A convergent subsequence of $\theta_n(x+iy)$ in a half strip

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Abstract

For $\frac{1}{2} < x < 1$, $y > 0$ and $n \in \mathbb{N}$, let $\theta_n(x+iy) = \sum_{i=1}^{n} \frac{\operatorname{sgn} q_i}{q_i^{x+iy}}$, where $Q = \{q_1, q_2, q_3, \cdots\}$ is the set of finite product of distinct odd primes and $\operatorname{sgn} q = (-1)^k$ if q is the product of k distinct primes. In this paper we prove that there exists an ordering on Q such that $\theta_n(x+iy)$ has a convergent subsequence.

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1 Introduction

Let $\mathbb{N}$ be the set of natural numbers and P be the set of odd primes.

Definition 1.1. For an ordering on $P = \{p_1, p_2, p_3, \cdots\}$ and $m \in \mathbb{N}$, let

$$P_m = \{p_1, p_2, \cdots, p_m\}.$$

Definition 1.2. Let Q be the set of finite products of distinct odd primes.

$$Q = \{p_1 p_2 \cdots p_k \mid k \in \mathbb{N} \text{ and } p_1, p_2, \cdots, p_k \text{ are distinct primes in } P\}$$

and, for each $m \in \mathbb{N}$, let

$$U_m = \{p_1 p_2 \cdots p_k \mid k \in \mathbb{N} \text{ and } p_1, p_2, \cdots, p_k \text{ are distinct primes in } P_m\}.$$

Notice that U_m depends on the choice of ordering on P and $U_m \subset U_{m+1}$.

Lemma 1.3. *The number of elements of U_m is $2^m - 1$.*

Proof. Since

$$U_m = \{p_1, \cdots, p_m, p_1p_2, \cdots, p_{m-1}p_m, p_1p_2p_3, \cdots\cdots, p_1p_2\cdots p_m\},$$

the number of elements of U_m is

$$\binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m} = 2^m - 1.$$

□

Definition 1.4. Let

$$Q_1 = U_1 \text{ and } Q_m = U_m - U_{m-1} \text{ for each } m = 2, 3, 4, \cdots .$$

Notice that

$$Q_m = \{p_m, p_mq \mid q \in U_{m-1}\}, \qquad \bigcup_{i=1}^{m} Q_m = U_m \tag{1}$$

and $Q_1, Q_2, Q_3, \cdots$ are mutually disjoint. Notice also that the number of elements of Q_m is

$$(2^m - 1) - (2^{m-1} - 1) = 2^{m-1}.$$

Example 1.5. In the increasing ordering on P, we have

$$p_1 = 3,\ p_2 = 5,\ p_3 = 7, \cdots .$$

Therefore

$$Q_1 = \{3\},\ Q_2 = \{5,\ 3\cdot 5\},\ Q_3 = \{7,\ 3\cdot 7,\ 5\cdot 7,\ 3\cdot 5\cdot 7\}, \cdots .$$

Definition 1.6. An ordering on P and the following two conditions (C1)-(C2) induce a unique ordering on $Q = \{q_1, q_2, q_3 \cdots\}$.

(C1) $i < j$ if $q_i < q_j$ and $q_i, q_j \in Q_m$ for some m.

(C2) $i < j$ if $q_i \in Q_m,\ q_j \in Q_n$ for some $m < n$

Note that any ordering on P induces a unique ordering on Q in this way.

Example 1.7. Suppose that P has the increasing ordering. In the induced ordering on Q, we have

$$q_1 = 3,\ q_2 = 5,\ q_3 = 15,\ q_4 = 7,\ q_5 = 21,\ q_6 = 35,\ q_7 = 105,\ q_8 = 11, \cdots .$$

Definition 1.8. For each $q = p_1p_2\cdots p_k \in Q$, let

$$\operatorname{sgn} q = (-1)^k$$

where $p_1, p_2, \cdots, p_k$ are distinct odd primes.

