One Tile Suffices

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Abstract

We have found for all k larger than two a possibility to tile the plane completely with k-gons. We use infinite many copies of a single tile. The proofs are not by written words, but by pictures. Amongst others, we use the well-known tiling with hexagons. We show for k larger than 4 new ways to cover the plane.

We think that it is useful to repeat the definition of a *simple polygon*.
A simple polygon with k vertices consists of k different points of the plane (x_1, y_1), (x_2, y_2), $\ldots (x_{k-1}, y_{k-1})$, (x_k, y_k), called *vertices*, and the straight lines between (x_i, y_i) and (x_{i+1}, y_{i+1}) for $1 \leq i \leq k-1$, called *edges*. Also the straight line between (x_k, y_k) and (x_1, y_1) belongs to the polygon. We demand that it is homeomorphic to a circle, and that there are no three consecutive collinear points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$ for $1 \leq i \leq k-2$. Also the three points $(x_k, y_k), (x_1, y_1), (x_2, y_2)$ and $(x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)$ are not collinear.
We call this just described simple polygon a *k-gon*.

Theorem 1. Let k be a natural number larger than 2. Than there exists a tiling of the plane $\mathbb{R}^2$ by k-gons. We need infinite copies of only a single tile.

Proof. This theorem is well-known. Please see [1], p. 11.
There is another proof. For $k = 3$ and $k = 4$ and $k = 6$ the theorem is trivial. For $k = 5$ please see Figure 1. We take a regular 6-gon and cut it into identical halves. See also the tiling in the case $k = 6$.

Now let k be a natural number larger than 6.

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• Possibility 1: $k \equiv 0$ *mod* 4.
The numbers k are $8, 12, 16, \ldots$
See Figure 2. As an example, we show one tile for the case $k = 12$. The measures for the big square are 4 x 4, while the two small squares have measures of 2 x 2.

• Possibility 2: $k \equiv 1$ *mod* 4.
The sequence of the numbers of k is $9, 13, 17, \ldots$
See Figure 3. There we show one tile for the case $k = 13$.
The big square has sidelengths of 4, while the small square has sidelengths of 2. The two horizontal edges on the left have lengths 4 and 2, respectively.They have a distance of 1. The two sloped edges both have a length of $\sqrt{2}$.

• Possibility 3: $k \equiv 2$ *mod* 4.
The sequence of the numbers of k is $10, 14, 18, \ldots$.
See Figure 4. We show a 14-gon. The square has a sidelength of 4, the rectangle has measures of 2 x 4. The triangle on the left has sidelengths 4, 2, and $\sqrt{20}$. The triangle on the right has sidelengths 4 and $\sqrt{32}$.

• Possibility 4: $k \equiv 3$ *mod* 4.
The sequence of the numbers of k is $7, 11, 15, \ldots$.
See Figure 5. Here we show a 15-gon.
The square also has a sidelength of 4, both rectangles have measures of 2 x 4. □

