Complete Integers. Extending integers to allow real powers have discontinuities in zero

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Abstract

In this paper we will define a superset of integers (the complete integers, $\mathbb{Z}_C$), which contains the dual of integers along parity (e.g. the odd zero, the even one, ...). Then we will see how they form a ring and how they can be used as exponents for real numbers powers, in order to write functions which have a discontinuity in zero (the function itself or one of its derivates), as for example $|x|$ and $\operatorname{sgn}(x)$.

1 Introduction

Doing calculus with polynomials is very easy, aside the fact they are all C^∞, you can apply some simple rules to resolve all the possible polynomials [Lan51]. Instead with functions defined by cases and with functions which are not C^∞, the analysis becomes more complicated: you usually have to study separately each case, and then combine them back.

It would be helpful to have a way to write those functions in a polynomial-like way. A lot of functions can be approximated with series (e.g. Taylor and Maclaurin), but just as many don't [TFW96].

In this paper we will explore a superset of the integers which introduces the dual of integers along parity, so we will define the "odd zero" (a number with null value, but the same behaviour of odd numbers), the "even one" (a number with unitary value, but the same behaviour of even numbers), and so on.

The peculiarity of those numbers is that they will allow us to define an exponentiation of reals $f(x) = x^z$ (where $x \in \mathbb{R}$, $z \in \mathbb{Z}_C$) which is not C^∞. More precisely we will see that a real number elevated to a complete integer is allowed to have a discontinuity or a non-differentiable point in zero.

All the theorems, lemmas and corollaries have been proved using Agda proof assistant [BDN09], based on Martin-Löf type theory [MS84]. The code of the proofs is available at the following git repository: `https://gitlab.com/DPDmancul/complete-integers-agda.git`. For the non trivial proofs a trail is reported in this paper.

2 Complete integers

In this section we will define the ring of complete integers ($\mathbb{Z}_C$) and we will show some useful properties of this numbers.

Definition 1 (Complete integers)**.** Let's define the set of the complete integer numbers as

$$\mathbb{Z}_C \coloneqq \mathbb{Z} \times \mathbb{F}_2$$

Where $\mathbb{F}_2$ is the filed of integers modulo 2 [LN97].

We will call the first component *value*, and the second *parity*, and write them as $\begin{bmatrix} value \\ parity \end{bmatrix}$.

The *value* states the quantity represented by the complete integer, whilst the *parity* represents its evenness; we will discuss it better in subsection 2.2.

Definition 2 (Ring of complete integers)**.** Let's define $\mathbb{Z}_C$ as a commutative ring with unit:

Given $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{Z}_C$

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \coloneqq \begin{bmatrix} a + c \\ b \oplus d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} \coloneqq \begin{bmatrix} a \cdot c \\ b \cdot d \end{bmatrix}$$

where $\oplus$ is the sum modulo 2 (known in boolean algebra as exclusive or) [LN97; Gre98].

It is easy to prove it forms a commutative ring with unit $\begin{bmatrix}1\\1\end{bmatrix}$ and zero $\begin{bmatrix}0\\0\end{bmatrix}$.

Lemma 1 (Powers of complete integers)**.**

$$\begin{bmatrix}v\\p\end{bmatrix}^n = \begin{bmatrix}v^n\\p\end{bmatrix} \qquad \forall\ n \in \mathbb{N}^+$$
$$\begin{bmatrix}v\\p\end{bmatrix}^0 = \begin{bmatrix}1\\1\end{bmatrix}$$

2.1 Cartesian plot

We can visualize the sum between complete integers as a vector sum on a bi-dimensional Cartesian plane with the value on x-axis and parity on y-axis (Figure 1).

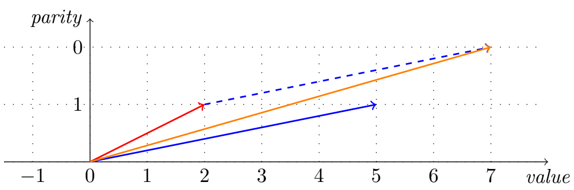

Figure 1: Sum of complete integers.
$\begin{bmatrix}2\\1\end{bmatrix}$; $\begin{bmatrix}5\\1\end{bmatrix}$; $\begin{bmatrix}7\\0\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}5\\1\end{bmatrix}$

Obviously $\mathbb{Z}_C$ is not a vector space because $\mathbb{Z}$ is not a field and is also a different set from $\mathbb{F}_2$. In fact on x-axis we could have only integer numbers and on y-axis we have only 0 and 1 cyclically repeated (integers modulo 2).