Definition 1.9. Suppose that an ordering is given on $Q = \{q_1, q_2, q_3, \cdots\}$. For $\frac{1}{2} < x < 1$, $y > 0$ and $n \in \mathbb{N}$, let

$$\theta_n(x+iy) = \sum_{i=1}^{n} \frac{\operatorname{sgn} q_i}{q_i^{x+iy}}$$

In this paper we prove

Theorem 1.10. *For each $\frac{1}{2} < x < 1$ and $y > 0$, there exists an ordering on P such that, under the induced ordering on Q, $\theta_n(x+iy)$ has a convergent subsequence.*

2 Preliminary Theorems

We need the following theorem in the proof of Theorem 1.10.

Theorem 2.1 ([1]). *Suppose that $y > 0$, $0 \le \alpha < 2\pi$ and $0 < K < 1$. Let P^+ be the set of primes p such that $\cos(y \ln p + \alpha) > K$ and P^- the set of primes p such that $\cos(y \ln p + \alpha) < -K$. Then we have*

$$\sum_{p \in P^+} \frac{1}{p} = \infty \quad \textit{and} \quad \sum_{p \in P^-} \frac{1}{p} = \infty.$$

From the argument in the proof of the Riemann rearrangement theorem, we have

Theorem 2.2 ([4],[5]). *For a series $\sum_{i=1}^{\infty} a_i$ of real numbers, suppose that*

$$\lim_{i \to \infty} a_i = 0$$

and let

$$a_i^+ = max\{a_i, 0\} \qquad \textit{and} \qquad a_i^- = -min\{a_i, 0\}. \tag{2}$$

If

$$\sum_{i=1}^{\infty} a_i^+ = \sum_{i=1}^{\infty} a_i^- = \infty$$

then there exists a rearrangement such that the series $\sum_{i=1}^{\infty} a_i$ is convergent.

We need the Lévy-Steinitz theorem which is a generalization of the Riemann rearrangement theorem and Theorem 2.2.

Lévy-Steinitz theorem ([5]). *The set of all sums of rearrangements of a given series of vectors*

$$\sum_{i=1}^{\infty} \boldsymbol{v}_i$$

in $\mathbf{R}^n$ is either the empty set or a translate of subspace i.e., a set of the form $\mathbf{v} + M$, where $\mathbf{v}$ is a vector and M is a subspace. If the following two conditions (a)-(b) are satisfied then it is nonempty i.e., it has convergent rearrangements.

(a) $\lim_{i\to\infty} \mathbf{v}_i = \mathbf{0}$

(b) For all vector $\mathbf{w}$ *in* $\mathbb{R}^n$*,*

$$\sum_{i=1}^{\infty}(\mathbf{v}_i, \mathbf{w})^+ \quad \textit{and} \quad \sum_{i=1}^{\infty}(\mathbf{v}_i, \mathbf{w})^-$$

are either both finite or both infinite, where we use the notations in eq. (2) and $(\mathbf{v}_i, \mathbf{w})$ *is the Euclidean inner product of* $\mathbf{v}_i$ *and* $\mathbf{w}$*.*

The Coriolis test is useful in the proof of Theorem 1.10..

Coriolis Test ([6]). *If* z_i *is a sequence of complex numbers such that*

$$\sum_{i=1}^{\infty} z_i \quad \textit{and} \quad \sum_{i=1}^{\infty} |z_i|^2$$

are convergent, then

$$\prod_{i=1}^{\infty}(1+z_i)$$

converges.

