A 2-Pitch Structure

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Abstract We have constructed a pitch structure. In this paper, we define a binary relation on the set of steps, thus the set become a circle set. And we define the norm of a key transpose. To apply the norm, we define a scale function on the circle set. Hence we may construct the 2-pitch structure over the circle set.

Contents

1. Introduction

In [2], we have constructed a pitch structure $\boldsymbol{M}$ over a circle set P. A pitch structure is a partial structure, and its underlying set is a finite set, see definition 2.11 for more details. A circle set is finite set equipped with the binary relation '$\olessthan$', cf. definitions 2.5 and 2.6 and proposition 3.1.

Let $P := \{p_0 \olessthan p_1 \olessthan \cdots \olessthan p_{n-1} \olessthan p_0\}$ be a circle set, and $\{\boldsymbol{M}_i\}$ a set of pitch structures over P such that $\tau_{M_i} = p_i$ for every p_i. Then we have that $\{\boldsymbol{M}_i\}$ and $\{SS(\boldsymbol{M}_i)\}$ are circle sets, see definitions 2.11 and 2.12 and proposition 3.1 for more details.

And we define a scale function $\widetilde{\lambda}$ in definition 3.3. Then we have that $\boldsymbol{M}_i \cong \boldsymbol{M}_j$ implies $\widetilde{\lambda}\big(SS(\boldsymbol{M}_i), SS(\boldsymbol{M}_j)\big) =$ ▬ for $\boldsymbol{M}_i \olessthan \boldsymbol{M}_j \in \{\boldsymbol{M}_i\}$, see proposition 3.2. If we assume that $\kappa_i \colon SS(\boldsymbol{M}_i) \rightsquigarrow SS(\boldsymbol{M}_j)$ is a regular key transpose for $\boldsymbol{M}_i \olessthan \boldsymbol{M}_j \in \{\boldsymbol{M}_i\}$, then we have that $\widetilde{\lambda}\big(SS(\boldsymbol{M}_i), SS(\boldsymbol{M}_j)\big) =$ ▬, see proposition 3.3 for more details.

We may construct a pitch structure $\widetilde{\boldsymbol{M}} := \langle \{SS(\boldsymbol{M}_i)\} \cup \mathbb{S}, \widetilde{\lambda}, \tau, \mathbb{S}, \olessthan \rangle$ over the circle set $\{SS(\boldsymbol{M}_i)\}$, see proposition 3.4 for more details. We call $\widetilde{\boldsymbol{M}}$ 2-pitch structure. And we may obtain an n-pitch structure in this way.

2. Preliminaries

2.1. Universal Algebra.

We recall some definitions in universal algebra.

Definition 2.1 ([1, 3])**.** An ordered pair $\langle L, \sigma \rangle$ is said to be a (first-order) **language** provided that

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- L is a nonempty set,
- $\sigma\colon L \to \mathbb{Z}$ is a mapping.

A language $\langle L, \sigma\rangle$ is denoted by $\mathfrak{L}$. If $f \in \mathfrak{L}$ and $\sigma(f) \geq 0$ then f is called an **operation symbol**, and $\sigma(f)$ is called the **arity** of f. If $r \in \mathfrak{L}$ and $\sigma(r) < 0$, then r is called a **relation symbol**, and $-\sigma(r)$ is called the **arity** of r. A language is said to be **algebraic** if it has no relation symbols.

Definition 2.2 ([1])**.** Let X be a nonempty class and n a nonnegative integer. Then an n-ary **partial operation** on X is a mapping from a subclass of X^n to X. If the domain of the mapping is X^n, then it is called an n-ary **operation**. And an n-ary **relation** is a subclass of X^n where $n > 0$. An operation(relation) is said to be **unary**, **binary** or **ternary** if the arity of the operation(relation) is 1, 2 or 3, respectively. And an operation is called **nullary** if the arity is 0.

Definition 2.3 ([1])**.** An ordered pair $\boldsymbol{A} := \langle A, \mathfrak{L}\rangle$ is said to be a **structure** of a language $\mathfrak{L}$ if A is a nonempty class and there exists a mapping which assigns to every n-ary operation symbol $f \in \mathfrak{L}$ an n-ary operation f^A on $\boldsymbol{A}$ and assigns to every n-ary relation symbol $r \in \mathfrak{L}$ an n-ary relation r^A on $\boldsymbol{A}$. If all operation on $\boldsymbol{A}$ are partial operations, then $\boldsymbol{A}$ is called a **partial structure**. A (partial)structure $\boldsymbol{A}$ is said to be a (**partial**)**algebra** if the language $\mathfrak{L}$ is algebraic.

