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Quadratic Phase Quaternion Domain Fourier Transform*

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Abstract. Based on the quaternion domain Fourier transform (QDFT) of 2016 and the quadratic-phase Fourier transform of 2018, we introduce the quadratic-phase quaternion domain Fourier transform (QPQDFT) and study some of its properties, like its representation in terms of the QDFT, linearity, Riemann-Lebesgue lemma, shift and modulation, scaling, inversion, Parseval type identity, Plancherel theorem, directional uncertainty principle, and the (direction-independent) uncertainty principle. The generalization thus achieved includes the special cases of QDFT, a quaternion domain (QD) fractional Fourier transform, and a QD linear canonical transform.

Keywords: Fourier transforms · quaternion algebra · quaternion domain functions · linear canonical transform · fractional Fourier transform · uncertainty

1 Introduction

Quaternions were introduced in the 19th century [10] and soon applied in physics, e.g. by J.C. Maxwell to electro-magnetism [18]. Nowadays, in theory and applications they are widely known and applied, e.g. in aero-space engineering [17], color image and signal processing [6], crystallography and material science [2, 20], and machine learning [22]. Quaternion analysis for holomorphic functions in the plane and space may be found in [9]. Quaternion based Fourier transforms are reviewed in [4] and [14]. In particular we refer to the quaternion domain Fourier transform (QDFT) introduced in 2016 [13] also described in Section 4.3.3 of [14]. A generalization to a special affine quaternion domain Fourier transform (SAQDFT) was undertaken in [15]. Independently, in 2018 the classical Fourier transform has been generalized to the quadratic-phase Fourier transform [5], with favorable new convolution identities. Most recently, the quadratic-phase Fourier transform (QPFT) has been extended to a new quaternion quadratic-phase Fourier transform (Q-QPFT) [3] for two-dimensional quaternionic signals in $L^2(\mathbb{R}^2;\mathbb{H})$, with the well-known QFT [6, 11] as a special case. Following up on

* I dedicate this paper to the late Ms. Aslaug Langaasdalen, missionary from Rjukan (Norway) to Fukui in Japan, for her true Christian practice of agape. The use of this paper is subject to the *Creative Peace License* [12].

these recent developments we extend in our current work the QPFT to quaternion domain function signals in $L^1(\mathbb{H};\mathbb{H})$ resulting in a quadratic-phase QDFT (QPQDFT).

The paper is organized as follows. Section 2 gives a brief introduction to quaternions and the QDFT, introducing some of its properties needed later in this work. Then Section 3 defines the QPQDFT and studies its basic properties, including its representation in terms of the QDFT, linearity, Riemann-Lebesgue lemma, shift and modulation, scaling, inversion, Parseval type identity and Plancherel theorem. Next, Section 4 investigates uncertainty relationships for (directed) effective spatial- and spectral (obtained from the QPQDFT) width of a quaternion domain signal. The paper concludes with Section 5, acknowledgments and references. Some proofs are given explicitly while others are only outlined.

2 Quaternions and the Quaternion Domain Fourier Transform

Gauss, Rodrigues and Hamilton's four-dimensional (4D) *quaternion algebra* $\mathbb{H}$ is defined over $\mathbb{R}$ with three imaginary units:

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},\ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j},$$
$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{1}$$

Every quaternion can be written explicitly as

$$q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} \in \mathbb{H}, \quad q_r, q_i, q_j, q_k \in \mathbb{R}, \tag{2}$$

and has a *quaternion conjugate*

$$\tilde{q} = q_r - q_i\boldsymbol{i} - q_j\boldsymbol{j} - q_k\boldsymbol{k}, \qquad \widetilde{pq} = \tilde{q}\tilde{p}. \tag{3}$$

This leads to the *norm* of $q \in \mathbb{H}$

$$|q| = \sqrt{q\tilde{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}, \qquad |pq| = |p||q|. \tag{4}$$

The *inverse* of a non-zero quaternion $q \in \mathbb{H}$ is

$$q^{-1} = \frac{\tilde{q}}{|q|^2}. \tag{5}$$

The *(symmetric) scalar part* of a quaternion is defined as

$$\langle q\rangle_0 = Sc(q) = q_r = \frac{1}{2}(q + \tilde{q}), \qquad Sc(pq) = Sc(qp) = Sc(\tilde{p}\tilde{q}), \tag{6}$$
$$Sc(pqr) = Sc(qrp) = Sc(rpq). \tag{7}$$

