \frac{\mathbf{x}_3}{\epsilon}) \wedge \cdots \\
&\qquad \wedge \vartheta_1^{n-1}(\frac{\mathbf{x}_{n-1}}{\epsilon} - \frac{\mathbf{x}_n}{\epsilon}) \wedge \vartheta_1^n(\frac{\mathbf{x}_n}{\epsilon} - \frac{\mathbf{y}_n}{\epsilon})
\end{aligned} \tag{2.13}$$

where $\vartheta_1^i(\frac{\mathbf{x}_i}{\epsilon} - \frac{\mathbf{x}_{i+1}}{\epsilon})$ is regarded as the pullback of the kernel of each R_ϵ^i (see 2.12) to the product space $\prod_{n+1} \mathbb{R}^m$. So the kernel ϱ_ϵ is a m-form on the product $\mathbb{R}^m \times \mathbb{R}^m$ where $\mathbf{x}_1, \mathbf{y}_n$ are the coordinates for the first and second factor respectively. Precisely, the smooth m-form with a compact support, $\varrho_1(\mathbf{x}_1, \mathbf{y}_n)$ is defined by the fibre integral

$$\begin{aligned}
&\int_{(\mathbf{x}_2, \cdots, \mathbf{x}_n) \in \prod_{n-1} \mathbb{R}^m} \vartheta_1^1(\mathbf{x}_1 - \mathbf{x}_2) \wedge \vartheta_1^1(\mathbf{x}_2 - \mathbf{x}_3) \wedge \cdots \\
&\qquad \vartheta_1^{n-1}(\mathbf{x}_{n-1} - \mathbf{x}_n) \wedge \vartheta_1^n(\mathbf{x}_n - \mathbf{y}_n).
\end{aligned} \tag{2.14}$$

Then the blow-up $\varrho_\epsilon(\mathbf{x}_1, \mathbf{y}_n)$ is just the blow-up of the compactly supported form $\varrho_1(\mathbf{x})$ with the pullback map

$$\begin{array}{ccc}
\mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^m \\
(\mathbf{x}_1, \mathbf{y}_n) & \to & \mathbf{x} = \mathbf{x}_1 - \mathbf{y}_n.
\end{array}$$

□

Example 2.7. *Let* $X = \mathbb{R}^m$ *have the standard coordinates* $\mathbf{x}$*. Let* μ *be the Lebesgue measure of* $\mathbb{R}^m$*. Let* $f(\mathbf{x})$ *be a* C^∞ *function of* $\mathbb{R}^m$ *with compact support in a ball of the origin such that*

$$\int_{\mathbb{R}^m} f(\mathbf{x}) d\mu = 1.$$

So $(\mathbb{R}^m, f)$ *is the de Rham data of* X*. For a positive number* ϵ*, the kernel* ϱ_ϵ *of the de Rham's regulator is just the blow-up*

$$s^*\left(\frac{1}{\epsilon^m} f(\frac{\mathbf{x}}{\epsilon}) d\mu\right).$$

where $d\mu$ *is the volume form and* s *is the map sending* $(\mathbf{x}_1, \mathbf{y}_1) \in \mathbb{R}^m \times \mathbb{R}^m$ *to* $\mathbf{x}_1 - \mathbf{y}_1 \in \mathbb{R}^m$*.*

3 Convergence of the regularization

Now we step back to focus on the particular types of currents: C^∞ regular chains, i.e the chains in $S^\infty(X)$. Notice that the convergence (1.3) only concerns the local Euclidean space. So we focus on an Euclidean space.

In general, we denote the Lebesgue measure on an Euclidean space $\mathbb{R}^l$ by $\mu_{\mathbf{w}}$ where $\mathbf{w}$ is the standard coordinate or a point. We abuse the notation to denote the volume form with the maximal degree in the coordinates and the volume element in the Lebesgue integral by the same expression $d\mu_{\mathbf{w}}$. In the context, the current of the integration over a set σ is also denoted by σ.