The previous representation is valid also for "scalar product" (sum exponentiation), which is the product by an integer (Figure 2).

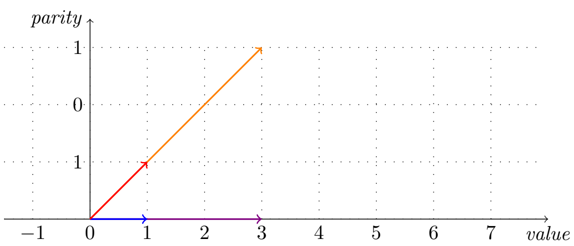

Figure 2: Sum exponentiation of complete integers.
$\begin{bmatrix}1\\1\end{bmatrix}$; $\begin{bmatrix}1\\0\end{bmatrix}$; $\begin{bmatrix}3\\1\end{bmatrix} = 3 \cdot \begin{bmatrix}1\\1\end{bmatrix}$; $\begin{bmatrix}3\\0\end{bmatrix} = 3 \cdot \begin{bmatrix}1\\0\end{bmatrix}$

Remark 1. Repeating parity on the negative side of the y-axis we notice that the "vectors" which goes up by one or goes down by one alongside y-axis are indeed the same (Figure 3). So we don't need to repeat parity (Figure 4).

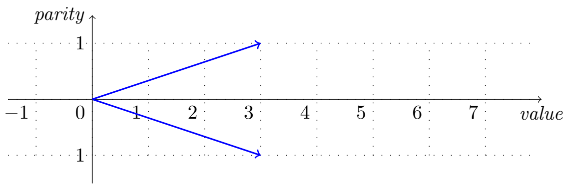

Figure 3: ±1 on the parity axis is the same.
$\begin{bmatrix}3\\1\end{bmatrix}$

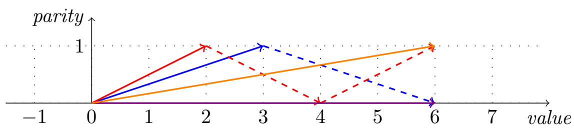

Figure 4: Sum exponentiation without repeating y-axis.
$\begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = 2 \cdot \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix}$; $3 \cdot \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}6\\1\end{bmatrix}$

2.2 Value and parity

In this subsection we will see two functions, which explain the role of the value and of the parity, and see their properties.

Definition 3 (Value function)**.**

$$\mathrm{val}\colon \mathbb{Z} \cup \mathbb{Z}_C \to \mathbb{Z}$$
$$\mathrm{val}(z) := z \qquad \forall\ z \in \mathbb{Z}$$
$$\mathrm{val}\left(\begin{bmatrix}v\\p\end{bmatrix}\right) := v \qquad \forall\ \begin{bmatrix}v\\p\end{bmatrix} \in \mathbb{Z}_C$$

The value function returns the quantity represented by a number. For an integer it is obviously itself, and for a complete integer it is its first component, as said before.

We can compare this function to the absolute value. In facts we can write[1] $\mathbb{Z} = \mathbb{N} \times \{+, -\}$ and the absolute value of a integer defined in this way would return the first component of it: the magnitude stripped by the sign. As the absolute value

[1]Paying attention to the fact $(0, +) = (0, -)$.

returns the magnitude of a number stripped by the information about its sign (all absolute values are positive), the value function returns the quantity (with the sign), but stripped by its information about the parity.