Definition 2.4 ([1, 3])**.** Let $\boldsymbol{A}, \boldsymbol{B}$ be (partial)structures of a language $\mathfrak{L}$. A mapping $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ is said to be a **homomorphism** provided that

$$\varphi(f^A(a_1, \dots, a_n)) = f^B(\varphi(a_1), \dots, \varphi(a_n)) \text{ for every } n\text{-ary operation } f;$$
$$r^A(a_1, \dots, a_n) \Longrightarrow r^B(\varphi(a_1), \dots, \varphi(a_n)) \text{ for every } n\text{-ary relation } r.$$

A homomorphism φ is called an **isomorphism** if φ is bijective.

2.2. **Pitch Structures.** We recall some important definitions, see paper [2] for more details.

Definition 2.5 ([2])**.** Suppose that P is a nonempty finite set. We may define a binary relation '$\otimes$' on P as follows. For every $s \in P$,

- there is exactly one $u \in P$ such that $u \otimes s$, and
- there is exactly one $v \in P$ such that $s \otimes v$.

Definition 2.6 ([2])**.** A **circle set** is a nonempty finite set equipped with the binary relation '$\otimes$' defined in definition 2.5. Let P be a circle set. Then a bijective mapping $\delta\colon P \to P$ is said to be a **shift** if δ preserves the order of P, i.e., $\delta(p_i) \otimes \delta(p_j)$ if and only if $p_i \otimes p_j$.

Definition 2.7 ([2])**.** Suppose that P is a circle set. Let $\tau := p$ for an arbitrary $p \in P$. We call τ a **tonic** of P.

Definition 2.8 ([2])**.** Suppose that P is a circle set. Let $\mathbb{S}$ be the set $\{\blacksquare, -, \otimes\}$. We may define a function $\lambda\colon P \times P \to \mathbb{S}$ given by

$$\lambda(p, p') = \begin{cases} \blacksquare \text{ or } - & \text{if } p \otimes p', \\ \otimes & \text{otherwise.} \end{cases} \tag{2.1}$$

And the elements of the set $\mathbb{S}$ is called **scales**.

Definition 2.9 ([2])**.** Suppose that P is a circle set. Let $\sharp$ be a unary relation on P such that

(1)
$$\lambda(\sharp(s), \sharp(p)) = \lambda(s, p);$$

(2)
$$\lambda(s, \sharp(p)) = \begin{cases} \rule{1.5em}{0.4em} & \text{if } \lambda(s,p) = \rule{1em}{0.15em}, \\ \otimes & \text{if } \lambda(s,p) = \rule{1.5em}{0.4em}; \end{cases}$$

(3)
$$\lambda(\sharp(s), p) = \begin{cases} \rule{1em}{0.15em} & \text{if } \lambda(s,p) = \rule{1.5em}{0.4em}, \\ \otimes & \text{if } \lambda(s,p) = \rule{1em}{0.15em} \end{cases}$$

for every $s, p \in P$ with $s \olessthan p$.

Definition 2.10 ([2])**.** Suppose that P is a circle set. Let $\flat$ be a unary relation on P such that

(1)
$$\lambda(\flat(s), \flat(p)) = \lambda(s, p);$$

(2)
$$\lambda(s, \flat(p)) = \begin{cases} \rule{1em}{0.15em} & \text{if } \lambda(s,p) = \rule{1.5em}{0.4em}, \\ \otimes & \text{if } \lambda(s,p) = \rule{1em}{0.15em}; \end{cases}$$

(3)
$$\lambda(\flat(s), p) = \begin{cases} \rule{1.5em}{0.4em} & \text{if } \lambda(s,p) = \rule{1em}{0.15em}, \\ \otimes & \text{if } \lambda(s,p) = \rule{1.5em}{0.4em} \end{cases}$$

for every $s, p \in P$ with $s \olessthan p$.

Remark 2.1. In fact, that $\sharp$ and $\flat$ are *not* real unary relations.