Lemma 3.1. *Let* Π_{m+r} *be an* $m+r$ *dimensional polyhedron in* $\mathbb{R}^{2m}$*. Let* ω_ϵ *be a blow-up family of forms of degree* m *in* $\mathbb{R}^{2m}$*. Then the currents*

$$\Pi_{m+r} \wedge \omega_\epsilon \tag{3.1}$$

converge weakly to a functional as $\epsilon \to 0$*.*

Proof. We may assume $\omega_1 = w_1(\mathbf{x}) d\mu_{\mathbf{x}}$ where w_1 is a smooth function with a compact support in $\mathbb{R}^m$ and $\mathbf{x}$ is the coordinate of $\mathbb{R}^m$. So we may also assume Π_{m+r} lies in $\mathbb{R}^m \times \mathbb{R}^r \subset \mathbb{R}^{2m}$ for some subspace $\mathbb{R}^r$ (otherwise $\Pi_{m+r} \wedge \omega_\epsilon$ is zero). We denote the coordinate of $\mathbb{R}^r$ by $\mathbf{z}$. Then let the test form on $\mathbb{R}^{2m}$ restricted to an form expressed as $\phi = \psi(\mathbf{x}, \mathbf{z}) d\mu_{\mathbf{z}}$ where $\psi(\mathbf{x}, \mathbf{z})$ is a smooth function on

$\mathbb{R}^{m+r}$ with a compact support. Also we denote the scalar multiplication map on the first factor

$$(\mathbf{x}, \mathbf{z}) \to (\epsilon\mathbf{x}, \mathbf{z})$$

by D_ϵ. Then we compute

$$\Big(\Pi_{m+r} \wedge \omega_\epsilon(\mathbf{w})\Big)[\phi] \tag{3.2}$$

$$= \int_{\Pi_{m+r}} \omega_1(\frac{\mathbf{x}}{\epsilon}) \wedge \phi(\mathbf{x}, \mathbf{z}) \tag{3.3}$$

$$(change\ variables:\ (\frac{\mathbf{x}}{\epsilon}, \mathbf{z}) \Rightarrow (\mathbf{x}, \mathbf{z}) \tag{3.4}$$

$$= \int_{D_{\epsilon^{-1}}(\Pi_{m+r})} w_1(\mathbf{x}) \wedge \psi(\epsilon\mathbf{x}, \mathbf{z}) d\mu_{(\mathbf{x},\mathbf{z})} \tag{3.5}$$

Recall $d\mu_{(\mathbf{x},\mathbf{z})}$ is the volume element for the Lebesgue integral in the Euclidean space $\mathbb{R}^m \times \mathbb{R}^r$. Next we have two steps.

Step 1. In the integral (3.5), since ω_1 and ψ are supported in a bounded set, the point $(\mathbf{x}, \mathbf{z})$ in the integral lies in the bounded set. Then, $\psi(\epsilon\mathbf{x}, \mathbf{z})$ uniformly converges to $\psi(\mathbf{0}, \mathbf{z})$ for all such points $(\mathbf{x}, \mathbf{z})$ in the bounded set. Then the difference

$$\left| \int_{D_{\epsilon^{-1}}(\Pi_{m+r})} w_1(\mathbf{x}) \wedge \psi(\epsilon\mathbf{x}, \mathbf{z}) d\mu_{(\mathbf{x},\mathbf{z})} - \int_{D_{\epsilon^{-1}}(\Pi_{m+r})} w_1(\mathbf{x}) \wedge \psi(\mathbf{0}, \mathbf{z}) d\mu_{(\mathbf{x},\mathbf{z})} \right|$$

is less than any number, provided ϵ is sufficiently small.

Step 2. Hence it suffices to consider the Lebesgue integral

$$\int_{D_{\epsilon^{-1}}(\Pi_{m+r})} w_1(\mathbf{x}) \wedge \psi(\mathbf{0}, \mathbf{z}) d\mu_{(\mathbf{x},\mathbf{0})}. \tag{3.6}$$

The convergence of it as $\epsilon \to 0$ will be implied by the weak convergence of the measures obtained as the restricted Lebesgue measures to the set $D_{\epsilon^{-1}}(\Pi_{m+r})$. Let's work with measures. Let $\mathcal{R}$ be a ray starting in the space. Since Π_{m+r} is a convex set, the intersection

$$\mathcal{R} \cap \Pi_{m+r}$$

is an interval on the ray. Hence

$$D_{\epsilon^{-1}}(\mathcal{R} \cap \Pi_{m+r}) \subset D_{(\epsilon')^{-1}}(\mathcal{R} \cap \Pi_{m+r}), \quad \text{for } \epsilon' < \epsilon$$