Theorem 1 (Properties of value)**.** *Given $x, y \in \mathbb{Z} \vee x, y \in \mathbb{Z}_C$ and $z \in \mathbb{Z}$ it is possible to show the following properties hold.*

1. *Value is an odd function.*

$$\operatorname{val}(-x) = -\operatorname{val}(x)$$

2. *Value is a linear function.*

$$\operatorname{val}(x+y) = \operatorname{val}(x) + \operatorname{val}(y)$$

And recalling that, since z is an integer number, $z \cdot x$ is the sum exponentiation (i.e. summing x for z times):
$z \cdot x = \operatorname{sgn}(z) \cdot (\underbrace{x + x + \cdots + x}_{|z| \text{times}})$

$$\operatorname{val}(z \cdot x) = z \operatorname{val}(x)$$

3. *Idempotence of the value.*

$$\operatorname{val} \circ \operatorname{val} = \operatorname{val}$$

4. *Completely multiplicative.*

$$\operatorname{val}(1) = \operatorname{val}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 1$$
$$\operatorname{val}(x \cdot y) = \operatorname{val}(x) \cdot \operatorname{val}(y)$$

Definition 4 (Parity function)**.**

$$\operatorname{par}\colon \mathbb{Z} \cup \mathbb{Z}_C \to \mathbb{F}_2$$
$$\operatorname{par}(z) := \begin{cases} 0 & z \text{ even} \\ 1 & z \text{ odd} \end{cases} \qquad \forall\, z \in \mathbb{Z}$$
$$\operatorname{par}\left(\begin{bmatrix}v\\p\end{bmatrix}\right) := p \qquad \forall \begin{bmatrix}v\\p\end{bmatrix} \in \mathbb{Z}_C$$

The parity function returns the evenness of its argument. We can read it as a boolean value: 0 means `false`, whilst 1 means `true`, as an answer to the question "is it even?". Another point of view is saying a number has the same evenness of its parity.

Obviously the parity of an even integer is the same of 0, whilst the parity of an odd integer is the same of 1. For what concerns complete integers we said in the definition the second component represents the parity of the number.

In the analogy we made before between value function and absolute value, the parity function would correspond to the sign: as the sign discards all the information about the magnitude of an integer, retaining only its sign, the parity function returns a representative of the parity, discarding the value of the number.

Theorem 2 (Properties of parity)**.** *Given $x, y \in \mathbb{Z} \vee x, y \in \mathbb{Z}_C$ and $z \in \mathbb{Z}$ we can prove the following properties of parity function.*

1. *Parity is an even function.*

$$\operatorname{par}(-x) = \operatorname{par}(x)$$

2. *Parity is a linear function.*
Since $\operatorname{par}(x) \in \mathbb{F}_2$ the sum operator must be replaced by exclusive or ($\oplus$).

$$\operatorname{par}(x+y) = \operatorname{par}(x) \oplus \operatorname{par}(y)$$

3. *Idempotence of the parity.*

$$\operatorname{par} \circ \operatorname{par} = \operatorname{par}$$

4. *Completely multiplicative.*

$$\operatorname{par}(1) = \operatorname{par}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 1$$
$$\operatorname{par}(x \cdot y) = \operatorname{par}(x) \cdot \operatorname{par}(y)$$

It is now easy to see how the parity of a complete integer is not a mere binary flag, but induces the same properties of even and odd numbers into $\mathbb{Z}_C$: complete integers with even parity act like even numbers and those with odd parity like odd numbers. This is summarized in Table 1 (recall 0 is a representative of even parity and 1 for odd parity).

Corollary 2.1 (Parity of powers)**.** *The parity of a power, with positive natural exponent, is the same of the base.*

$$\operatorname{par}(x^n) = \operatorname{par}(x) \quad \forall\, x \in \mathbb{Z} \cup \mathbb{Z}_C, n \in \mathbb{N}^+$$

$\text{par}(x)$	$\text{par}(y)$	$\text{par}(x+y)$	$\text{par}(xy)$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Table 1: The behaviour induced by parity in sum and multiplication.

3 Integers and dis-integers

In the previous section we have defined the set of complete integers and seen it forms a ring with the given operations. Despite in the introduction we said $\mathbb{Z}_C$ to be a superset of the integers we haven't stated such relation yet. In this section we will define an isomorphism between a subset of $\mathbb{Z}_C$ and $\mathbb{Z}$ to prove our first utterance.

Moreover we will analyze the other complete integers (the ones which are not involved in this isomorphism) and their properties.