Definition 2.11 ([2])**.** A partial structure $\boldsymbol{M} := \langle M, \mathfrak{L} \rangle$ of the language $\mathfrak{L}$ is called a **pitch structure** over a circle set P provided that the underlying set $M = P \cup \mathbb{S}$ where P equipped with $\olessthan$ is a circle set[definition 2.6], and the language is defined to be the set $\mathfrak{L} := \{\lambda, \tau, \mathbb{S}, \olessthan\}$ where λ is a partial binary operation defined in definition 2.8, $\olessthan$ is a binary relation defined in definition 2.5, τ is a nullary operation defined in definition 2.7, and $\mathbb{S} = \{\rule{1.5em}{0.4em}, \rule{1em}{0.15em}, \otimes\}$ is the set of nullary operations defined in definition 2.8.

Definition 2.12 ([2])**.** Let $\boldsymbol{M}$ be a pitch structure over a circle set P, $|P| = n$, and $\tau := m_0$ for $m_0 \in P$. Then we define $SS_{\tau_M}(\boldsymbol{M})$ to be the following sequence

$$\{\lambda(m_0, m_1), \lambda(m_1, m_2), \ldots, \lambda(m_{n-2}, m_{n-1}), \lambda(m_{n-1}, m_0)\}, \tag{2.2}$$

if we have $m_0 \olessthan m_1 \olessthan m_2 \olessthan \cdots \olessthan m_{n-2} \olessthan m_{n-1} \olessthan m_0 \in P$. And the sequence $SS_{\tau_M}(\boldsymbol{M})$ is called a **step** of the pitch structure $\boldsymbol{M}$ at the tonic m_0.

Definition 2.13 ([2])**.** Suppose that $\boldsymbol{M}$ is a pitch structure over a circle set P, and the tonic $\tau = m_0$. Then the ordered pair $\langle \tau_M, SS_{\tau_M}(\boldsymbol{M}) \rangle$ is called the **key** of $\boldsymbol{M}$.

Definition 2.14 ([2])**.** Suppose that $\boldsymbol{M}, \boldsymbol{N}$ are pitch structures over a circle set P, and $\tau_M = m_i$, $\tau_N = m_j$ for $m_i, m_j \in P$. Let δ be a shift[definition 2.6] which assigns m_j to m_i. Then a bijective mapping $\kappa\colon SS_{\tau_M}(\boldsymbol{M}) \rightsquigarrow SS_{\tau_N}(\boldsymbol{N})$ is called a **key transpose** along δ provided that κ assigns $\lambda_N(\delta(m), \delta(m'))$ to $\lambda_M(m, m')$ for every m, $m' \in P$ with $m \olessthan m'$.

3. A 2-Pitch Structure

Definition 3.1. Let $\nu\colon \mathbb{S}\times\mathbb{S}\to\{-1,0,1\}$ be a function defined as follows:

$$\nu(x,y)=\begin{cases}1 & \text{if } \langle x,y\rangle=\langle \text{—},\text{▬}\rangle\\ -1 & \text{if } \langle x,y\rangle=\langle \text{▬},\text{—}\rangle\\ 0 & \text{otherwise,}\end{cases}\tag{3.1}$$

where the set $\mathbb{S}$ is defined in definition 2.8.

Definition 3.2. Suppose that $\boldsymbol{M},\boldsymbol{N}$ are two pitch structures over a circle set P, and $P := \{p_0 \otimes p_1 \otimes \cdots \otimes p_{n-1} \otimes p_0\}$. Let $\kappa\colon SS(\boldsymbol{M}) \rightsquigarrow SS(\boldsymbol{N})$ be a key transpose along a shift δ. Then the integer

$$\|\kappa\| = \sum_{i=0}^{n-1}\nu\Big(\lambda_M\big(p_i,p_{(i+1)\bmod n}\big),\lambda_N\big(\delta(p_i),\delta(p_{(i+1)\bmod n})\big)\Big)\tag{3.2}$$

is called **norm** of κ, where λ is defined in definition 2.8, ν is defined in definition 3.1, δ is defined in definition 2.6, $SS(\boldsymbol{M})$ is defined in definition 2.12, and κ, that is defined in definition 2.14, assigns $\lambda_N\big(\delta(p_i),\delta(p_{(i+1)\bmod n})\big)$ to $\lambda_M\big(p_i,p_{(i+1)\bmod n}\big)$.