Now taking the union of all rays, we obtain

$$D_{\epsilon^{-1}}(\Pi_{m+r}) \subset D_{(\epsilon')^{-1}}(\Pi_{m+r}), \quad \text{for } \epsilon' < \epsilon. \tag{3.7}$$

Taking the union $\cup_{\epsilon \in (0,1]}\Big(D_{\epsilon^{-1}}(\Pi_{m+r})\Big)$, we obtain the measurable set

$$\mathcal{D}_0 := \lim_{\epsilon \to 0} D_{\epsilon^{-1}}(\Pi_{m+r}).$$

We denote the Lebesgue measure restricted to the set $\left(D_{\epsilon^{-1}}(\Pi_{m+r})\right)$ by μ_ϵ, to $\mathcal{D}_0$ by μ_0. Then (3.7) implies the measure μ_ϵ converges to the measure μ_0 set-wisely. By the Portemanteau theorem (Theorem 13.16, [3]), the set-wise convergence implies the weak convergence of measures. Hence (3.5) converges.

□

Next proposition goes further to address a C^∞ cell, i.e. a cell whose differential map can be extended defferentially to a neighborhood of the polyhedron.

Lemma 3.2. *Let $c \in S^\infty(\mathbb{R}^{2m})$, Let ω_ϵ be a blow-up of degree m as in Lemma 3.1. Then currents, $c \wedge \omega_\epsilon$ converge weakly to a functional as $\epsilon \to 0$.*

Proof. By the linearlity of c, it suffices to deal with the case when c is a single cell. So assume

$$c : \Pi_{m+r} \to \mathbb{R}^{2m} \tag{3.8}$$

can be extended to an diffeomorphism-to-image in a neighborhood of Π_{m+r}. Let $c \wedge \omega_\epsilon$ be the functional

$$\begin{array}{c} \phi \\ \downarrow \\ \int_c \omega_\epsilon \wedge \phi \end{array} \tag{3.9}$$

where ϕ is a test form of the degree r, and c represents the image $c(\Pi_{m+r})$. Similarly as in Lemma 3.1, we may assume the smooth form ω_1 in $\mathbb{R}^m \subset \mathbb{R}^{2m}$ is written as $\omega_1 = w_1(\mathbf{x})d\mu_{\mathbf{x}}$ for some volume form in coordinate $\mathbf{x}$ of $\mathbb{R}^m$ of degree m. Let $\Pi_{m+r} \subset \mathbb{R}^m \times \mathbb{R}^r$ where $\mathbb{R}^r$ is a different Euclidean space. We denote their coordinates by $\mathbf{u}, \mathbf{v}$. Then

$$\int_c \omega_\epsilon \wedge \phi = \int_{\Pi_{m+r}} \epsilon^{-m} w_1(\epsilon^{-1}\mathbf{x}(\mathbf{u}, \mathbf{v})))\psi(\mathbf{u}, \mathbf{v}) d\mu_{(\mathbf{u},\mathbf{v})} \tag{3.10}$$

where $\psi(\mathbf{u}, \mathbf{v})$ is some smooth function with a compact support induced from the test form ϕ and coordinate's change. Substitute $\mathbf{u}$ by $\epsilon\mathbf{u}$ to have

$$(3.10) = \int_{D_{\epsilon^{-1}}(\Pi_{m+r})} w_1(\epsilon^{-1}\mathbf{x}(\epsilon\mathbf{u}, \mathbf{v}))\psi(\epsilon\mathbf{u}, \mathbf{v}) d\mu_{(\mathbf{u},\mathbf{v})}. \tag{3.11}$$

