An analytical treatment of rotations in Euclidean space

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Abstract

An analytical treatment of rotations in the Euclidean plane and 3-dimensional Euclidean space, using differential equations, is presented. Fundamental geometric results, such as the linear transformation for rotations, the invariance of the Euclidean norm, a proof of the Pythagorean theorem, and the existence of a period of rotations, are derived from a set of fundamental equations. Basic Euclidean geometry is also constructed from these equations.

1 Introduction

Rotations of Cartesian coordinate systems in Euclidean space are usually studied in terms of linear transformations. For instance, in the Euclidean plane, the transformation [1] to obtain a new coordinate system $(\bar{x}, \bar{y})$ from an old one (x, y), by rotating the latter by an angle θ is

$$\bar{x} = x\cos\theta + y\sin\theta, \tag{1a}$$

$$\bar{y} = y\cos\theta - x\sin\theta. \tag{1b}$$

The fact that the distance of any point from the origin (the Euclidean norm) is preserved under a rotation is instated by the relation

$$\bar{x}^2 + \bar{y}^2 = x^2 + y^2. \tag{2}$$

In 3-dimensional Euclidean space, the transformation [2] to obtain a new coordinate system $(\bar{x}, \bar{y}, \bar{z})$ from an old one (x, y, z), by rotating the latter by an angle θ about an axis pointing along the unit vector (u_1, u_2, u_3) is

$$\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix} = \begin{bmatrix}\cos\theta + u_1^2(1-\cos\theta) & u_1u_2(1-\cos\theta) + u_3\sin\theta & u_1u_3(1-\cos\theta) - u_2\sin\theta\\ u_1u_2(1-\cos\theta) - u_3\sin\theta & \cos\theta + u_2^2(1-\cos\theta) & u_2u_3(1-\cos\theta) + u_1\sin\theta\\ u_1u_3(1-\cos\theta) + u_2\sin\theta & u_2u_3(1-\cos\theta) - u_1\sin\theta & \cos\theta + u_3^2(1-\cos\theta)\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}, \tag{3}$$

and the preservation of the Euclidean norm is expressed by stating that

$$\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = x^2 + y^2 + z^2. \tag{4}$$

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However, in arriving at these equations, one usually employs geometric techniques. The same results (and more) may be derived in an analytical manner by using differential equations, which is the focus of this work. To this extent, the coordinates x, y and z of a point are taken to be (analytic) functions of the angle θ. A coordinate system (in 3-dimensional space) is then given by $(x(\theta), y(\theta), z(\theta))$. The choice of θ for the initial coordinate system is arbitrary, since only the amount of rotation matters. Since distinct values of θ result in distinctly rotated coordinate systems, θ may be said to "label" a coordinate system.

2 Rotations in the Euclidean plane

2.1 The fundamental equations

Consider the system of equations (with primes denoting differentiation with respect to θ)

$$x' = y, \tag{5a}$$
$$y' = -x. \tag{5b}$$

Successive differentiation yields

$$x'' = -x, \tag{6a}$$
$$y'' = -y. \tag{6b}$$

A Taylor series solution [3] for x is

$$x(\theta) = \sum_{n=0}^{\infty} \frac{(\theta - \theta_0)^n}{n!} x^{(n)}(\theta_0). \tag{7}$$

Since successive differentiation of (6a) implies $x^{(n+2)} = -x^{(n)}$, the solution simplifies to

$$x(\theta) = x(\theta_0) \sum_{n=0}^{\infty} (-1)^n \frac{(\theta - \theta_0)^{2n}}{(2n)!} + x'(\theta_0) \sum_{n=0}^{\infty} (-1)^n \frac{(\theta - \theta_0)^{2n+1}}{(2n+1)!}. \tag{8}$$

The solution for y is similar. Defining

$$f_1(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!}, \tag{9a}$$
$$f_2(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}, \tag{9b}$$

and noting that $x'(\theta_0) = y(\theta_0)$ and $y'(\theta_0) = -x(\theta_0)$, we get

$$x(\theta) = x(\theta_0) f_1(\theta - \theta_0) + y(\theta_0) f_2(\theta - \theta_0), \tag{10a}$$
$$y(\theta) = y(\theta_0) f_1(\theta - \theta_0) - x(\theta_0) f_2(\theta - \theta_0). \tag{10b}$$

One may have already recognized f_1 and f_2 as the trigonometric cosine and sine functions, respectively. However, we shall continue to use the developed notation to derive their properties independently of geometric techniques. Equations (10) are strikingly similar to (1), and describe a rotation by an angle $\theta - \theta_0$, starting from a coordinate system θ_0.