Figure 1:
$k = 5$ and $k = 6$

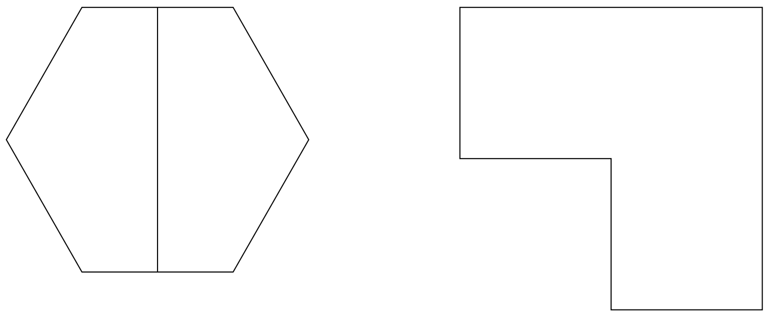


Figure 2:
$k = 12$

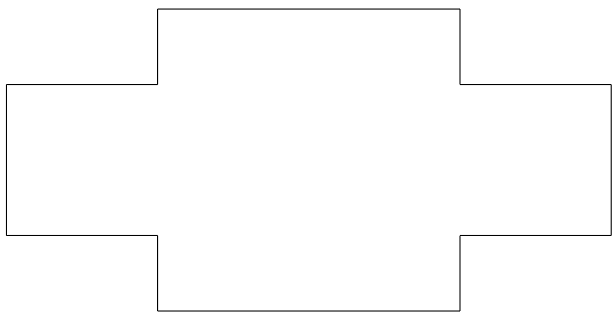


Figure 3:
$k = 13$

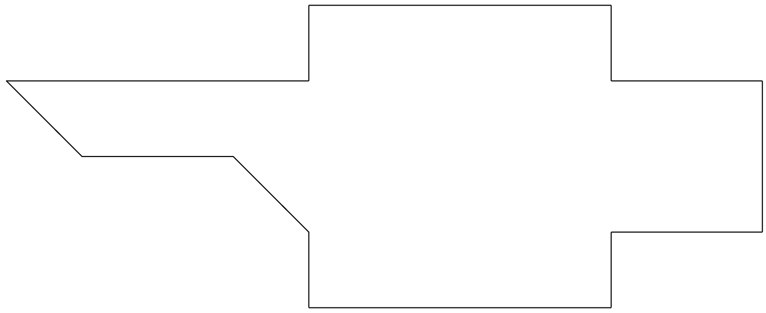


Figure 4:
$k = 14$

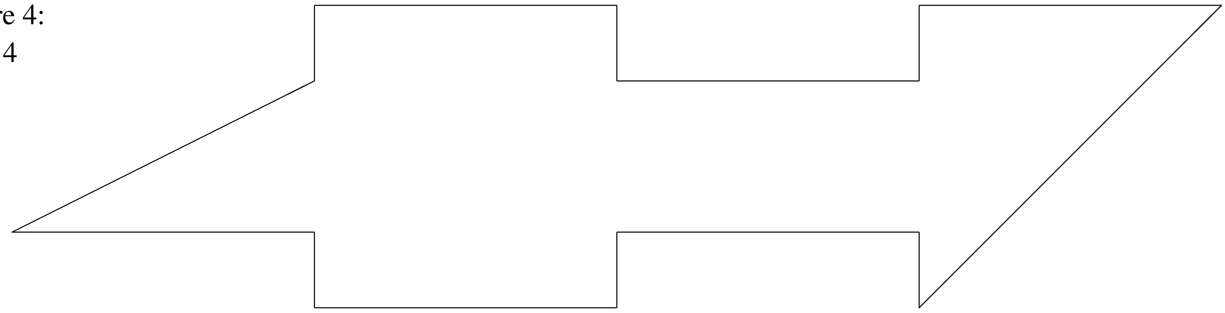
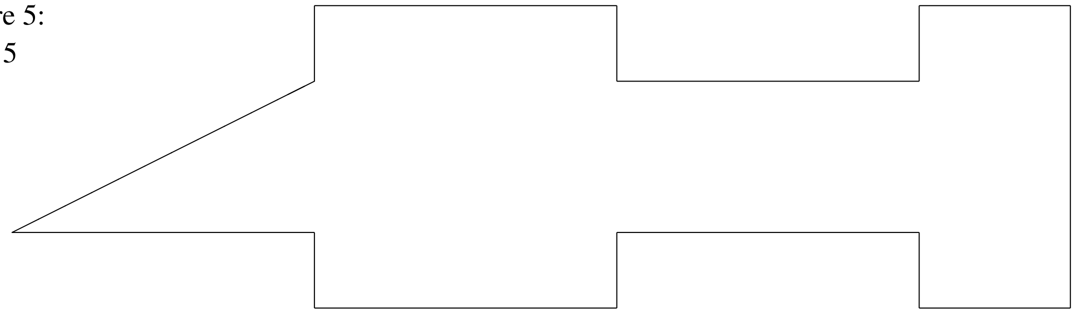


Figure 5:
$k = 15$



References

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[2] http://www.mathematische-Basteleien.de/parkett2.htm

[3] Ehrhard Behrends: *Parkettierungen der Ebene*, Springer (2019)

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