Every quaternion $a \in \mathbb{H}$, $a \neq 0$, can be written as scalar part plus *(pure) vector part*

$$a = a_r + a_i\boldsymbol{i} + a_j\boldsymbol{j} + a_k\boldsymbol{k} = a_r + \mathbf{a} = |a|(\cos\alpha + \frac{\mathbf{a}}{|a|}\sin\alpha) = |a|e^{\hat{\mathbf{a}}\alpha}, \quad (8)$$

with $\hat{\mathbf{a}} = \mathbf{a}/|a|$, $\cos\alpha = a_r/|a|$, $\alpha \in [0, \pi)$.

A *scalar product* of quaternions can be defined for $x, y \in \mathbb{H}$ as

$$x \cdot y = Sc(\tilde{x}y) = x_r y_r + x_i y_i + x_j y_j + x_k y_k, \qquad x \cdot x = \tilde{x}x = |x|^2. \quad (9)$$

Every *quaternion valued quaternion domain function* f maps $\mathbb{H} \to \mathbb{H}$, and its four coefficient functions f_r, f_i, f_j, f_k, are in turn real valued quaternion domain functions:

$$f : x \mapsto f(x) = f_r(x) + f_i(x)\boldsymbol{i} + f_j(x)\boldsymbol{j} + f_k(x)\boldsymbol{k} \in \mathbb{H}. \quad (10)$$

Quaternion valued quaternion domain functions have been historically studied in [23, 8, 24, 21], and applications are described in [9].

We define for two functions $f, g : \mathbb{H} \to \mathbb{H}$ the following *quaternion valued inner product*[1]

$$(f, g) = \int_{\mathbb{H}} f(x)\tilde{g}(x)d^4x \quad (11)$$

with $d^4x = dx_r dx_i dx_j dx_k \in \mathbb{R}$.

Let $\mathcal{S}$ be the Schwartz space, and $C_0(\mathbb{H})$ the Banach space of all continuous quaternion domain functions that vanish at infinity, with the supremum norm $||\cdot||_\infty$. In $L^1(\mathbb{H};\mathbb{H})$ we use the norm defined by

$$||f||_1 := \frac{1}{(2\pi)^2}\int_{\mathbb{H}} |f(x)|d^4x, \quad (12)$$

where $1/(2\pi)^2$ is for convenience later on. For $1 < p < \infty$ the space $L^p(\mathbb{H};\mathbb{H})$ has the norm

$$||f||_p = \Big(\int_{\mathbb{H}} |f(x)|^p d^4x\Big)^{\frac{1}{p}}. \quad (13)$$

Definition 1 (Quaternion Domain Fourier Transform (QDFT)[13]). *The quaternion domain Fourier transform*[2] *(QDFT) for $h \in L^2(\mathbb{H};\mathbb{H})$ is defined as*

$$\mathcal{F}_{QDFT}\{h\}(\omega) = \hat{h}(\omega) = \frac{1}{(2\pi)^2}\int_{\mathbb{H}} h(x)e^{-I x\cdot\omega}d^4x, \quad (14)$$

[1] We note that (11) is quaternion valued, but by construction $(f, f) = ||f||_2^2$ is real valued and positive for $f \neq 0$.

[2] We also assume always that $\int_{\mathbb{H}} |h(x)|d^4x$ exists as well. But we do not explicitly write this condition again in the rest of the paper. Strictly speaking, the integral definition of Def. 1 only works for $h \in L^1(\mathbb{H};\mathbb{H})$. But one can first define the QDFT on the dense subset $L^1(\mathbb{H};\mathbb{H}) \bigcap L^2(\mathbb{H};\mathbb{H})$, and then use the continuity of the Fourier transform on $L^1(\mathbb{H};\mathbb{H}) \bigcap L^2(\mathbb{H};\mathbb{H})$, due to Plancherel's theorem for the QDFT, see equations (4.19) to (4.201) in [13], to define the QDFT on $L^2(\mathbb{H};\mathbb{H})$, see e.g. [7].

with $x, \omega \in \mathbb{H}$*, and some constant pure unit quaternion*[3] $I \in \mathbb{H}$, $I^2 = -1$.

The QDFT has the following inverse transform [13].