We also get from (5)

$$xx' + yy' = 0, \tag{11}$$

which on integrating gives

$$x(\theta)^2 + y(\theta)^2 = \text{constant}. \tag{12}$$

This proves the invariance of the (squared) Euclidean norm under rotations.

Setting $y(0) = 0$ (as we are free to do so), we arrive at from (12)

$$x(\theta)^2 + y(\theta)^2 = x(0)^2. \tag{13}$$

In this case, we have from (10) (with $\theta_0 = 0$)

$$x(\theta) = x(0) f_1(\theta), \tag{14a}$$

$$y(\theta) = -x(0) f_2(\theta), \tag{14b}$$

so that (13) results in

$$f_1(\theta)^2 + f_2(\theta)^2 = 1. \tag{15}$$

We thus observe that f_1 and f_2 are both restricted to the interval $[-1, 1]$ and are complementary in nature.

Now, additionally, let $x(\theta_0) = 0$ for some θ_0. We then have from (10)

$$x(\theta) = y(\theta_0) f_2(\theta - \theta_0), \tag{16a}$$

$$y(\theta) = y(\theta_0) f_1(\theta - \theta_0). \tag{16b}$$

From (13), we obtain $y(\theta_0)^2 = x(0)^2$. The case $y(\theta_0) = -x(0)$ produces from (14) and (16)

$$f_1(\theta) = -f_2(\theta - \theta_0), \tag{17a}$$

$$f_2(\theta) = f_1(\theta - \theta_0). \tag{17b}$$

Replacing θ by $\theta + \theta_0$, we get

$$f_1(\theta + \theta_0) = -f_2(\theta), \tag{18a}$$

$$f_2(\theta + \theta_0) = f_1(\theta). \tag{18b}$$

Successive application of (18) yields

$$f_1(\theta + 2\theta_0) = -f_1(\theta), \qquad f_2(\theta + 2\theta_0) = -f_2(\theta), \tag{19a}$$

$$f_1(\theta + 3\theta_0) = f_2(\theta), \qquad f_2(\theta + 3\theta_0) = -f_1(\theta), \tag{19b}$$

$$f_1(\theta + 4\theta_0) = f_1(\theta), \qquad f_2(\theta + 4\theta_0) = f_2(\theta). \tag{19c}$$

Equations (19c) show that f_1 and f_2 both repeat after intervals of (integral multiples of) $4\theta_0$. The smallest of these intervals is the period, say, Θ. Consequently, from (10), we find that x and y are also periodic with the same period Θ.

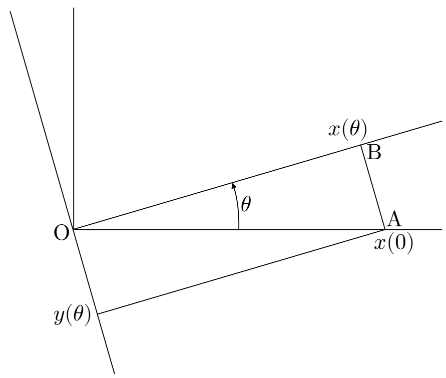

Figure 1: Two coordinate systems, one of whose axes are rotated by an angle θ.