Definition 5 (Integers prime)**.** Let us define the set of integers prime as the subset of $\mathbb{Z}_C$ where the parity equals the parity of the value.

$$\mathbb{Z}' := \left\{ \begin{bmatrix} v \\ p \end{bmatrix} \in \mathbb{Z}_C : p = \text{par}(v) \right\} = \\ = \left\{ \begin{bmatrix} v \\ \text{par}(v) \end{bmatrix} : v \in \mathbb{Z} \right\}$$

Definition 6 (Dis-integers)**.** Let us define the set of dis-integers as the subset of $\mathbb{Z}_C$ where the parity is different from the parity of the value.

$$\mathbb{Z}_D := \left\{ \begin{bmatrix} v \\ p \end{bmatrix} \in \mathbb{Z}_C : p \neq \text{par}(v) \right\}$$

Remark 2. $\{\mathbb{Z}', \mathbb{Z}_D\}$ is a partition of $\mathbb{Z}_C$.

$$\mathbb{Z}' \sqcup \mathbb{Z}_D = \mathbb{Z}_C$$

Theorem 3 (Integers and integers prime are isomorphic)**.** *The function $f_{\mathbb{Z}} \colon \mathbb{Z} \to \mathbb{Z}'$ defined as*

$$f_{\mathbb{Z}}(z) = \begin{bmatrix} z \\ \text{par}(z) \end{bmatrix}$$

is an isomorphism.

Since $\mathbb{Z}'$ is isomorphic to $\mathbb{Z}$, and so the two cannot be distinguished, we won't write $\mathbb{Z}'$ anymore and we will use the notation $\begin{bmatrix} v \\ \text{par}(v) \end{bmatrix}$ to denote elements in $\mathbb{Z}$ too.

More precisely we will write, with an abuse of notation, $\mathbb{Z}' = \mathbb{Z}$ and $\begin{bmatrix} v \\ \text{par}(v) \end{bmatrix} = v$ meaning respectively $\mathbb{Z}' = f_{\mathbb{Z}}(\mathbb{Z})$ and $\begin{bmatrix} v \\ \text{par}(v) \end{bmatrix} = f_{\mathbb{Z}}(v)$.

3.1 Dis-integers

We said in the introduction that dis-integers are the dual of integers along parity, in fact in $\mathbb{Z}_D$ we have $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which has null value and odd parity, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which has unitary value, but even parity, and in general $\begin{bmatrix} v \\ p \end{bmatrix}$ where the parity p is the dual of parity of the integer v.

Definition 7 (Odd zero)**.** Let's call $o := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the *odd zero*, since it has null value and odd parity.

Lemma 2 (Swap parity)**.** *Summing the odd zero to a complete integer its parity changes, but not its value.*

$$\text{val}(z + o) = \text{val}(z) \quad \forall z \in \mathbb{Z}_C$$

$$\text{par}(z + o) \neq \text{par}(z) \quad \forall z \in \mathbb{Z}_C$$

more precisely

$$\text{par}(z + o) = 1 - \text{par}(z) = \overline{\text{par}(z)} \quad \forall z \in \mathbb{Z}_C$$

Definition 8 (Even unit)**.** Let's call $l := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ the *even unit*, since it has unitary value and even parity.

Lemma 3 (Change only value)**.** *Summing the even unit to a complete integer only it's value changes, not its parity.*

$$\text{val}(z + l) = \text{val}(z) + 1 \qquad \forall z \in \mathbb{Z}_C$$
$$\text{par}(z + l) = \text{par}(z) \qquad \forall z \in \mathbb{Z}_C$$

Now we can link integers and dis-integers

Lemma 4 (Dis-integer as integer plus even unit)**.** *Each dis-integer can be written as the sum of an integer with l.*

$$\forall\ z \in \mathbb{Z}_D\ \exists\ y \in \mathbb{Z} \colon z = y + l$$

Proof. We can write $z = \begin{bmatrix} v \\ \mathrm{par}(v) \end{bmatrix}$, since dis-integers have parity different from the parity of the value and in $\mathbb{F}_2$ we have only two elements.

Picking $y = \begin{bmatrix} v-1 \\ \mathrm{par}(v-1) \end{bmatrix} \in \mathbb{Z}$ we have

$$\mathrm{par}(v-1) = \mathrm{par}(v) \oplus \mathrm{par}(-1) = \mathrm{par}(v) \oplus 1 = \overline{\mathrm{par}(v)}$$

$$\therefore$$

$$y + l = \begin{bmatrix} v-1 \\ \mathrm{par}(v) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = z$$

□

4 Real powers

In this section we will define an exponentiation function with real bases and complete integer exponents.