Since Π_{m+r} is bounded, the variable $\mathbf{v}$ in the integral is bounded. On the other hand, if we view $\epsilon^{-1}\mathbf{x}(\epsilon\mathbf{u}, \mathbf{v})$ as a diffeomorphic map

$$\mathbf{u} \in \mathbb{R}^m \to \mathbf{x} \in \mathbb{R}^m$$

parametrized by $\mathbf{v}$, it sends a bounded set to a bounded set uniformly with respect of all $\mathbf{v}$. The converse also sends a bounded set to a bounded set uniformly. Then there is a bounded set $K \subset R^m \times \mathbb{R}^r$ independent of ϵ such that

$$(3.11) = \int_{D_{\epsilon^{-1}}(\Pi_{m+r}) \cap K} w_1(\epsilon^{-1}\mathbf{x}(\epsilon\mathbf{u},\mathbf{v}))\psi(\epsilon\mathbf{u},\mathbf{v})d\mu_{(\mathbf{u},\mathbf{v})}. \tag{3.12}$$

Since $\mathbf{u}$ is bounded in $\mathbb{R}^m$, we apply the chain rule to $\lim_{\epsilon\to 0}\epsilon^{-1}\mathbf{x}(\epsilon\mathbf{u},\mathbf{v})$. We then obtain that the limit converges uniformly for all $\mathbf{u},\mathbf{v}$ to $B_{\mathbf{v}}(\mathbf{u})$ where $B_{\mathbf{v}}$ is a constant Jacobian $\frac{\partial\mathbf{x}(\mathbf{u},\mathbf{v})}{\partial\mathbf{u}}|_{\mathbf{u}=\mathbf{0}}$. Then

$$\begin{aligned}
(3.12) &= \lim_{\epsilon\to 0}\int_{D_{\epsilon^{-1}}(\Pi_{m+r})} w_1(\epsilon^{-1}\mathbf{w}(\epsilon\mathbf{u},\mathbf{v}))\psi(\epsilon\mathbf{u},\mathbf{v})d\mu_{(\mathbf{u},\mathbf{v})} \\
&= \lim_{\epsilon\to 0}\int_{D_{\epsilon^{-1}}(\Pi_{m+r})} w_1(B_{\mathbf{v}}(\mathbf{u}))\psi(\mathbf{0},\mathbf{v})d\mu_{(\mathbf{u},\mathbf{v})} \\
&= \lim_{\epsilon\to 0}\int_{\Pi_{m+r}} \omega'_\epsilon \wedge \psi(\mathbf{0},\mathbf{v})d\mu_{\mathbf{v}} \\
&= (\Pi_{m+r}\wedge\omega'_\epsilon)\Big[\psi(\mathbf{0},\mathbf{v})d\mu_{\mathbf{v}}\Big]
\end{aligned} \tag{3.13}$$

where ω'_ϵ is the blow-up of the differential form $w_1(B_{\mathbf{v}}(\mathbf{u}))d\mu_{\mathbf{u}}$. Then the convergence follows from Lemma 3.1. We complete the proof.

□

The following example is a well-known local regularization in cohomology theory (see [3]).

Example 3.3. *Let c have the dimension $2m$ and contain the origin, and blow-up forms ω_ϵ have the top degree $2m$. Then $c \wedge \omega_\epsilon$ converge weakly to a constant multiple of the delta function at the origin.*

Proposition 3.4. *Let X be a manifold of dimension m. For chains σ_1, σ_2 in $S^\infty(X)$ such that $dim(\sigma_1) + dim(\sigma_2) \geq m$, the exterior product*

$$\sigma_1 \wedge R_\epsilon(\sigma_2)$$

weakly converges to a current as $\epsilon \to 0$, where the de Rham data for the regularization is uniformed.

Proof. (1) Let ϕ be a test form. Then

$$\big(\sigma_1 \wedge R_\epsilon(\sigma_2)\big)[\phi] = \int_{\sigma_1} R_\epsilon\sigma_2 \wedge \phi = \int_{\sigma_1\times\sigma_2} \varrho_\epsilon(\mathbf{x},\mathbf{y}) \wedge \phi(\mathbf{y}), \tag{3.14}$$

where $\mathbf{x}, \mathbf{y}$ are the local coordinates for the first and second X in $X \times X$. By Theorem 2.6, the kernel $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ of R_ϵ is a local blow-up whose local structure consists of locally finite open covering subsets U of X and a subspace

$$V \simeq \mathbb{R}^m \subset U \times U$$

such that

$$\varrho_\epsilon(\mathbf{x}, \mathbf{y})|_{U \times U} = \varrho_1(\frac{\mathbf{x}}{\epsilon}, \frac{\mathbf{y}}{\epsilon})|_{U \times U} = \pi^*(\theta(\frac{\mathbf{v}}{\epsilon})), \tag{3.15}$$

