Change of Variable in Integration

Jouni S. Puuronen

2.10.2023

Abstract

We find one proof for one form of the change of variable in integration result with Lebesgue integrals.

The main theorem *We assume that $N \in \{1, 2, 3, \ldots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ and $V \subset \mathbb{R}^N$ are open sets. We assume that $\varphi : U \to V$ is a bijective mapping that is continuously differentiable in its domain, and whose Jacobian $D\varphi(x)$ is non-singular for all $x \in U$. We assume that $A \subset U$ is a Lebesgue measurable set. Then $\varphi(A)$ is Lebesgue measurable. If $f : \varphi(A) \to [-\infty, \infty]$ is a Lebesgue measurable function, then*

$$\int\limits_{\varphi(A)} f(y) dm_N(y) \;=\; \int\limits_{A} (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x),$$

where the integrals are Lebesgue integrals. This means that if an integral on one side of the equation exists as a member of $[-\infty, \infty]$, then also the other side exists with the same value. If an integral on one side of the equation fails to exist, then also the integral on the other side fails to exist.

Theorem 1 *We assume that $N \in \{1, 2, 3, \ldots\}$ is some number, and that $I \subset \mathbb{R}^N$ is a bounded interval. We fix some $\varepsilon > 0$. Then there exists a finite collection $C_1, C_2, \ldots, C_K \subset \mathbb{R}^N$ of cubes such that*

$$I \;=\; \bigcup_{k=1}^{K} C_k \qquad \textit{and} \qquad \sum_{k=1}^{K} m_N(C_k) \;<\; m_N(I) \;+\; \varepsilon.$$

Here m_N means the N-dimensional Lebesgue measure. By cubes we mean intervals whose widths in all axis directions are the same. We omit the proof of Theorem 1.

Theorem 2 *We assume that $N \in \{1, 2, 3, \ldots\}$ is some number, and that $I_1, I_2, I_3, \ldots \subset \mathbb{R}^N$ is a sequence of intervals. Then there exists a sequence $J_1, J_2, J_3, \ldots \subset \mathbb{R}^N$ of intervals such that $J_1, J_2, J_3, \ldots$ are almost disjoint and*

$$\bigcup_{k=1}^{\infty} I_k \;=\; \bigcup_{k=1}^{\infty} J_k.$$

By the intervals $J_1, J_2, J_3, \ldots$ being almost disjoint we mean that $m_N(J_k \cap J_{k'}) = 0$ for all $k, k' \in \{1, 2, 3, \ldots\}$ such that $k \neq k'$. So the boundaries of the intervals can overlap. We omit the proof of Theorem 2.

Theorem 3 *We assume that $N \in \{1, 2, 3, \ldots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ is an open set such that $m_N(U) < \infty$. We fix some $\varepsilon > 0$. Then there exists a sequence $C_1, C_2, C_3, \ldots \subset \mathbb{R}^N$ of cubes such that*

$$U = \bigcup_{k=1}^{\infty} C_k \qquad \textit{and} \qquad \sum_{k=1}^{\infty} m_N(C_k) < m_N(U) + \varepsilon.$$

Proof According to the definition of Lebesgue measure there exists a sequence $I_1, I_2, I_3, \ldots \subset \mathbb{R}^N$ of intervals such that

$$U \subset \bigcup_{k=1}^{\infty} I_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} m_N(I_k) < m_N(U) + \frac{\varepsilon}{2}.$$

We can assume it to be known that since U is open, there exists a sequence $J_1, J_2, J_3, \ldots \subset \mathbb{R}^N$ of intervals such that

$$U = \bigcup_{k=1}^{\infty} J_k.$$

According to Theorem 2 there also exists a sequence $J_1', J_2', J_3', \ldots \subset \mathbb{R}^N$ of intervals such that $J_1', J_2', J_3', \ldots$ are almost disjoint and

$$U = \bigcup_{k=1}^{\infty} J_k = \bigcup_{k=1}^{\infty} J_k'.$$

Now $(I_k \cap J_{k'}')_{k,k' \in \{1,2,3,\ldots\}}$ is a countable collection of intervals and empty sets such that

$$U = \bigcup_{k,k'=1}^{\infty} (I_k \cap J_{k'}')$$

and

$$\sum_{k,k'=1}^{\infty} m_N(I_k \cap J_{k'}') = \sum_{k=1}^{\infty} m_N\Big(I_k \cap \Big(\bigcup_{k'=1}^{\infty} J_{k'}'\Big)\Big) \leq \sum_{k=1}^{\infty} m_N(I_k)$$
$$< m_N(U) + \frac{\varepsilon}{2}.$$

Let's denote that $I_1', I_2', I_3', \ldots$ is the same collection of intervals as $(I_k \cap J_{k'}')_{k,k' \in \{1,2,3,\ldots\}}$. We can omit the empty sets from being present in