To help us come up with a good definition we can split on the exponent z:

1. If z is an integer, this operation is already defined as x^z for $x \in \mathbb{R}$;

2. If z is a dis-integer, we know from Lemma 4 that there exist an integer y s.t. $z = y + l$; supposing that our function respects exponent rules (which we will prove in Theorem 4), we can write $x^z = z^{y+l} = z^y \cdot z^l$.

So all we have to do is to define the value of x^l. If we pick an $x \in \mathbb{R}$ we can intuitively say that x^l should be equal to $|x|$ because:

1. being the parity of l even, x^l should be an even function of x;

2. being the value of l one, x^l should be a somewhat linear function.

So our definition, for $x \in \mathbb{R}$ and $z \in \mathbb{Z}_C$, would be:

$$x^z = \begin{cases} \text{usual } x^z & z \in \mathbb{Z} \\ x^y \cdot x^l = x^y \cdot |x| & z \in \mathbb{Z}_D \end{cases}$$

with $y = z - l \in \mathbb{Z}$

We will instead use another definition, which could be proven to be equal.

Definition 9 (Real exponentiation to complete integers)**.** For $x \in \mathbb{R}$ and $z \in \mathbb{Z}_C$, we define

$$x^z = x^{\mathrm{val}(z)-k}|x|^k$$

with $k = \mathrm{par}\,(\mathrm{val}(z)) \oplus \mathrm{par}(z)$.

Remark 3. Integers have a value of $k = 0$, while dis-integers have a value of $k = 1$. So we can call k the *dis-integerness* of a complete integer.

Remark 4. We can easily see how the squared norm of the vector representing a complete integer in the Cartesian plane has the same parity of its *dis-integerness* k.

In other words given $\begin{bmatrix} v \\ p \end{bmatrix} \in \mathbb{Z}_C$

$$\mathrm{par}\left(\left\|\begin{pmatrix} v \\ p+2a \end{pmatrix}\right\|^2\right) = \mathrm{par}\,(v) \oplus p = k \quad \forall a \in \mathbb{Z}$$

Lemma 5 (Dis-integerness of sum)**.** *The* dis-integerness k_{z+w} *of the sum of two complete integers is the sum modulo two of the corresponding* dis-integernesses*: given* $z, w \in \mathbb{Z}_C$ *and* $k_y = \mathrm{par}(\mathrm{val}(y)) \oplus \mathrm{par}(y)$

$$k_{z+w} = k_z \oplus k_w = k_z + k_w - 2k_z k_w$$

Remark 5. We can also prove the parity of the dot product of two vectors representing complete integers to be the *dis-integerness* k_{zw} of the product of the two complete integers.

In other words given $z = \begin{bmatrix} v_z \\ p_z \end{bmatrix}, w = \begin{bmatrix} v_w \\ p_w \end{bmatrix} \in \mathbb{Z}_C$

$$\mathrm{par}\left(\begin{pmatrix} v_z & p_z + 2a \end{pmatrix} \begin{pmatrix} v_w \\ p_w + 2b \end{pmatrix}\right) =$$
$$= \mathrm{par}\,(v_z v_w) \oplus p_z p_w = k_{zw} \quad \forall a, b \in \mathbb{Z}$$

Remark 4 and 5 suggest another reading of the *dis-integerness*: it is the parity of the energy of a vector, or of the mutual energy of two vectors [EL07], representing a complete integer.

Theorem 4 (Exponent rules)**.** *Definition 9 respects exponent rules, i.e. for* $x, y \in \mathbb{R}$ *and* $z, w \in \mathbb{Z}_C$

$$x^{z+w} = x^z \cdot x^w; \quad (x \cdot y)^z = x^z \cdot y^z; \quad (x^z)^w = x^{zw}$$

Proof. With $\begin{bmatrix} v_z \\ p_z \end{bmatrix} = z, \begin{bmatrix} v_w \\ p_w \end{bmatrix} = w$ and $k_y = \mathrm{par}(v_y) \oplus p_y$