Lemma 1 (Inverse QDFT). *For* $h, \mathcal{F}_{QDFT} \in L^2(\mathbb{H};\mathbb{H})$*, we obtain the inverse transform as*

$$h(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} \mathcal{F}_{QDFT}\{h\}(\omega) e^{+Ix\cdot\omega} d^4\omega, \quad d^4\omega = d\omega_r d\omega_i d\omega_j d\omega_k. \tag{15}$$

We will also need the *directional uncertainty principle* for the QDFT of (4.24) in [13].

Theorem 1 (Directional QDFT Uncertainty Principle). *For unit norm signals* $f \in L^2(\mathbb{H};\mathbb{H})$, $||f|| = 1$*, and constant quaternions* $a, b \in \mathbb{H}$*, we have*

$$\Delta x_a \Delta\omega_b \geq \frac{|a \cdot b|}{2}, \tag{16}$$

with (directed) effective spatial and spectral widths

$$\Delta x_a = ||(x \cdot a) f||_2 = \sqrt{\int_{\mathbb{H}} (x \cdot a)^2 |f(x)|^2 d^4x},$$
$$\Delta\omega_b = ||(\omega \cdot b)\mathcal{F}_{QDFT}\{f\}||_2 = \sqrt{\int_{\mathbb{H}} (\omega \cdot b)^2 |\mathcal{F}_{QDFT}\{f\}(\omega)|^2 d^4\omega}. \tag{17}$$

3 The Quadratic-Phase Quaternion Domain Fourier Transform

Generalizing (1.1) of [5] to quaternionic variables, for parameters $a, b, c \in \mathbb{R}$ (with $b \neq 0$) and $d, e \in \mathbb{H}$, we define the quadratic phase function for $x, \omega \in \mathbb{H}$,

$$Q(x, \omega) := a|x|^2 + bx \cdot \omega + c|\omega|^2 + d \cdot x + e \cdot \omega. \tag{18}$$

Remark 1. Note that in (18) the entities d, e need to be quaternions and not scalars in order to construct a scalar phase function $Q(x, \omega)$. This means the parameter dimension of $Q(x, \omega)$ consists of three real and two quaternionic degrees of freedom corresponding to a total of 11 real degrees of freedom.

Definition 2. *The quadratic-phase quaternion domain Fourier transform*[4] *(QPQDFT) for* $h \in L^2(\mathbb{H};\mathbb{H})$ *is defined as*

$$\mathcal{F}\{h\}(\omega) = \hat{h}(\omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} h(x) e^{-IQ(x,\omega)} d^4x, \tag{19}$$

[3] The QPQDFT of Def. 2 inherits this choice of constant pure unit quaternion $I \in \mathbb{H}$, $I^2 = -1$. We thank one of the reviewers to draw our attention to [1], which appears to allow for another use of pure quaternions in the kernel factor of (14).

[4] We refer the reader to footnote 2 for the density argument that also applies for the QPQDFT, where we note also the computation of the QPDFT in terms of the QDFT according to Lemma 2.

with $x, \omega \in \mathbb{H}$*, some constant pure unit quaternion* $I \in \mathbb{H}$*,* $I^2 = -1$*, and phase* $Q(x, \omega)$ *of* (18)*.*

Remark 2 (Special QPQDFT Cases).

(i) For the parameter values $a = c = d = e = 0$ and $b = \pm 1$ we obtain the QDFT and its inverse transform.
(ii) For parameters $d = e = 0$, the QPQDFT includes linear canonical transforms and fractional Fourier transforms for quaternion domain functions, up to constant factors, like $\sqrt{-i}$ for the linear canonical transform, and $\sqrt{(1 - i\cot(\alpha)/2\pi}$ for the fractional Fourier transform.

Defining the function

$$g(x) := h(x)e^{-I(a|x|^2 + d\cdot x)}, \tag{20}$$

it is possible to compute the QPQDFT of $h \in L^1(\mathbb{H};\mathbb{H})$ in terms of the QDFT of g.

Lemma 2.

$$\mathcal{F}\{h\}(\omega) = \mathcal{F}_{QDFT}\{g\}(b\omega)e^{-I(c|\omega|^2 + e\cdot\omega)}. \tag{21}$$

Then we obtain the following lemmata.