Now, consider figure 1, which describes a rotation between two coordinate systems by an angle θ. Equation (13) shows that

$$\mathrm{OB}^2 + \mathrm{BA}^2 = \mathrm{OA}^2, \tag{20}$$

which proves the Pythagorean theorem. Defining $r = \sqrt{x^2 + y^2}$, we obtain from (12)

$$r(\theta) = \text{constant}. \tag{21}$$

Equation (13) also describes the locus of all points whose distance from the origin is $r(\theta) = |x(0)|$. Consequently, it represents a circle of radius r, centred at the origin. For $x(0) \geq 0$, we then obtain from (14)

$$x(\theta) = r(\theta) f_1(\theta), \tag{22a}$$
$$y(\theta) = -r(\theta) f_2(\theta). \tag{22b}$$

Now, eliminating $x(0)$ from (14) and defining $m = -f_2/f_1$ results in

$$y(\theta) = m(\theta) x(\theta). \tag{23}$$

The initial values of f_1 and f_2 (determined from (14)) are $f_1(0) = 1$ and $f_2(0) = 0$. Hence, for $\theta = 0$, (23) produces the straight line $y(0) = 0$. For any arbitrary value of θ, it represents the same straight line in a rotated coordinate system, as is evident from figure 1. Here, m represents the slope of the line.

2.2 More on the functions f_1 and f_2

The equations governing f_1 and f_2 (obtained by substituting (14) in (5)) are

$$f_1' = -f_2, \tag{24a}$$
$$f_2' = f_1. \tag{24b}$$

From (24), we get

$$f_1'' = -f_1, \tag{25a}$$
$$f_2'' = -f_2. \tag{25b}$$

The characteristic equation [4] for both these equations is $\lambda^2 = -1$. Defining a quantity $i = \sqrt{-1}$, we get $\lambda = \pm i$. The general solution to (25) is

$$f_1(\theta) = A_1 e^{i\theta} + A_2 e^{-i\theta}, \tag{26a}$$
$$f_2(\theta) = B_1 e^{i\theta} + B_2 e^{-i\theta}, \tag{26b}$$

where

$$A_1 + A_2 = f_1(0) = 1, \tag{27a}$$
$$i(A_1 - A_2) = f_1'(0) = -f_2(0) = 0, \tag{27b}$$
$$B_1 + B_2 = f_2(0) = 0, \tag{27c}$$
$$i(B_1 - B_2) = f_2'(0) = f_1(0) = 1. \tag{27d}$$

Therefore,

$$f_1(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \tag{28a}$$
$$f_2(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \tag{28b}$$

We observe that f_1 is an even function of θ and f_2 is an odd function of θ. Equivalently expressing $e^{i\theta}$ in terms of $f_1(\theta)$ and $f_2(\theta)$, we arrive at

$$e^{i\theta} = f_1(\theta) + i f_2(\theta). \tag{29}$$

It is to be noted that an equality of the form $a_1 + ib_1 = a_2 + ib_2$, where a_1, b_1, a_2 and b_2 are real numbers, implies $a_1 - a_2 = i(b_2 - b_1)$. The left-hand side expresses a real number, whereas the right-hand side does not, unless $a_1 = a_2$ and $b_1 = b_2$.

Supposing $\theta = \theta_1 + \theta_2$ and substituting in (29), we obtain after some simplification

$$f_1(\theta_1 + \theta_2) = f_1(\theta_1)f_1(\theta_2) - f_2(\theta_1)f_2(\theta_2), \tag{30a}$$
$$f_2(\theta_1 + \theta_2) = f_1(\theta_1)f_2(\theta_2) + f_2(\theta_1)f_1(\theta_2). \tag{30b}$$

Further results may be derived by similar application of these equations. Henceforth, we resume with the common notation for the trigonometric functions.

2.3 A geometric deduction of (5)

For a geometric deduction of (5), consider figure 2. θ refers to an arbitrarily oriented coordinate system with origin O. The axes are rotated by an angle $\Delta\theta$ to form a new coordinate system $\theta + \Delta\theta$. Let P be a point referred to both the coordinate systems. From the figure,

$$\mathrm{OR} = \mathrm{OQ} + \mathrm{QR}, \tag{31}$$

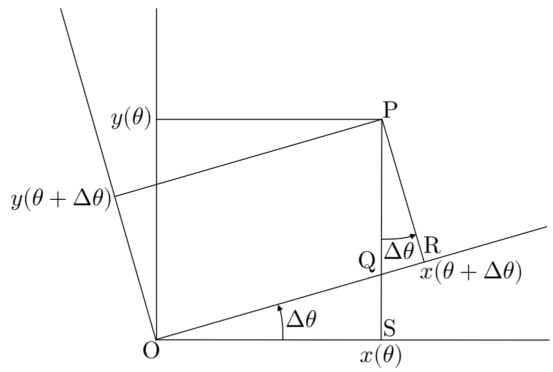

Figure 2: Two coordinate systems, one of whose axes are rotated by an angle $\Delta\theta$.