- $x^{z+w} = x^z \cdot x^w$

$$x^{z+w} = x^{z+w-k_{z+w}}|x|^{k_{z+w}}$$

Which, using Lemma 5, becomes

$$x^{z+w-k_z-k_w+2k_zk_w}|x|^{k_z+k_w}|x|^{-2k_zk_w} =$$
$$= x^{z+w-k_z-k_w\cancel{+2k_zk_w}}|x|^{k_z+k_w}\cancel{x^{-2k_zk_w}} =$$
$$= x^{z-k_z}|x|^{k_z}x^{w-k_w}|x|^{k_w} = x^zx^w$$

- $(x \cdot y)^z = x^z \cdot y^z$

$$(xy)^z = (xy)^{v_z-k_z}|xy|^{k_z} =$$
$$= x^{v_z-k_z}|x|^{k_z}y^{v_z-k_z}|y|^{k_z} = x^zy^z$$

- $(x^z)^w = x^{zw}$

$$(x^z)^w = x^{(v_z-k_z)w}|x|^{k_zw}$$

and calling $y \coloneqq v_z - k_z$ we have

$$x^{v_{yw}-k_{yw}}|x|^{k_{yw}}|x|^{v_{k_zw}\cancel{-k_{k_zw}}}\cancel{|x|^{k_{k_zw}}} =$$
$$= x^{v_{yw}-k_{yw}}|x|^{k_{yw}+v_{k_zw}} =$$
$$= x^{v_{yw}-k_{yw}}|x|^{k_{yw}+v_{k_zw}-k_{zw}}|x|^{k_{zw}}$$

Knowing $k_{yw} + v_{k_zw} - k_{zw}$ is even we obtain

$$x^{v_{yw}\cancel{-k_{yw}+k_{yw}}+v_{k_zw}-k_{zw}}|x|^{k_{zw}} =$$
$$x^{((v_z-\cancel{k_z})+\cancel{k_z})v_w-k_{zw}}|x|^{k_{zw}} = x^{zw}$$

Finally $k_{yw} + v_{k_zw} - k_{zw}$ is indeed even, since its parity is zero:

$$\text{par}(k_{yw} + v_{k_zw} - k_{zw}) =$$
$$= \underbrace{p_yp_{v_w} \oplus p_yp_w}_{p_yk_w} \oplus \underbrace{p_{k_z}p_{v_w} \oplus p_{v_z}p_{v_w}}_{p_zp_{v_w}} \oplus p_zp_w =$$
$$= p_yk_w \oplus p_zk_w = (p_y \oplus p_z)k_w = 0k_w = 0$$

□

We will now see how two well known functions, which are not C^∞, can be written as a real exponentiation to the power of a complete integer.

Remark 6 (Absolute value). As defined above, a real elevated to the power of the even unit is its absolute value: $x^l = |x|$. This function is C^0, but not C^1, hence cannot be expressed by a polynomial.

Remark 7 (Sign function). A real number elevated to the power of the odd zero is the sign function: $x^o = \text{sgn}(x)$. This function, being the derivative of $|x|$, is not C^0 (continuous), and so cannot by written as a polynomial.

4.1 Calculus

Now we will briefly see how to differentiate and integrate the aforementioned function

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^z \quad z \in \mathbb{Z}_C$$

Lemma 6 (Derivative)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}x^z = \text{val}(z)x^{z-1} \quad \forall z \in \mathbb{Z}_C$$

Proof. With $\begin{bmatrix} v \\ p \end{bmatrix} = z$ and $k = \text{par}(v) \oplus p$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^z = \frac{\mathrm{d}}{\mathrm{d}x}x^{v-k}|x|^k$$

- $k = 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{v-0}|x|^0 = \frac{\mathrm{d}}{\mathrm{d}x}x^v = vx^{v-1}$$

From Remark 3 we know z to be an integer, and so, by definition, $z = v$

$$\therefore \quad vx^{v-1} = \text{val}(z)x^{z-1}$$

- $k = 1$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{v-1}|x| = |x|\frac{\mathrm{d}}{\mathrm{d}x}x^{v-1} + x^{v-1}\frac{\mathrm{d}|x|}{\mathrm{d}x} =$$
$$= (v-1)\underbrace{x^{v-1-1}|x|}_{x^{z-1}} + x^{v-1}\,\text{sgn}(x)$$

which, for Remark 7, becomes

$$(v-1)x^{z-1} + x^{v-1}x^o = (v-1)x^{z-1} + x^{v+o-1}$$

Finally, by Remark 3 and Lemma 2, $z = v + o$. In facts z is a dis-integer, so $\text{par}(v) = \overline{p}$, but $\text{val}(v+o) = v$ and $\text{par}(v+o) = \overline{\text{par}(v)} = p$.