Lemma 3. *The* L^2*-norms of* $\mathcal{F}\{h\}$ *and* $\mathcal{F}_{QDFT}\{h\}$ *are related by*

$$||\mathcal{F}\{h\}||_2 = \frac{1}{b^2}||\mathcal{F}_{QDFT}\{h\}||_2. \tag{22}$$

Proof.

$$\begin{aligned}
||\mathcal{F}\{h\}||_2 &= \Big[\int_{\mathbb{H}} |\mathcal{F}\{h\}(\omega)|^2 d^4\omega\Big]^{\frac{1}{2}} \\
&= \Big[\int_{\mathbb{H}} |\mathcal{F}_{QDFT}\{h\}(b\omega)|^2 d^4\omega\Big]^{\frac{1}{2}} \\
&= \Big[\int_{\mathbb{H}} \frac{1}{b^4}|\mathcal{F}_{QDFT}\{h\}(\omega')|^2 d^4\omega'\Big]^{\frac{1}{2}} \\
&= \frac{1}{b^2}\Big[\int_{\mathbb{H}} |\mathcal{F}_{QDFT}\{h\}(\omega)|^2 d^4\omega\Big]^{\frac{1}{2}} \\
&= \frac{1}{|b|^2}||\mathcal{F}_{QDFT}\{h\}||_2,
\end{aligned} \tag{23}$$

Lemma 4 (Riemann-Lebesgue lemma). *If* $h \in L^1(\mathbb{H};\mathbb{H})$ *then* $\mathcal{F}\{h\} \in C_0(\mathbb{H})$*, and* $||\mathcal{F}\{h\}|| \leq ||h||_1$*.*

Proof. Because $|e^{-IQ(x,\omega)}| = 1$, we have

$$\begin{aligned}
||\mathcal{F}\{h\}||_\infty &= \sup_{\omega\in\mathbb{H}} |\mathcal{F}\{h\}(\omega)| = \sup_{\omega\in\mathbb{H}} \frac{1}{(2\pi)^2}\Big|\int_{\mathbb{H}} h(x)e^{-IQ(x,\omega)}d^4\omega\Big| \\
&\leq \sup_{\omega\in\mathbb{H}} \frac{1}{(2\pi)^2}\Big|\int_{\mathbb{H}} |h(x)||e^{-IQ(x,\omega)}|d^4\omega\Big| = ||h||_1.
\end{aligned} \tag{24}$$

Furthermore, the function $g(x)$ of (20) is in $L^1(\mathbb{H};\mathbb{H})$ if and only if $h \in L^1(\mathbb{H};\mathbb{H})$. Hence the classic Riemann-Lebesgue lemma results in

$$|\mathcal{F}\{h\}(\omega)| = \frac{|e^{-I(c|\omega|^2+e\cdot\omega)}|}{(2\pi)^2}\Big|\int_{\mathbb{H}} e^{-Ibx\cdot\omega}g(x)d^4x\Big| = \frac{1}{(2\pi)^2}\Big|\int_{\mathbb{H}} e^{-Ibx\cdot\omega}g(x)d^4x\Big| \to 0, \tag{25}$$

as $|\omega| \to \infty$, completing the sketch of the proof.

The QPQDFT has the following linearity properties.

Theorem 2 (Linearity). *The QPQDFT is left linear with respect to coefficients $\alpha_1, \alpha_2 \in \mathbb{H}$ for $h_1, h_2 \in L^1(\mathbb{H};\mathbb{H})$*

$$\mathcal{F}\{\alpha_1 h_1 + \alpha_2 h_2\}(\omega) = \alpha_1\mathcal{F}\{h_1\}(\omega) + \alpha_2\mathcal{F}\{h_2\}(\omega). \tag{26}$$

It is right linear for coefficients $\beta_1, \beta_2 \in \mathbb{H}$ that commute with the unit pure quaternion I of Definition 19.

$$\begin{aligned}&\mathcal{F}\{h_1\beta_1 + h_2\beta_2\}(\omega) = \mathcal{F}\{h_1\}(\omega)\beta_1 + \mathcal{F}\{h_2\}(\omega)\beta_2,\\ &\forall \beta_1, \beta_2 \in \mathbb{H}: \quad \beta_1 I = I\beta_1, \beta_2 I = I\beta_2.\end{aligned} \tag{27}$$

The QPQDFT has the following shift-, modulation-, and scaling properties obtained by straightforward computation.