where $\pi : U \times U \to V$ is a C^∞ map to $V \simeq \mathbb{R}^m$, θ is a C^∞ m-form on V and $\mathbf{v}$ is the point in V. By a partition of unity it suffices to focus on one open set U. Precisely it suffices to show the convergence of the real numbers

$$\int_{\sigma_1|_U \times \sigma_2|_U} \pi^*(\theta(\frac{\mathbf{v}}{\epsilon})) \wedge \phi \tag{3.16}$$

where $\sigma_1|_U \times \sigma_2|_U$ is the restriction to $U \times U \simeq \mathbb{R}^{2m}$. Notice that $\pi^*(\theta(\frac{\mathbf{v}}{\epsilon}))$ is a blow-up and $\sigma_1|_U \times \sigma_2|_U$ is a regular chain of dimension $m+r$. Then according to Lemma 3.2, the convergence follows. We complete the proof.

(2) Let ϕ be an element of a subset of $\mathscr{D}(X)$ bounded to any orders. Applying a partition of unity, we may address it on the sufficiently small local chart U only. By observing the local expression (3.12), we obtain that

$$\int_{\sigma_1} R_\epsilon(\sigma_2) \wedge \phi$$

is bounded by a multiple of $||\phi||_\infty$. Hence globally,

$$\lim_{\epsilon \to 0} \int_{\sigma_1} R_\epsilon \sigma_2 \wedge \phi$$

is bounded for ϕ in the bounded set. This shows the weak limit

$$\lim_{\epsilon \to 0} \big(\sigma_1 \wedge R_\epsilon(\sigma_2)\big)$$

is a continuous functional, thus a current.

□

4 The supportive intersection

Definition 4.1. *Let σ_1, σ_2 be two chains in $S^\infty(X)$ where X is a differentiable manifold equipped with an uniformed de Rham data. We define*

$$[\sigma_1 \wedge \sigma_2]$$

to be the weak limit

$$\lim_{\epsilon\to 0}\big(\sigma_1\wedge R_\epsilon\sigma_2\big)$$

where $R_\epsilon\sigma_2$ is the de Rham's regularization associated to the given de Rham data.

Remark The notation does not specify the de Rham data which plays an important role in the determination of the supportive intersection. Because of this role, the supportive intersection is not invariant of any structures, but its existence is a C^∞ invariant.

Property 4.2.

Let X a differentiable manifold of dimension m equipped with an uniformed de Rham data. For chains σ_1, σ_2 in $S^\infty(X)$, the intersection $[\sigma_1\wedge\sigma_2]$ satisfies:

(1) (Supportivity)

$$supp([\sigma_1\wedge\sigma_2])\subset supp(\sigma_1)\cap supp(\sigma_2). \tag{4.1}$$

(2) (Closedness) The intersection current $[\sigma_1\wedge\sigma_2]$ is closed if σ_1, σ_2 are.

(3) (Cohomologicity) According to the de Rham's theory in [1], the homology of the complex of currents coincides with the singular cohomology with real coefficients. Hence we use $\langle\sigma\rangle$ to denote the singular cohomology class represented by a closed current σ, and $\smile$ the cup-product. If σ_1, σ_2 are closed, then in singular cohomology

$$\langle[\sigma_1\wedge\sigma_2]\rangle = \langle\sigma_1\rangle\smile\langle\sigma_2\rangle \tag{4.2}$$

(4) (Leibniz rule) If $deg(\sigma_1)=p$, then the differential map of chains follows Leibniz rule,

$$d[\sigma_1\wedge\sigma_2] = [d\sigma_1\wedge\sigma_2] + (-1)^p[\sigma_1\wedge d\sigma_2], \tag{4.3}$$

where the differential map d is the operator $(-1)^{p+1}b$ for the boundary operator b acting on chains of the codimension p.