3 Proof of Theorem 1.10

Definition 3.1. Suppose that P has the increasing ordering. For $\frac{1}{2} < x < 1$ and $y > 0$, let

$$\begin{aligned}
\rho(x+iy) &= \frac{1}{2^{x+iy}} + \sum_{i=1}^{\infty} \frac{1}{p_i^{x+iy}} \\
&= \frac{\cos(y\ln 2) - i\sin(y\ln 2)}{2^x} + \sum_{i=1}^{\infty} \frac{\cos(y\ln p_i) - i\sin(y\ln p_i)}{p_i^x}
\end{aligned}$$

Lemma 3.2. $\rho(x+iy)$ *has a convergent rearrangement and therefore*

$$\sum_{i=1}^{\infty} \frac{1}{p_i^{x+iy}} \tag{3}$$

has a convergent rearrangement, too. In other words, P *has an ordering such that eq. (3) is convergent.*

Proof. Recall that $\frac{1}{2} < x < 1$ and $y > 0$. Let

$$\mathbf{v}_1 = \left(\frac{\cos(y\ln 2)}{2^x}, \ -\frac{\sin(y\ln 2)}{2^x}\right)$$

and, for $i \in \mathbb{N}$, let

$$\mathbf{v}_{i+1} = \left(\frac{\cos(y \ln p_i)}{p_i^x}, \ -\frac{\sin(y \ln p_i)}{p_i^x} \right).$$

Since P has the increasing ordering, we have

$$\lim_{i\to\infty} \mathbf{v}_i = \mathbf{0}. \tag{4}$$

Let

$$\mathbf{w} = r(\cos\alpha, \ \sin\alpha)$$

be a vector in $\mathbb{R}^2$, where $r \geq 0$ and $0 \leq \alpha \leq 2\pi$. If $r = 0$ then $(\mathbf{v}_i, \mathbf{w}) = 0$ for all $i \in \mathbb{N}$ and therefore

$$\sum_{i=1}^{\infty} (\mathbf{v}_i, \mathbf{w})^+ = \sum_{i=1}^{\infty} (\mathbf{v}_i, \mathbf{w})^- = 0. \tag{5}$$

Suppose that $r > 0$. We have

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{w} &= \frac{r\cos(y\ln 2)\cos\alpha - r\sin(y\ln 2)\sin\alpha}{2^x} \\ &= \frac{r\cos(y\ln 2 + \alpha)}{2^x} \end{aligned}$$

and

$$\begin{aligned} \mathbf{v}_{i+1} \cdot \mathbf{w} &= \frac{r\cos(y\ln p_i)\cos\alpha - r\sin(y\ln p_i)\sin\alpha}{p_i^x} \\ &= \frac{r\cos(y\ln p_i + \alpha)}{p_i^x} \end{aligned}$$

Let P^+ be the set of primes p such that $\cos(y \ln p + \alpha) > \frac{1}{2}$ and P^- the set of primes p such that $\cos(y \ln p + \alpha) < -\frac{1}{2}$. From Theorem 2.1, we have

$$\begin{aligned} \sum_{i=1}^{\infty} (\mathbf{v}_i, \mathbf{w})^+ &\geq \sum_{p\in P^+} \frac{r\cos(y\ln p + \alpha)}{p^x} \\ &\geq \frac{r}{2} \sum_{p\in P^+} \frac{1}{p^x} \geq \frac{r}{2} \sum_{p\in P^+} \frac{1}{p} = \infty \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{\infty} (\mathbf{v}_i, \mathbf{w})^- &\geq -\sum_{p\in P^-} \frac{r\cos(y\ln p + \alpha)}{p^x} \\ &\geq \frac{r}{2} \sum_{p\in P^-} \frac{1}{p^x} \geq \frac{r}{2} \sum_{p\in P^-} \frac{1}{p} = \infty. \end{aligned}$$

Therefore

$$\sum_{i=1}^{\infty}(\mathbf{v}_i,\mathbf{w})^+ = \sum_{i=1}^{\infty}(\mathbf{v}_i,\mathbf{w})^- = \infty. \tag{6}$$

From eq. (4), (5), (6) and Lévy-Steinitz theorem, we know that the series of vectors in $\mathbb{R}^2$

$$\sum_{i=1}^{\infty}\mathbf{v}_i$$

has a convergent rearrangement, and therefore $\rho(x+iy)$ has a convergent rearrangement. □