Theorem 3 (Shift). *For $h \in L^1(\mathbb{H};\mathbb{H})$, $x, \omega \in \mathbb{H}$ and constant quaternion $s \in \mathbb{H}$ we have*

$$\mathcal{F}\{h(x-s)\}(\omega) = \mathcal{F}\{h(x)\}(\omega - \frac{2a}{b}s)e^{-I(\frac{4ac}{b}\omega\cdot s - \frac{4a^2c}{b^2}|s|^2 + \frac{2a}{b}e\cdot s)}. \tag{28}$$

Remark 3. Alternative ways of expressing the shift property are

$$\mathcal{F}\{h(x-s)\}(\omega) = \mathcal{F}\{h(x)e^{-2Iax\cdot s}\}(\omega)\, e^{-I(bs\cdot\omega + a|s|^2 + d\cdot s)} \tag{29}$$

or

$$\mathcal{F}\{h(x-s)\}(\omega) = \frac{1}{(2\pi)^2}\int_{\mathbb{H}} h(x)\, e^{-IQ'(x,\omega)}d^4x e^{-I(a|s|^2+d\cdot s)}, \tag{30}$$

with

$$Q'(x,\omega) := a|x|^2 + bx\cdot\omega + c|\omega|^2 + d'\cdot x + e'\cdot\omega, \quad d' = d + 2as, \quad e' = e + bs. \tag{31}$$

Theorem 4 (Modulation). *For $h \in L^1(\mathbb{H};\mathbb{H})$, $x, \omega \in \mathbb{H}$ and constant quaternionic frequency $\mu \in \mathbb{H}$ we have*

$$\mathcal{F}\{h(x)e^{Ix\cdot\mu}\}(\omega) = \mathcal{F}\{h(\mathbf{x})\}(\omega - \frac{\mu}{b})e^{-I\left(2\frac{c}{b}\omega\cdot\mu + \frac{1}{b}e\cdot\mu - \frac{c}{b^2}|\mu|^2\right)}. \tag{32}$$

Theorem 5 (Quaternionic Scaling). *For $h \in L^1(\mathbb{H};\mathbb{H})$, $x, \omega \in \mathbb{H}$ and constant quaternionic scaling factor $p \in \mathbb{H}$, $h_p(x) = h(px)$, we have*

$$\mathcal{F}\{h_p(x)\}(\omega) = \frac{1}{(2\pi)^2|p|^4}\int_{\mathbb{H}} h(x)e^{-I\frac{1}{|p|^2}Q'(x,p\omega)}d^4x, \tag{33}$$

with

$$Q'(x,\omega) = a|x|^2 + bx\cdot\omega + c|\omega|^2 + d'\cdot x + e'\cdot\omega, \quad d' = pd, \quad e' = pe. \tag{34}$$

Corollary 1 (Real Scaling). *For $h \in L^1(\mathbb{H};\mathbb{H})$, $x, \omega \in \mathbb{H}$ and constant real scaling factor $r \in \mathbb{R}$, $h_r(x) = h(rx)$, we have*

$$\mathcal{F}\{h_r(x)\}(\omega) = \frac{1}{(2\pi)^2 r^4} \int_{\mathbb{H}} h(x) e^{-I \frac{1}{r^2} Q'(x, r\omega)} d^4x, \tag{35}$$

with

$$Q'(x,\omega) = a|x|^2 + bx \cdot \omega + c|\omega|^2 + d' \cdot x + e' \cdot \omega, \quad d' = rd, \quad e' = re. \tag{36}$$

The QPQDFT has the following inverse.

Theorem 6 (Inverse QPQDFT). *For $h, \mathcal{F} \in L^2(\mathbb{H};\mathbb{H})$, we obtain the inverse QPQDFT transform as*

$$h(x) = \frac{b^4}{(2\pi)^2} \int_{\mathbb{H}} \mathcal{F}\{h\}(\omega) e^{+IQ(x,\omega)} d^4\omega. \tag{37}$$

Proof.

$$\begin{aligned}
&\frac{b^4}{(2\pi)^2} \int_{\mathbb{H}} \mathcal{F}\{h\}(\omega) e^{+IQ(x,\omega)} d^4\omega \\
&= \frac{b^4}{(2\pi)^2} \int_{\mathbb{H}} \mathcal{F}_{QDFT}\{g\}(b\omega) e^{-I(c|\omega|^2 + e\cdot\omega)} e^{+I(c|\omega|^2 + e\cdot\omega)} e^{I(a|x|^2 + bx\cdot\omega + d\cdot x)} d^4x \\
&= \frac{b^4}{(2\pi)^2} \int_{\mathbb{H}} \mathcal{F}_{QDFT}\{g\}(b\omega) e^{Ibx\cdot\omega} d^4\omega e^{I(a|x|^2 + d\cdot x)} \\
&= \frac{b^4}{(2\pi)^2 b^4} \int_{\mathbb{H}} \mathcal{F}_{QDFT}\{g\}(\mu) e^{Ix\cdot\mu} d^4\mu e^{I(a|x|^2 + d\cdot x)} \\
&= h(x) e^{-I(a|x|^2 + d\cdot x)} e^{I(a|x|^2 + d\cdot x)} = h(x),
\end{aligned} \tag{38}$$

