Nervous Equivariant Holonomy

Ryan J. Buchanan

December 20, 2023

Abstract

One of the possible explanations for entanglement is a sort of perverse holonomy which acts on sheaves whose germs are eigenvectors for a tuple of local variables. We take baby steps towards realizing this model by introducing an equivariant form of holonomy. As a test category, we take U(1)-bundles whose outbound fibrations are Koszul nerves of degree (p+q)=n.

1 Introduction

Denote by *sSets* the category of simplicial sets, and write Δ for a particular simplicial set. We will use the convention $\Delta^0 = x_0 \in M$ for some point x_0. Fix an ∞-category (quasi-category), C^∞, once and for all. Denote, for an n-cell $C^\infty \rightrightarrows sSets$, the nth homotopy coherent nerve, $\mathcal{N}^n_M$.

Lemma 1. *For the nth homotopy coherent nerve, $\mathcal{N}^n_M$ of a manifold M, there is a decomposition into charts:*

$$\varphi_n(M) \circ ... \circ \varphi_0^{-1}(M)$$

Let Σ^∞_M denote the infinity-fold suspension of M.

Proposition 1. *The map*

$$\Sigma^\infty_M \hookrightarrow \Sigma^\infty_{LM}$$

where LM denotes the free loop space of M, is trivial.

From this, we deduce that $H_n(M) \simeq H_n(LM)$, where H_n is the nth homology group. Thus, the sequence

$$\varphi_0^{-1}(M) \longrightarrow \varphi_1^{-1}(M) \longrightarrow ... \longrightarrow \varphi_n(M)$$

has, as its adjoint, the inverse sequence:

$$\varphi_0^{-1}(LM) \longleftarrow \varphi_1^{-1}(LM) \longleftarrow ... \longleftarrow \varphi_n(M)$$

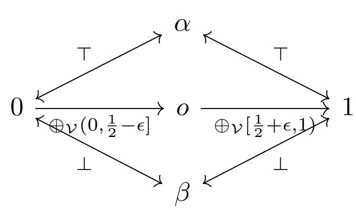
These sequences may be written $Sp(M)$ and $Sp(M)^{ad}$, respectively, with the convention $Sp(M)^{ad} \cong Sp(LM)^{-1}$. We then define sk_M to be the principle fiber bundle

$$Sp(M) \longrightarrow Sp(M)^{ad} \longrightarrow sSets \cong LM \longrightarrow M \longrightarrow \Delta$$

consisting of restrictions of charts to objects in a simplicially enriched category. For such a category, one fixes a Grothendieck universe $\mathcal{V}$, and, for objects well-inside $\mathcal{V}$, we say they are $\mathcal{V}$-small. By "well-inside", we mean that the relationship

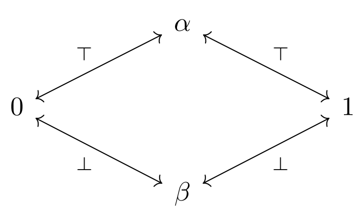
$$o \ll \mathcal{V}$$

is obeyed. This means that, for a principle bundle of $\mathcal{V}$, there is a set-enriched $\mathcal{V}$-scheme taking its values in a set whose terminal and initial objects are cofinal with o. Shown below is a diagram of a $\mathcal{V}$-category, with $\oplus_{\mathcal{V}}$ as the group action passing through o:



Proposition 2. *$\mathcal{V} \setminus o$ is a groupoid.*

Proof. Clearly, every morphism in the diagram



is invertible. $\square$

2 Main results

In this paper, we wish to treat the holonomy groupoid $Hol_{\Delta}^{\mathscr{G}}$ as an *operad* which takes as its input an n-tuple of skelata, and produces as an output, a nerve $\mathcal{N}_{sk_{\Delta}}^{n} : sk_0 \longrightarrow sk_n$ which is fibered in groupoids. At the same time, we wish to do so in a $\mathscr{G}$-equivariant way. That is to say, for a specific inertia group $\Lambda(x_0)$ for a point x_0 in a manifold M, we want the n-cells $x_0 \rightrightarrows x_n$ to be bijective, so that the formula

$$Map(fib(0,n),\mathcal{I}) \simeq \mathcal{N}^n_{sk_\Delta}$$

holds, where by $\mathcal{I}$ we mean the interval $[x_0, x_n]$.

To accomplish this, we will let $\mathcal{U}(x)$ be a $U(n)$-bundle, with $x = \int_{k=0}^n x_k$. We will denote by Γ_x the collection of sections of $\mathcal{U}(x)$. Our main result is as follows:

Theorem 1. *For a $U(n)$-bundle $\mathcal{U}(x)$, the set of deloopings $\mathcal{N} = \Omega_\varphi(B\mathcal{N})$ may be written as a sequence of transition maps*

$$Sp(M_0) \longrightarrow ... \longrightarrow Sp(M_n)$$

We will not seek to directly prove this theorem here, but we hope that a proof will manifest itself with diligent investigation. For starters, we will note that the above chain is essentially a chain of the form:

$$M_0 \longrightarrow M_1 \overset{\longrightarrow}{\longrightarrow} M_2 \overset{\overset{\longrightarrow}{\longrightarrow}}{\longrightarrow} M_3 ...$$

such that, for every submanifold $M_n \subset M$, there is an n-cell $\mathcal{N}^n_M : M_{n-1} \overset{n}{\to} M_n$. Right away, it is evident that the proper formalization for the nervous equivariant holonomy groupoid is an ∞-groupoid.[1]

2.1 Moore Paths

The following is recalled from [2]. Denote by $Path(M)$ the path category of M. A *Moore path* is a pair (r, γ), where $r > 0$ and γ is the interval $[0, r]$.

