Newton Flows of $\tanh(\ln(1+t^2))$ and Semigroups of Holomorphic Mappings

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Abstract

This paper investigates the Newton flow for a general function $f(t)$, highlighting a closed-form solution when $f(t) = S(t) = \tanh(\ln(1+t^2))$. The study derives the differential equation of the Newton flow $z(t)$, identifying four solutions when $f(t) = S(t)$ that converge to fixed values as $t \to \infty$. Additionally, Newton flows are connected to semigroups of holomorphic mappings, showing that the flow acts as an infinitesimal generator within this framework. This relationship provides a new perspective on the application of continuous Newton's method in complex analysis.

The differential equation for the Newton flow $z(t)$ of $f(t)$ is given by

$$\dot{z}(t) = -\frac{f(z(t))}{\frac{d}{dt} f(z(t))} = -\frac{f(z(t))}{\dot{f}(z(t))} \tag{1}$$

If $f(t) = S(t) = \tanh\left(\ln\left(1+t^2\right)\right.$ and we let

$$b(a) = 2\left(1+a^2\right) + a^4 \tag{2}$$

then define

$$g(t,a) = e^t - 1 + \frac{2}{b(a)} \tag{3}$$

and

$$h(t,a) = \sqrt{e^{2t} - \frac{a^4\left(2+a^2\right)^2}{b(a)^2}} \tag{4}$$

then there are 4 solutions of $z(t)$ for the Newton flow given by

$$z(t,a) = \pm\sqrt{\pm\frac{g(t,a)+h(t,a)}{g(t,a)}} \tag{5}$$

$$\lim_{t\to\infty} z(t,a) \in \{0, \pm i\sqrt{2}\}\ \forall a \in \mathbb{C}$$

and $S(\lim_{t\to\infty} z(t,a)) = 0$. The root at zero is a double root and the roots at $\pm i\sqrt{2}$ are simple roots.

Newton Flows Are Infinitesimal Generators of Semigroups of Holomorphic Mappings

The Newton flow corresponding to the solution of the continuous Newton's method can be considered as an infinitesimal generator of a semigroup of holomorphic mappings. To understand this, we need to briefly discuss the continuous Newton's method and the concept of semigroups of holomorphic mappings.

Continuous Newton's method is a generalization of the classical Newton's method in which the iterations are replaced by a continuous flow. For a given holomorphic function $f(z)$, the continuous Newton's method is described by the following ordinary differential equation (ODE):

$$\frac{dz}{dt} = -\frac{f(z)}{f'(z)}, \tag{6}$$

with an initial condition $z(0) = z_0$.

The solution of this ODE is a continuous trajectory $z(t)$ in the complex plane, which converges to the zero of the function $f(z)$ as t approaches infinity.

A semigroup of holomorphic mappings is a family of mappings $T(t): D \to D$, where D is an open subset of the complex plane and t belongs to a certain interval $[0, t_{\max})$, with the following properties:

1. $T(0)$ is the identity map on D.
2. For all $s, t \geq 0$, if $s + t < t_{\max}$, then $T(s) \circ T(t) = T(s + t)$, where $\circ$ denotes the composition of mappings.

An infinitesimal generator A of a semigroup of holomorphic mappings $T(t)$ is a linear operator that satisfies the following condition:

$$\lim_{t \to 0} \frac{T(t)\, z(t) - z(t)}{t} = A\, z(t)\, \forall z(t) \in D \tag{7}$$

Now, consider the Newton flow as a semigroup of holomorphic mappings $T(t)$ with an infinitesimal generator A. It can be shown that the Newton flow generated by the continuous Newton's method indeed satisfies the properties of a semigroup, and its infinitesimal generator A can be represented by the linear operator

$$-\frac{f(z)}{f'(z)} = \dot{z}(t) = -\frac{f(z(t))}{\frac{d}{dt} f(z(t))} = -\frac{f(z(t))}{\dot{f}(z(t))} \tag{8}$$

Therefore, the Newton flow corresponding to the solution of the continuous Newton's method can be considered as an infinitesimal generator of a semigroup of holomorphic mappings.[1][2]

Bibliography

[1] M. B. Kim, J. W. Neuberger, and W. P. Schleich. A perfect memory makes the continuous newton method look ahead. *Physica Scripta*, 92(8):85201, Aug 2017.

[2] J W Neuberger, C Feiler, H Maier, and W P Schleich. Newton flow of the riemann zeta function: separatrices control the appearance of zeros. *New Journal of Physics*, 16(10):103023, oct 2014.