where in the first equality we used Lemma 2 with g given by (20), and for the third we substituted $\mu := b\omega$, $d^4\mu = b^4 d^4\omega$, and in the fourth we used the inverse QDFT of Lemma 1 and (20).

Theorem 7 (Parseval-Type Identity, Plancherel Theorem). *(i) For any $f, h \in L^2(\mathbb{H};\mathbb{H})$, the following identity holds*

$$(\mathcal{F}\{f\}, \mathcal{F}\{h\}) = \frac{1}{b^4}(f, h). \tag{39}$$

In the special case of $f = h$, we have

$$||\mathcal{F}\{f\}||_2^2 = \frac{1}{b^4}||f||_2^2. \tag{40}$$

(ii) Plancherel Theorem. If $b = \pm 1$, then $\mathcal{F}$ defines a unitary operator in $L^2(\mathbb{H};\mathbb{H})$.

Proof. By simple computations we have

$$
\begin{aligned}
&(\mathcal{F}\{f\}, \mathcal{F}\{h\}) \\
&= \frac{1}{(2\pi)^4} \int_{\mathbb{H}} \int_{\mathbb{H}} \int_{\mathbb{H}} f(x) e^{-IQ(x,\omega)} e^{+IQ(y,\omega)} \widetilde{g}(y) d^4x d^4y d^4\omega \\
&= \frac{1}{(2\pi)^4} \int_{\mathbb{H}} \int_{\mathbb{H}} \int_{\mathbb{H}} f(x) e^{-I(c|\omega|^2 + e\cdot\omega)} e^{I(c|\omega|^2 + e\cdot\omega)} e^{-Ibx\cdot\omega} e^{+Iby\cdot\omega} d^4\omega \\
&\qquad\qquad e^{-I(a|x|^2 - d\cdot x)} e^{+I(a|y|^2 - d\cdot y)} \widetilde{g}(y) d^4x d^4y \\
&= \frac{1}{b^4} \int_{\mathbb{H}} \int_{\mathbb{H}} f(x) \delta(x-y) e^{-I(a|x|^2 - d\cdot x)} e^{+I(a|y|^2 - d\cdot y)} \widetilde{g}(y) d^4y d^4x \\
&= \frac{1}{b^4} \int_{\mathbb{H}} f(x) \widetilde{g}(x) d^4x = \frac{1}{b^4}(f, g), \qquad (41)
\end{aligned}
$$

where we have applied that (see Appendix A for more details)

$$
\frac{1}{(2\pi)^4} \int_{\mathbb{H}} e^{-Ibx\cdot\omega} e^{+Iby\cdot\omega} d^4\omega = \delta(b(x-y)) = \frac{1}{b^4}\delta(x-y). \qquad (42)
$$

This proves proposition (i). For $f = g$ we have (40), and for $b = \pm 1$ we obtain proposition (ii).

4 QPQDFT and Uncertainty

Theorem 8 (Directional Uncertainty). *Let $h \in L^2(\mathbb{H}; \mathbb{H})$ with QPQFT $\mathcal{F}\{h\}$. Assume that $||h||_2 < \infty$, then the following inequality holds for arbitrary constant quaternions $v, w \in \mathbb{H}$:*

$$
||(x\cdot v)h||_2 \, ||(\omega\cdot w)\mathcal{F}\{h\}||_2 \geq \frac{1}{|b|} \frac{|v\cdot w|}{2} ||h||_2 \, ||\mathcal{F}\{h\}||_2. \qquad (43)
$$