□

Lemma 7 (Antiderivative)**.**

$$\int x^z \,\mathrm{d}x = \frac{x^{z+1}}{\text{val}(z)+1} + c \quad \textit{with } c \in \mathbb{R};\ \forall z \in \mathbb{Z}_C$$

Proof. By Lemma 6 we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{z+1}}{\text{val}(z)+1} + c\right) = \text{val}(z+1)\frac{x^z}{\text{val}(z)+1} =$$
$$= \cancel{(\text{val}(z)+1)}\frac{x^z}{\cancel{\text{val}(z)+1}}$$

□

We can happily observe how those two power rules are very similar to the power rules for integer exponents [Lan51]. The only difference is that as coefficient we have to take the value of the exponent. This make sense since in real numbers talking about parity is meaningless.

5 Conclusions

We have defined the field of complete integers over a set containing integers and dis-integers: the dual of integers along parity. To understand better what we have done we defined the concepts of value and parity and showed the analogy between absolute value and value and that between sign and parity to compare the extension of naturals into integers to our extension from integers to complete integers. As in natural numbers there is already the concept of sign, but we don't have all combinations of absolute value / sign (i.e. we have only positive numbers), also in integers we have already the concept of parity, but not all combinations of value / parity (i.e. we have only the even zero, the odd one, and so on). So the complete integers cover the missing combinations.

The aim of this extension is to define an exponentiation of the real numbers which allows discontinuities in zero: in this way we can easily study some functions like the sign and the absolute value, which are not C^∞, as they were classical powers. In fact the differentiation and integration rules are very similar to classical power rules. As we will see in the first proposed future work (subsection 6.1) this could be further extended to all the functions by cases for which each case could be represented by a polynomial.

In the meanwhile we also showed how to represent complete integers on a Cartesian plane and took note of the relation between the dot product of the vectors induced by this representation and the *dis-integerness* of the complete integers represented.

6 Future work

The contents presented in this paper have no practical utility by themselves, but could be a trampoline for other studies. In this section two possible future works are presented.

6.1 Functions by cases in a polynomial-like form

Remark 7 tell us that using complete integers as exponents we can write functions with a discontinuity in zero. We can use this fact to define an extension of polynomials, which have complete integer exponents. With those "polynomials" we could write all functions which are interpolable with a polynomial for $x \leqslant 0$ and with another for $x \geqslant 0$.

We can further extend this concept summing some of those extended "polynomials", translated on the x axis (evaluated in $x - \bar{x}$): every of those "polynomials" has a different discontinuity point $\bar{x}$. So those hyper "polynomials" could represent all functions by cases for which each case can by represented with a polynomial.

For example, as can be seen in Figure 5, $\frac{(x-a)^o-(x-b)^o}{2}$ is always zero, except in the interval (a, b), and so can be used as a "mask" for that case.

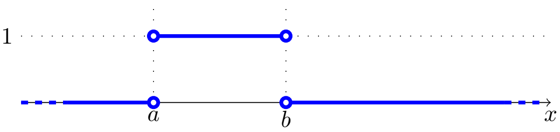

Figure 5: Selector function $\frac{(x-a)^o-(x-b)^o}{2}$.

6.2 Complete rationals

As we have defined the commutative ring $\mathbb{Z}_C = \mathbb{Z} \times \mathbb{F}_2$, we can define also a commutative ring over $\mathbb{Q}_H = \mathbb{Q} \times \mathbb{F}_2$. In this case however we cannot prove $\mathbb{Q} \subset \mathbb{Q}_H$. Defining $\mathbb{Q}_{\bar{H}} = \left\{ \frac{q}{l} : q \in \mathbb{Q}_H \right\}$ and $\mathbb{Q}_C = \mathbb{Q}_H \sqcup \mathbb{Q}_{\bar{H}}$ we could prove:

1. $\mathbb{Q} \subset \mathbb{Q}_C$;
2. The division of two complete integers is in $\mathbb{Q}_C$;
3. $\mathbb{Q}_C = \{x/y : x, y \in \mathbb{Z}_C ; y \neq 0\}$.

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