Proposition 3.1. *Let $P := \{p_0 \otimes p_1 \otimes \cdots \otimes p_{n-1} \otimes p_0\}$ be a circle set. Suppose that $\{\boldsymbol{M}_i\}_{0\le i\le n-1}$ is a set of pitch structures over the circle set P, and $\tau_{M_i} = p_i$ for every $i \in \{0,\dots,n-1\}$ where τ is defined in definition 2.11. Then the sets $\{\boldsymbol{M}_i\}_{0\le i\le n-1}$ and $\{SS(\boldsymbol{M}_i)\}_{0\le i\le n-1}$ are circle sets.*

Proof. We define $\boldsymbol{M}_i \otimes \boldsymbol{M}_j$, $SS(\boldsymbol{M}_i) \otimes SS(\boldsymbol{M}_j)$ if $p_i \otimes p_j \in P$. Then this is an immediate consequence of definitions 2.5 and 2.6. □

Definition 3.3. Let the notations be as in proposition 3.1, and $\widetilde{P} := \{SS(\boldsymbol{M}_i)\}_{0\le i\le n-1}$. Suppose that $SS(\boldsymbol{M}_i) \otimes SS(\boldsymbol{M}_j)$, and $\kappa_i\colon SS(\boldsymbol{M}_i) \rightsquigarrow SS(\boldsymbol{M}_j)$ is a key transpose along the shift δ_i which assigns τ_{M_j} to τ_{M_i} for all $i,j \in \{0,\dots,n-1\}$ with $j=(i+1) \bmod n$. Then let $\widetilde{\lambda}\colon \widetilde{P}\times\widetilde{P}\to\mathbb{S}$ be a function given by

$$\widetilde{\lambda}(SS(\boldsymbol{M}_i),SS(\boldsymbol{M}_j))=\begin{cases}\text{▬} & \text{if } \|\kappa_i\|\ge 0\\ \text{—} & \text{if } \|\kappa_i\|<0,\end{cases}\tag{3.3}$$

where $\|\kappa_i\|$ is as defined in definition 3.2.

Proposition 3.2. *Let the notations be as in definition 3.3. We have that $\boldsymbol{M}_i \cong \boldsymbol{M}_j$ implies $\widetilde{\lambda}\big(SS(\boldsymbol{M}_i),SS(\boldsymbol{M}_j)\big) = \text{▬}$ for $\boldsymbol{M}_i \otimes \boldsymbol{M}_j \in \{\boldsymbol{M}_i\}_{0\le i\le n-1}$.*

Proof. By [2, proposition 3.1], we have that $SS(\boldsymbol{M}_i) = SS(\boldsymbol{M}_j)$. It follows that $\|\kappa_i\| = 0$ by equations (3.1) and (3.2). Therefore, we have that $\widetilde{\lambda}\big(SS(\boldsymbol{M}_i),SS(\boldsymbol{M}_j)\big) = \text{▬}$ by equation (3.3). □

Proposition 3.3. *Let the notations be as in definition 3.3. If κ_i is regular*[2, definition 3.12]*, then $\widetilde{\lambda}\big(SS(\boldsymbol{M}_i),SS(\boldsymbol{M}_j)\big) = \text{▬}$, for $SS(\boldsymbol{M}_i) \otimes SS(\boldsymbol{M}_j) \in \widetilde{P}$.*

Proof. Since [2, lemmas 3.1 and 3.2], we have that if κ_i is regular, then

- there exist a $\flat$-shrink if and only if there exists a $\flat$-stretch, and
- there exist a $\sharp$-shrink if and only if there exists a $\sharp$-stretch.

Hence there exists $\kappa_i\colon \text{▬} \mapsto\!\!\rightsquigarrow \text{—}$ if and only if there exists $\kappa_i\colon \text{—} \mapsto\!\!\rightsquigarrow \text{▬}$. This implies $\|\kappa_i\| = 0$ by equation (3.2). This completes the proof by equation (3.3). □

Remark. It is obvious that the converses of propositions 3.2 and 3.3 do *not* hold.

Proposition 3.4. *Let the notations be as in definition 3.3. And let* $\tau = SS(\boldsymbol{M}_i)$ *for an* $i \in \{0, \ldots, n-1\}$*. Then the partial structure* $\widetilde{\boldsymbol{M}} := \langle \widetilde{P} \cup \mathbb{S}, \widetilde{\lambda}, \tau, \mathbb{S}, \circledcirc \rangle$ *is a pitch structure over the circle set* $\widetilde{P}$*. And the pitch structure* $\widetilde{\boldsymbol{M}}$ *is called* 2*-pitch structure.*

Proof. The proposition follows from definitions 2.11 and 3.3 and proposition 3.1. ◻

References

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