or,

$$x(\theta + \Delta\theta) = x(\theta)\sec\Delta\theta + y(\theta + \Delta\theta)\tan\Delta\theta. \tag{32}$$

In the infinitesimal limit of $\Delta\theta$, we get

$$x(\theta + d\theta) = x(\theta) + y(\theta)\, d\theta, \tag{33}$$

thus resulting in (5a). Similarly,

$$\mathrm{SP} = \mathrm{SQ} + \mathrm{QP}, \tag{34}$$

or,

$$y(\theta) = x(\theta)\tan\Delta\theta + y(\theta + \Delta\theta)\sec\Delta\theta, \tag{35}$$

whose infinitesimal limit yields (5b).

One may gain a better understanding by referring to figure 3, which depicts the rotation between two coordinate systems in the infinitesimal limit. It is readily seen that $dx = y\, d\theta$ and $-dy = x\, d\theta$.

3 Rotations in 3-dimensional Euclidean space

Let u_1, u_2 and u_3 be real numbers such that $u_1^2 + u_2^2 + u_3^2 = 1$. Consider now the system of equations

$$x' = u_3 y - u_2 z, \tag{36a}$$
$$y' = u_1 z - u_3 x, \tag{36b}$$
$$z' = u_2 x - u_1 y. \tag{36c}$$

Successive differentiation yields

$$x''' = -x', \quad y''' = -y', \quad z''' = -z'. \tag{37}$$

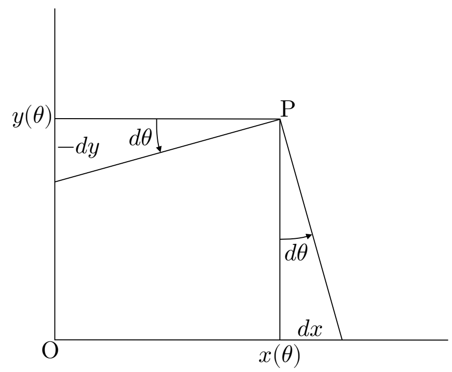

Figure 3: Two coordinate systems infinitesimally rotated with respect to each other.

The solution for x is

$$x(\theta) = C_1 \cos\theta + C_2 \sin\theta + C_3, \tag{38}$$

where

$$C_1 + C_3 = x(0), \tag{39a}$$
$$C_2 = x'(0) = u_3 y(0) - u_2 z(0), \tag{39b}$$
$$C_1 = -x''(0) = (1 - u_1^2)x(0) - u_1 u_2 y(0) - u_1 u_3 z(0). \tag{39c}$$

The solutions for y and z may be obtained similarly. The solution to (36) is

$$\begin{bmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta + u_1^2(1-\cos\theta) & u_1u_2(1-\cos\theta) + u_3\sin\theta & u_1u_3(1-\cos\theta) - u_2\sin\theta \\ u_1u_2(1-\cos\theta) - u_3\sin\theta & \cos\theta + u_2^2(1-\cos\theta) & u_2u_3(1-\cos\theta) + u_1\sin\theta \\ u_1u_3(1-\cos\theta) + u_2\sin\theta & u_2u_3(1-\cos\theta) - u_1\sin\theta & \cos\theta + u_3^2(1-\cos\theta) \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}. \tag{40}$$

Equation (40) describes a rotation by an angle θ about an axis pointing along the unit vector (u_1, u_2, u_3) (note the similarity with (3)). It may be checked that (36) reduces to (5) for $(u_1, u_2, u_3) = (0, 0, 1)$. Eliminating u_1, u_2 and u_3 from (36) results in

$$xx' + yy' + zz' = 0, \tag{41}$$

which on integrating gives

$$x(\theta)^2 + y(\theta)^2 + z(\theta)^2 = \text{constant}. \tag{42}$$

This proves the invariance of the (squared) Euclidean norm under rotations.

References

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