Proposition 3. *If $Mor_{Path(M)}$ is the full space of morphisms, then the source and target maps, $s, t : Mor_{Path(M)} \longrightarrow M$ are Serre fibrations with contractible fiber.*

Proposition 4. *The geometric realization of the path category has the weak homotopy type of M.*

$$|Path(M)| \simeq M$$

The fiber spectrum E^ϕ_x at a point $x \in M$ is $\phi(x)$, and if $\gamma : x \longrightarrow y$ is a path in M between points x and y, then on the level of morphisms, $\phi(\gamma) : E^\phi_x \longrightarrow E^\phi_y$ is an equivalence.[2] In our language, this means that the class of diffeomorphisms of M contains all of the necessary sections of the tangent spaces of path-connected points. Viz.:

$$Diff_M \supset \Gamma(T_x\mathcal{Z}) \;\; \forall \mathcal{Z} \in \gamma$$

[1] See [1] for more details.
[2] Ibid, pg. 16

2.2 U(1)-bundles

The simplest case in which we can begin our investigation is that of a $U(1)$-bundle over a manifold M. This gives us a natural stratification:

$$Strat_M^{\mathcal{U}} \longrightarrow (U(1) \in M) \times M \longrightarrow N \subset M$$

into a submanifold N which acts as a portable $\mathcal{D}$-module. We can then induce a flat embedding:

$$\mathcal{D} \overset{\flat}{\hookrightarrow} Post(Path(N/\sim))$$

into the Postnikov tower of submanifolds of M, which are defined up to modular isomorphism with N. This gives us the *effective Thom spectrum*, $Thom_{Eff}$ of our base manifold. We have:

$$\Omega^\infty(M) = \Sigma^\infty(\pi_\infty(M)) \simeq \mathcal{N}_M^\infty$$
$$\simeq QCoh(\mathcal{B}^d) \bigwedge QCoh(\mathcal{B}^d \cup \#\mathcal{B}^d)$$
$$= Dehn_{\hat{Q}}(N)$$
$$= D(Bun_G) \times Coh(X^{cofib})$$

where X is a p,q-tensor:

$$X = \prod_{p=0} q\mathcal{N}_N^p$$

on the Nisnevich site, meaning that $\tilde{G}$ has finite cohomological dimension.[3] We generally would like to think of X as a germ on the presheaf $Pshv_G$, such that the map

$$X \in Pshv_G \overset{\sim}{\longrightarrow} \tilde{G}$$

is an isomorphism for all im(X), and where $\tilde{G}$ is a stack.[4]

Proposition 5. *If sk_x is symmetric monoidal, then $sk_x \otimes Bun_{U(1)} = \widetilde{Bun_{U(1)}}_+$ is a Cartesian closed category.*

Let us assume that the above proposition holds, and let $\mathfrak{P} = \widetilde{Bun_{U(1)}}$. Then, we obtain a wordline $\mathfrak{W}_{\mathfrak{P}}$, which is equivalent to

$$\mathbb{L}^{p-q} \otimes (Taut(Strat_M^{\{*\}}) \times Dehn(\mathcal{F}_{\mathfrak{P}} \cdot \mathbb{A}^1))$$

with $\mathcal{F}_{\mathfrak{P}}$ some preferred foliation and $\mathbb{A}^1$ the affine line, as used in motivic homotopy theory.[5] It is trivial to show that this wordline is $\mathscr{G}$-equivariant, for

[3] As of the time of this writing, it is unknown whether X can be calculated (or generalized) for etale cohomology.

[4] Note that if X is perfect, and $\tilde{G}$ is etale, then the above map is a totally lossless projection. Thus (speaking *level-wise*), if X is perfect, then every $\Gamma(im(X))$ is perfect as well.

[5] See [3]

some isotropy group $\mathscr{G}$; indeed, this is equivalent to the statement that the ambient space is an H-space, and

$$\Omega^\infty(\mathfrak{W}_\mathfrak{P}) \supset \{*\}$$

assuming the space to be presentable. Letting this be the kernel of an index $\mathfrak{I}$, we have a collection of Ehresmann connections

$$\mathfrak{I} = \sum_{p<q}^{p=q} \overset{\Gamma}{\nabla}(p,q)$$

which are ramified by the Tate circle.[6] This "basically" means that, for any geodesic γ encompassed by Ω_0^r, the maximum possible τ-value a point-like object can attain is q. So, for any induced periodic flow whose coefficient is $\mathfrak{I}$, the maximal degree at which a homotopy is killed is q. Thus, there is a relationship

$$pRCurv_X$$

where by R we mean "is basically", such that for a p-adic cochain generated by p, the mean curvature of X is basically equivalent to (and is in fact generated by) p.

3 Future Work

In the future, the author hopes to extend this discussion by generalizing from the case of $U(1)$-bundles to $U(n)$-bundles for any n, and specifically to the adic case of $U(p)$-bundles for fixed primes p.

One route for exploring this would be to review Edward Witten's work on the Dirac index of a loop space operator. It is tempting to make the connection between the $\mathfrak{I}$ that has been established here, and the Dirac index of a loop space. This would further the aim of unification between the work of G. Segal and P. Dirac. In particular, employing contact geometry (and specifically the notion of an overtwist) would give some insight into the Hopfian structure of bordisms, especially given the relationship between Ω-structures and cobordisms.

[6] Ibid

4 References

[1] R.J. Buchanan, P. Emmerson, *Worldlines and Bordisms II*, (2023)
[2] R.L. Cohen, J.D.S. Jones, *Gauge Theory and String Topology*, (2013)
[3] V. Voevodsky, et al., *Motivic Homotopy Theory [Lecture]*, (date unknown)