Proof. By direct computation we obtain

$$
\begin{aligned}
||(\omega\cdot w)\mathcal{F}\{h\}||_2 &= \Big[\int_{\mathbb{H}} (\omega\cdot w)^2 |\mathcal{F}\{h\}(\omega)|^2 d^4\omega\Big]^{\frac{1}{2}} \\
&= \Big[\int_{\mathbb{H}} (\omega\cdot w)^2 |\mathcal{F}_{QDFT}\{h\}(b\omega)|^2 d^4\omega\Big]^{\frac{1}{2}} \\
&= \Big[\int_{\mathbb{H}} \frac{1}{b^4} (\omega'\cdot w')^2 |\mathcal{F}_{QDFT}\{h\}(\omega')|^2 d^4\omega'\Big]^{\frac{1}{2}} \\
&= \frac{1}{b^2}\Big[\int_{\mathbb{H}} (\omega\cdot w')^2 |\mathcal{F}_{QDFT}\{h\}(\omega)|^2 d^4\omega\Big]^{\frac{1}{2}} \\
&= \frac{1}{|b|^3}\Big[\int_{\mathbb{H}} (\omega\cdot w)^2 |\mathcal{F}_{QDFT}\{h\}(\omega)|^2 d^4\omega\Big]^{\frac{1}{2}} \\
&= \frac{1}{|b|^3} ||(\omega\cdot w)\mathcal{F}_{QDFT}\{h\}||_2, \qquad (44)
\end{aligned}
$$

where for the second equality we applied Lemma 2, for the third we substituted $\omega' = b\omega$, $d^4\omega' = b^4 d^4\omega$, $w' = \frac{1}{b}w$ (then $\omega \cdot w = \omega' \cdot w'$), for the fourth equality we renamed $\omega' \to \omega$, for the fifth we inserted $w' = bw$ again, and the last equality applied the definition of $||\cdot||_2$ of (13) for $p = 2$. According to the directional uncertainty principle for the QDFT of Theorem 1 we have (not assuming unit norm signals)

$$||(x \cdot v)h||_2 \, ||(\omega \cdot w)\mathcal{F}_{QDFT}\{h\}||_2 \geq \frac{v \cdot w}{2} ||h||_2 \, ||\mathcal{F}_{QDFT}\{h\}||_2, \tag{45}$$

and with the norm relation of Lemma 3 we finally obtain

$$||(x \cdot v)h||_2 \, ||(\omega \cdot w)\mathcal{F}\{h\}||_2 \geq \frac{1}{|b|} \frac{|v \cdot w|}{2} ||h||_2 \, ||\mathcal{F}\{h\}||_2. \tag{46}$$

Remark 4. For $b = \pm 1$ and unit norm signals, i.e., $||h||_2 = ||\mathcal{F}\{h\}||_2 = 1$, we obtain the familiar form of the directional uncertainty principle, relating the (directed) effective spatial and spectral widths by

$$\Delta x_v \Delta \omega_w \geq \frac{|v \cdot w|}{2}, \tag{47}$$

where

$$\Delta x_v = ||(x \cdot a)h||_2 = \sqrt{\int_{\mathbb{H}} (x \cdot v)^2 |h(x)|^2 d^4x},$$
$$\Delta \omega_w = ||(\omega \cdot a)\mathcal{F}\{h\}||_2 = \sqrt{\int_{\mathbb{H}} (\omega \cdot w)^2 |\mathcal{F}\{h\}(\omega)|^2 d^4\omega}. \tag{48}$$

Corollary 2 (Uni-directional Uncertainty Principle). *For the single direction $w = \pm v$, $|v| = 1$, we get the following uni-directional uncertainty principle*

$$||(x \cdot v)h||_2 \, ||(\omega \cdot v)\mathcal{F}\{h\}||_2 \geq \frac{1}{2|b|} ||h||_2 \, ||\mathcal{F}\{h\}||_2. \tag{49}$$

Remark 5. In (49) equality holds for Gaussian wave packets

$$G(x) = Ae^{-k|x|^2}, \tag{50}$$

with $x \in \mathbb{H}$, and constants $A \in \mathbb{H}$, $k \in \mathbb{R}$, $k > 0$.