Proof. (1) Suppose

$$\mathbf{a}\notin supp(\sigma_1)\cap supp(\sigma_2).$$

Then $\mathbf{a}$ must be either outside of $supp(\sigma_1)$ or outside of $supp(\sigma_2)$. Let's assume first it is not in $supp(\sigma_2)$. Since the support of a currents is closed, we choose a small neighborhood $U_{\mathbf{a}}$ of $\mathbf{a}$ in X, but disjoint from $supp(\sigma_2)$. Let ϕ be a C^∞-form of X with compact support in $U_{\mathbf{a}}$. According to Definition 2.1, when ϵ is small enough $R_\epsilon(\sigma_2)$ is zero in $U_{\mathbf{a}}$. Hence

$$[\sigma_1\wedge\sigma_2][\phi] = 0. \tag{4.4}$$

Hence $\mathbf{a} \notin supp([\sigma_1 \wedge \sigma_2])$. If $\mathbf{a} \notin supp(\sigma_1)$, $U_{\mathbf{a}}$ can be chosen disjoint with $supp(\sigma_1)$. Then since $\phi \in \mathscr{D}(U_{\mathbf{a}})$ is a C^∞-form of X with compact support in $U_{\mathbf{a}}$ disjoint with $supp(\sigma_1)$, the restriction of ϕ to σ_1 is zero. Hence

$$[\sigma_1 \wedge \sigma_2][\phi] = 0.$$

Then $\mathbf{a} \notin supp([\sigma_1 \wedge \sigma_2])$. Thus

$$\mathbf{a} \notin supp(\sigma_1) \cap supp(\sigma_2)$$

will always imply

$$\mathbf{a} \notin supp([\sigma_1 \wedge \sigma_2]).$$

This completes the proof.

(2) Let ϕ be a test form. By the definition

$$\begin{aligned} & b[\sigma_1 \wedge \sigma_2][\phi] \\ & = \lim_{\epsilon \to 0} \sigma_1[R_\epsilon \sigma_2 \wedge d\phi] \\ & = \pm \lim_{\epsilon \to 0} \sigma_1[dR_\epsilon \sigma_2 \wedge \phi] \\ & = \pm \lim_{\epsilon \to 0} \sigma_1[bR_\epsilon \sigma_2 \wedge \phi] \end{aligned} \tag{4.5}$$

According to the homotopy formula (2.1)

$$bR_\epsilon \sigma_2 - b\sigma_2 = bbA_\epsilon \sigma_2 - bA_\epsilon b\sigma_2 \tag{4.6}$$

Because σ_2 is closed,

$$bR_\epsilon \sigma_2 = 0.$$

So $[\sigma_1 \wedge \sigma_2]$ is closed.

(3) Let ϕ be a closed C^∞ form of degree $deg(\sigma_1) + deg(\sigma_2)$, and has compact support. Denote the cohomology class by $\langle \cdot \rangle$. The intersection number,

$$deg\Big(\big\langle [\sigma_1 \wedge \sigma_2] \big\rangle \smile \langle \phi \rangle \Big) \tag{4.7}$$

is a well-defined real number that is equal to

$$\lim_{\epsilon \to 0} \sigma_1[R_\epsilon(\sigma_2) \wedge \phi]. \tag{4.8}$$

By the definition in §20, [1], (4.7) is the de Rham's symbol

$$\Big(\sigma_1 \wedge (\sigma_2 \wedge \phi) \Big)[1].$$

which by de Rham is the intersection number

$$deg\Big((\langle\sigma_1\rangle \smile \langle\sigma_2\rangle) \smile \langle\phi\rangle \Big). \tag{4.9}$$

The formulas (4.7) and (4.9) yield

$$\big\langle [\sigma_1 \wedge \sigma_2] \big\rangle = \langle\sigma_1\rangle \smile \langle\sigma_2\rangle.$$