Lemma 3.3. *Let $z = x+iy$. For all $m \in \mathbb{N}$, we have*

$$\prod_{i=1}^{m}\left(1-\frac{1}{p_i^z}\right)-1 = \sum_{q\in Q_1}\frac{\operatorname{sgn} q}{q^z}+\sum_{q\in Q_2}\frac{\operatorname{sgn} q}{q^z}+\cdots+\sum_{q\in Q_m}\frac{\operatorname{sgn} q}{q^z}.$$

Proof. We use induction on m. If $m=1$, it is clear. Suppose that it is true for $m=k-1$. We will show that it is true for $m=k$. From eq. (1), we have

$$\begin{aligned}
\prod_{i=1}^{k}\left(1-\frac{1}{p_i^z}\right) &= \left(\prod_{i=1}^{k-1}\left(1-\frac{1}{p_i^z}\right)\right)\left(1-\frac{1}{p_k^z}\right)\\
&= \left(1+\sum_{q\in Q_1}\frac{\operatorname{sgn} q}{q^z}+\cdots+\sum_{q\in Q_{k-1}}\frac{\operatorname{sgn} q}{q^z}\right)\left(1-\frac{1}{p_k^z}\right)\\
&= \left(1+\sum_{q\in Q_1}\frac{\operatorname{sgn} q}{q^z}+\cdots+\sum_{q\in Q_{k-1}}\frac{\operatorname{sgn} q}{q^z}\right)\\
&\qquad -\frac{1}{p_k^z}\left(1+\sum_{q\in Q_1}\frac{\operatorname{sgn} q}{q^z}+\cdots+\sum_{q\in Q_{k-1}}\frac{\operatorname{sgn} q}{q^z}\right)\\
&= \left(1+\sum_{q\in Q_1}\frac{\operatorname{sgn} q}{q^z}+\cdots+\sum_{q\in Q_{k-1}}\frac{\operatorname{sgn} q}{q^z}\right)-\frac{1}{p_k^z}\left(1+\sum_{q\in U_{k-1}}\frac{\operatorname{sgn} q}{q^z}\right)\\
&= 1+\sum_{q\in Q_1}\frac{\operatorname{sgn} q}{q^z}+\cdots+\sum_{q\in Q_{k-1}}\frac{\operatorname{sgn} q}{q^z}+\sum_{q\in Q_k}\frac{\operatorname{sgn} q}{q^z}
\end{aligned}$$

□

Now we can prove Theorem 1.10.

Proof of Theorem 1.10

By Lemma 3.2, we can choose an ordering on P such that

$$\sum_{i=1}^{\infty} \frac{1}{p_i^{x+iy}}$$

is convergent. From now on, we assume that P has the chosen ordering, and Q has the induced ordering.

Since $\frac{1}{2} < x < 1$,

$$\sum_{i=1}^{\infty} \left| \frac{1}{p_i^{x+iy}} \right| = \sum_{i=1}^{\infty} \frac{1}{p_i^{2x}}$$

is convergent. Therefore, by the Coriolis test,

$$\prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^{x+iy}} \right)$$

is convergent. By Lemma 3.3, Lemma 1.3 and eq. (1), we have

$$\begin{aligned} \prod_{i=1}^{m} \left(1 - \frac{1}{p_i^{x+iy}} \right) - 1 &= \sum_{q \in Q_1} \frac{\operatorname{sgn} q}{q^{x+iy}} + \sum_{q \in Q_2} \frac{\operatorname{sgn} q}{q^{z}} + \cdots + \sum_{q \in Q_m} \frac{\operatorname{sgn} q}{q^{x+iy}} \\ &= \sum_{q \in U_m} \frac{\operatorname{sgn} q}{q^{x+iy}} \\ &= \sum_{i=1}^{2^m - 1} \frac{\operatorname{sgn} q_i}{q_i^{x+iy}}. \end{aligned}$$

Therefore $\theta_{2^m-1}(x+iy)$ is a convergent subsequence of $\theta_n(x+iy)$. □

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