Corollary 3 (Uncertainty and Orthogonal Directions). *For orthogonal v and w, i.e., $v \cdot w = 0$, the uncertainty can be zero*

$$||(x \cdot v)h||_2 \, ||(\omega \cdot w)\mathcal{F}\{h\}||_2 \geq 0. \tag{51}$$

Finally, we can extend the directional uncertainty principle to the direction-independent QPQDFT uncertainty principle

Theorem 9 (QPQDFT Uncertainty Principle). *Let $h \in L^2(\mathbb{H};\mathbb{H})$ with QPQFT $\mathcal{F}\{h\}$. Assume that $||h||_2 < \infty$, then the following inequality holds:*

$$||xh||_2 \, ||\omega\mathcal{F}\{h\}||_2 \geq \frac{1}{|b|} ||h||_2 \, ||\mathcal{F}\{h\}||_2. \tag{52}$$

5 Conclusion

This paper first gave a brief introduction to quaternions and the quaternion domain Fourier transform (QDFT). Then the quadratic-phase QDFT (QPQDFT) was defined and its basic properties established, including its representation in terms of the QDFT, linearity, Riemann-Lebesgue lemma, shift and modulation, scaling, inversion, Parseval type identity and Plancherel theorem. Finally, the uncertainty relationships for (directed) effective spatial- and spectral (obtained from the QPQDFT) width of a quaternion domain signal were investigated.

Following [5], it may be interesting to see how far the favorable convolution properties of the scalar quadratic-phase Fourier transform can be extended to the quaternion domain function case, to study related Young type inequalities, the asymptotic behavior of quaternionic oscillatory integrals and solvability of quaternionic convolution integral equations. Future research should also look into establishing quadratic-phase quaternion domain wavelets and their application in science and technology. Rich applications are expected in fields like physics, electro-magnetism, aero-space engineering, color image and signal processing, crystallography and material science, and machine learning, and quaternion analysis for holomorphic functions in the plane and space, etc.

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A Quaternion Domain Intergration and Dirac Delta Function

We want to look at equation (42) in some more detail:

$$\frac{1}{(2\pi)^4}\int_{\mathbb{H}} e^{-Ibx\cdot\omega}e^{+Iby\cdot\omega}d^4\omega = \delta(b(x-y)) = \frac{1}{b^4}\delta(x-y). \tag{53}$$

We do have the variables $b \in \mathbb{R}$, $b \neq 0$, and the three quaternion variables $x, y, \omega \in \mathbb{H}$. But they only appear in the scalar product, i.e.

$$x\cdot\omega = Sc(\tilde{x}\omega) = x_r\omega_r + x_i\omega_i + x_j\omega_j + x_k\omega_k \in \mathbb{R}, \tag{54}$$

$$y\cdot\omega = Sc(\tilde{y}\omega) = y_r\omega_r + y_i\omega_i + y_j\omega_j + y_k\omega_k \in \mathbb{R}. \tag{55}$$

Furthermore we have the pure unit quaternion $I \in \mathbb{H}$, $I^2 = -1$. This means that the arguments of the exponential functions commute and therefore we can rewrite the product of the two exponential factors as

$$\begin{aligned} e^{-Ibx\cdot\omega}e^{+Iby\cdot\omega} &= e^{-Ib[(x-y)\cdot\omega]} \\ &= e^{-Ib(x_r-y_r)\omega_r}e^{-Ib(x_i-y_i)\omega_i}e^{-Ib(x_j-y_j)\omega_j}e^{-Ib(x_k-y_k)\omega_k}. \end{aligned} \tag{56}$$

We further note that

$$d^4\omega = d\omega_r d\omega_i d\omega_j d\omega_k. \tag{57}$$

Therefore the integral has simplified to

$$\begin{aligned} &\frac{1}{(2\pi)^4}\int_{\mathbb{H}} e^{-Ibx\cdot\omega}e^{+Iby\cdot\omega}d^4\omega \\ &= \frac{1}{(2\pi)^4}\int_{\mathbb{H}} e^{-Ib(x_r-y_r)\omega_r}e^{-Ib(x_i-y_i)\omega_i}e^{-Ib(x_j-y_j)\omega_j}e^{-Ib(x_k-y_k)\omega_k}d\omega_r d\omega_i d\omega_j d\omega_k \\ &= \Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-Ib(x_r-y_r)\omega_r}d\omega_r\Big)\Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-Ib(x_i-y_i)\omega_i}d\omega_i\Big) \\ &\quad\times\Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-Ib(x_j-y_j)\omega_j}d\omega_j\Big)\Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-Ib(x_k-y_k)\omega_k}d\omega_k\Big) \\ &= \delta(b(x_r-y_r))\,\delta(b(x_i-y_i))\,\delta(b(x_j-y_j))\,\delta(b(x_k-y_k)). \end{aligned} \tag{58}$$

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