(4) (Leibniz Rule) Let $\phi \in \mathscr{D}(\mathcal{X})$ be a test form. Let

$$deg(T_1) = p, deg(T_2) = q.$$

Then

$$\begin{aligned}
& b[\sigma_1 \wedge \sigma_2](\phi) \\
& = \lim_{\epsilon\to 0} \int_{\sigma_1} R_\epsilon \sigma_2 \wedge d\phi \\
& (\text{ Leibniz Rule for } C^\infty \text{ forms }) \\
& = \lim_{\epsilon\to 0} \int_{\sigma_1} (-1)^q d(R_\epsilon\sigma_2 \wedge \phi) + (-1)^{q+1} dR_\epsilon\sigma_2 \wedge \phi \\
& = \lim_{\epsilon\to 0} (-1)^q \int_{b\sigma_1} R_\epsilon\sigma_2 \wedge \phi + \lim_{\epsilon\to 0} (-1)^{q+1} \int_{\sigma_1} dR_\epsilon\sigma_2 \wedge \phi \\
& \lim_{\epsilon\to 0} (-1)^q \int_{b\sigma_1} R_\epsilon\sigma_2 \wedge \phi + \lim_{\epsilon\to 0} (-1)^{q+1} \int_{\sigma_1} R_\epsilon(d\sigma_2) \wedge \phi \\
& = (-1)^q [b\sigma_1 \wedge \sigma_2][\phi] + (-1)^{q+1} [\sigma_1 \wedge d\sigma_2][\phi]
\end{aligned}$$

Hence

$$b[\sigma_1 \wedge \sigma_2] = (-1)^q [b\sigma_1 \wedge \sigma_2] + (-1)^{q+1} [\sigma_1 \wedge d\sigma_2]. \tag{4.10}$$

After change the sign, we found (4.10) is the same as (4.3).

□

Example 4.3. *Let $\mathcal{X} = \mathbb{R}^2$, and $\mathcal{U}$ the de Rham data consisting of single chart $\mathbb{R}^2$ with the bump function function f satisfying*

$$\int_{\mathbb{R}^2} f(x_1, x_2) dx_1 \wedge dx_2 = 1 \tag{4.11}$$

where x_1, x_2 are coordinates of $\mathbb{R}^2$. Let σ be a piece of the parabola

$$x_1 = x_2^2 \tag{4.12}$$

containing the origin $\mathbf{0}$*. Since* σ *has dimension 1,* $[\sigma\wedge\sigma]$ *exists as a* 0*-dimensional current. Let* $\phi(x)$ *be a test function. Denote the coordinates for the second copy of* $\mathbb{R}^2$ *by* y_1, y_2*. Then the regularization is the fibre integral (integration along the* y_1, y_2*)*

$$R_\epsilon(\sigma) = \frac{1}{\epsilon^2}\int_{(y_1,y_2)\in\sigma} f(\frac{x_1-y_1}{\epsilon}, \frac{x_2-y_2}{\epsilon})(dx_1-dy_1)\wedge(dx_2-dy_2) \tag{4.13}$$

which is a C^∞ *1-form in variables* x_1, x_2*. Then we calculate*

$$\begin{array}{c} \int_{[\sigma\wedge\sigma]}\phi \\ \| \\ \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_{(x_1,x_2)\in\sigma}\int_{(y_1,y_2)\in\sigma} f(\frac{x_1-y_1}{\epsilon}, \frac{x_2-y_2}{\epsilon})\phi(x_1,x_2)(dx_1-dy_1)\wedge(dx_2-dy_2) \end{array} \tag{4.14}$$

Then the functional

$$\begin{array}{c} \phi \\ \downarrow \\ \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_{(x_2,y_2)\in I\times I} f(\frac{x_2^2-y_2^2}{\epsilon}, \frac{x_2-y_2}{\epsilon})\phi(x_2^2, x_2)(2y_2-2x_2)dx_2\wedge dy_2 \end{array} \tag{4.15}$$

is the current $[\sigma\wedge\sigma]$ *of degree* 0*, where* ϕ *is a test function on* $\mathbb{R}^2$ *and* I *is the given interval. According to Lemma 3.2, the limit in (4.13) exists and is equal to a Lebesgue integral over a measurable set.*

Remark This example also shows the intersection $[\bullet\wedge\bullet]$ depends on de Rham data $\mathcal{U}$.

