The infinite series $1 + 2 + 3 + 4 + \cdots$ is strictly divergent

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Abstract

In this paper , I prove that the infinite series $\sum_{n=1}^{+\infty} n = 1 + 2 + 3 + 4 + \cdots$ is strictly divergent or simply $\sum_{n=1}^{+\infty} n = 1 + 2 + 3 + 4 + \cdots = +\infty$ applying an argument by contradiction .

Notation and reminder

$\mathbb{N}^* := \{1,2,3,4, \ldots\}$ *the set of natural numbers .*

$\sum_{n=1}^{+\infty} a_n = +\infty$ *means that the infinite series* $\sum_{n=1}^{+\infty} a_n$ *is divergent .*

$\mathbb{R}$: *denoted the set of real numbers*.

$\ln(x)$: *denoted the natural logarithm of x .*

Introduction

In analysis , a divergent series is an infinite series that is not convergent, meaning that the infinite sequence of the partial sums of the series does not have a finite limit. Before the 19th century, divergent series were widely used by Leonhard Euler and others, but often led to confusing and contradictory results. A major problem was Euler's idea that any divergent series should have a natural sum, without first defining what is meant by the sum of a divergent series. Augustin-Louis Cauchy eventually gave a rigorous definition of the sum of a (convergent) series, and for some time after this, divergent series were mostly excluded from mathematics . They reappeared in 1886 with Henri Poincaré's work on asymptotic series. In 1890, Ernesto Cesàro realized that one could give a rigorous definition of the sum of some divergent series, and defined Cesàro summation. Among the infinite series best known in mathematics , number theory and analysis we find the harmonic series $\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ and the infinite series whose terms are the natural numbers $\sum_{n=1}^{+\infty} n = 1 + 2 + 3 + 4 + \cdots$.

$$1 + 2 + 3 + 4 + \cdots = +\infty$$

Lemma(Harmonic Series). $\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty$.
In other words , the infinite series $\sum_{n=1}^{+\infty} \frac{1}{n}$ *is divergent (slowly divergent).*
Proof. $\forall\, m \in \mathbb{N}^*$ we have $\int_1^{m+1} \frac{dt}{t} \leq \sum_{n=1}^{m} \frac{1}{n} \Rightarrow \ln(m+1) \leq \sum_{n=1}^{m} \frac{1}{n}$
and $\lim\limits_{m\to+\infty} \ln(m+1) = +\infty \Rightarrow \lim\limits_{m\to+\infty} \sum_{n=1}^{m} \frac{1}{n} = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$.

Lemma. *We have* $\sum_{n=1}^{+\infty} \frac{1}{n} \leq \sum_{n=1}^{+\infty} n$.
Proof. Indeed , $\forall\, n \in \mathbb{N}^*$ we have $\frac{1}{n} \leq n$, then $\forall\, m \in \mathbb{N}^*$ we have
$\sum_{n=1}^{m} \frac{1}{n} \leq \sum_{n=1}^{m} n \Rightarrow \lim\limits_{m\to+\infty} \sum_{n=1}^{m} \frac{1}{n} \leq \lim\limits_{m\to+\infty} \sum_{n=1}^{m} n$.

Main Theorem. $\sum_{n=1}^{+\infty} n = 1 + 2 + 3 + 4 + \cdots = +\infty$.
In other words , the infinite series $\sum_{n=1}^{+\infty} n$ *is strictly divergent* .

Proof. An argument by contradiction . Suppose that $\sum_{n=1}^{+\infty} n = \ell \in \mathbb{R}$ or the infinite series $\sum_{n=1}^{+\infty} n$ is convergent . The fame indian mathematician Srinivasa Ramanujan showed that $\ell = -\frac{1}{12}$, he wrote about this in his letter to G. H. Hardy british mathematician , dated 27 February 1913 . On the other hand , $\sum_{n=1}^{+\infty} \frac{1}{n} \leq \sum_{n=1}^{+\infty} n \Rightarrow +\infty \leq \ell$, and we get a contradiction .

References

[1] Henrik Stenlund . On a Method for Solving Infinite Series . arXiv:1405.7633v2 [math.GM] .

[2] Frank Thorne . 1 + 2 + 3 + 4 + · · · .

[3] Dimitris Koukoulopoulos . Introduction à la théorie des nombres .

[4] Thomas John I'Anson Bromwich . An introduction to the theory of infinite series .

[5] John H. Heinbockel . Special Values for the Riemann Zeta Function. Journal of Applied Mathematics and Physics, 2021, 9, 1108-1120 .

[6] Konrad Knopp . Theory And Application Of Infinite Series .

[7] Tim Smits . Integration and Infinite Series.

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