Example 4.4. *We give multiple cases where the supportive intersections are independent of de Rham data. All of them are known as Kronecker index in [1]. Let* $\mathcal{X} = \mathbb{R}^2$*. Let* $\mathcal{U}$ *be the de Rham data consisting of the single chart* $\mathbb{R}^2$ *with the convolution function* f *satisfying*

$$\int_{\mathbb{R}^2} f(x_1,x_2)dx_1\wedge dx_2 = 1 \tag{4.16}$$

where x_1, x_2 *are the coordinates of* $\mathbb{R}^2$*.*

Case 1: Let σ_1 *be a line through the origin* $\mathbf{0}$ *and* σ_2 *is another line segment through the origin. Then*

$$[\sigma_1\wedge\sigma_2] = \delta_{\mathbf{0}}$$

if the order of "$\wedge$*" matches to the orientation of* $\mathbb{R}^2$*, where* $\delta_{\mathbf{0}}$ *is the delta-function at the point* $\mathbf{0}$*.*

Case 2: Continuing from the setting in case 1, let σ_2 be the line $x_1 = 0$. Let σ_1 be a piece of parabola

$$x_1 = x_2^2, x_2 \in (-1, 1). \tag{4.17}$$

Denote the second copy of $\mathbb{R}^2$ for the de Rham's regularization by (y_1, y_2). The regularization is the fibre integral along y_2,

$$R_\epsilon(\sigma_2) = \frac{1}{\epsilon^2} \int_{y_2 \in \mathbb{R}} f(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} - \frac{y_2}{\epsilon}) dy_2 \wedge dx_1. \tag{4.18}$$

To calculate $[\sigma_1 \wedge \sigma_2]$, let $\phi(x)$ be a test function supported in a neighborhood of the origin. Then

$$\begin{aligned} &\int_{[\sigma_1 \wedge \sigma_2]} \phi \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{x_1 \in \sigma_1} \int_{y_2 \in \mathbb{R}} f(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} - \frac{y_2}{\epsilon}) \phi(x_1, x_2) dy_2 \wedge dx_1. \end{aligned} \tag{4.19}$$

Let

$$f_1(x_1) = \int_{y_2 \in \mathbb{R}} f(x_1, -y_2) dy_2.$$

Now we continue to have

$$\begin{gathered} \int_{[\sigma_1 \wedge \sigma_2]} \phi \\ \| \\ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{(x_1, x_2) \in T_1} f_1(\frac{x_1}{\epsilon}) \phi(x_1, x_2) dx_1 \\ \| \\ \phi(\mathbf{0}) \left(\int_{+\infty}^{0} f_1(x_1) dx + \int_{0}^{+\infty} f_1(x_1) dx_1 \right) = 0, \end{gathered} \tag{4.20}$$

So

$$[\sigma_1 \wedge \sigma_2] = 0.$$

Case 3: Continuing from the setting in case 2, let σ_2 be the line $x_1 = 0$. Let σ_1 be a piece of the cubic curve

$$x_1 = x_2^3, x_2 \in (-1, 1). \tag{4.21}$$

The same calculation in case 2 shows if order of σ_1, σ_2 is concordant with the orientation of $\mathbb{R}^2$, then

$$[\sigma_1 \wedge \sigma_2] = \delta_{\mathbf{0}} \tag{4.22}$$

where $\delta_{\mathbf{0}}$ is the δ-function at the origin.

Appendix

A Kernel

In [1] G. de Rham defined the notion of "regularizing operator" which includes the de Rham's regulator R_ϵ. Let X, Y be two manifolds. Let $\mathscr{T} \in \mathscr{D}'(X \times Y)$. Then $\mathscr{T}$ derives an operator

$$\Lambda ; \mathscr{D}(X) \quad \rightarrow \quad \mathscr{D}'(Y) \tag{A.1}$$

We call $\mathscr{T}$ the kernel of Λ. Conversely given a homomorphism Λ, there is a kernel $\mathscr{T}$ on $X \times Y$. Notice

$$\begin{array}{cc} \mathscr{D}(X), & \mathscr{E}(Y) \\ \cap & \cap \\ \mathscr{E}'(X), & \mathscr{D}'(Y) \end{array} \tag{A.2}$$

where $\mathscr{E}(\bullet)$ is the set of C^∞ forms, and $'$ is the topological dual.

Definition A.1. *If operator Λ has an extension*

$$\Lambda : \mathscr{E}'(X) \quad \rightarrow \quad \mathscr{E}(Y) \tag{A.3}$$

we say Λ is regularizing.

Theorem A.2. *(G. de Rham)*
Λ is regularizing if and only if the kernel $\mathscr{T}$ is a C^∞ form on $X \times Y$. In particular R_ϵ is regularizing.

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