$I'_1, I'_2, I'_3, \ldots$. Then

$$U \;=\; \bigcup_{k=1}^{\infty} I'_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} m_N(I'_k) \;<\; m_N(U) \;+\; \frac{\varepsilon}{2}.$$

Then by Theorem 1 for all $k \in \{1, 2, 3, \ldots\}$ there exists a collection $C_{k,1}, C_{k,2}, \ldots, C_{k,K_k}$ of cubes such that

$$I'_k \;=\; \bigcup_{k'=1}^{K_k} C_{k,k'} \qquad \text{and} \qquad \sum_{k'=1}^{K_k} m_N(C_{k,k'}) \;<\; m_N(I'_k) \;+\; \frac{\varepsilon}{2^{k+1}}.$$

Let's denote that $C_1, C_2, C_3, \ldots$ is the same collection of cubes as $(C_{k,k'})_{k \in \{1,2,3,\ldots\}, k' \in \{1,2,\ldots,K_k\}}$. Then

$$U \;=\; \bigcup_{k=1}^{\infty} C_k$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} m_N(C_k) \;&=\; \sum_{k=1}^{\infty} \sum_{k'=1}^{K_k} m_N(C_{k,k'}) \;<\; \sum_{k=1}^{\infty} \left(m_N(I'_k) \;+\; \frac{\varepsilon}{2^{k+1}} \right) \\ &=\; \sum_{k=1}^{\infty} m_N(I'_k) \;+\; \frac{\varepsilon}{2} \;<\; m_N(U) \;+\; \varepsilon. \end{aligned}$$

□

Theorem 4 *We assume that $N \in \{1, 2, 3, \ldots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ is an open set, and that $X \subset U$ is a set such that $m^*_N(X) < \infty$. We fix some $\varepsilon > 0$. Then there exists a sequence $C_1, C_2, C_3, \ldots \subset U$ of cubes such that*

$$X \;\subset\; \bigcup_{k=1}^{\infty} C_k \qquad \textit{and} \qquad \sum_{k=1}^{\infty} m_N(C_k) \;<\; m^*_N(X) \;+\; \varepsilon.$$

Here m^*_N means the Lebesgue outer measure.

Proof According to the definition of Lebesgue outer measure there exists a sequence $I_1, I_2, I_3, \ldots \subset \mathbb{R}^N$ of intervals such that

$$X \;\subset\; \bigcup_{k=1}^{\infty} I_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} m_N(I_k) \;<\; m^*_N(X) \;+\; \frac{\varepsilon}{2}.$$

We can assume it to be known that since U is open, there exists a sequence $J_1, J_2, J_3, \ldots \subset \mathbb{R}^N$ of intervals such that

$$U \;=\; \bigcup_{k=1}^{\infty} J_k.$$

According to Theorem 2 there also exists a sequence $J'_1, J'_2, J'_3, \ldots \subset \mathbb{R}^N$ of intervals such that $J'_1, J'_2, J'_3, \ldots$ are almost disjoint and

$$U \;=\; \bigcup_{k=1}^{\infty} J_k \;=\; \bigcup_{k=1}^{\infty} J'_k.$$

Now $(I_k \cap J'_{k'})_{k,k' \in \{1,2,3,\ldots\}}$ is some countable collection of intervals and empty sets such that

$$X \;\subset\; \bigcup_{k,k'=1}^{\infty} (I_k \cap J'_{k'}) \qquad \text{and} \qquad I_k \cap J'_{k'} \;\subset\; U \quad \forall k, k' \in \{1,2,3,\ldots\}.$$

Let's denote that $I'_1, I'_2, I'_3, \ldots \subset U$ is the same collection of intervals as $(I_k \cap J'_{k'})_{k,k' \in \{1,2,3,\ldots\}}$. We can omit the empty sets from being present in $I'_1, I'_2, I'_3, \ldots$. Then by Theorem 1 for all $k \in \{1,2,3,\ldots\}$ there exists a collection $C_{k,1}, C_{k,2}, \ldots, C_{k,K_k} \subset U$ of cubes such that

$$I'_k \;=\; \bigcup_{k'=1}^{K_k} C_{k,k'} \qquad \text{and} \qquad \sum_{k'=1}^{K_k} m_N(C_{k,k'}) \;<\; m_N(I'_k) \;+\; \frac{\varepsilon}{2^{k+1}}.$$

Let's denote that $C_1, C_2, C_3, \ldots \subset U$ is the same collection of cubes as $(C_{k,k'})_{k \in \{1,2,3,\ldots\}, k' \in \{1,2,\ldots,K_k\}}$. Then

$$X \;\subset\; \bigcup_{k=1}^{\infty} C_k$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty} m_N(C_k) \;&=\; \sum_{k=1}^{\infty} \sum_{k'=1}^{K_k} m_N(C_{k,k'}) \;<\; \sum_{k=1}^{\infty} \Big(m_N(I'_k) \;+\; \frac{\varepsilon}{2^{k+1}} \Big) \\
&=\; \sum_{k,k'=1}^{\infty} m_N(I_k \cap J'_{k'}) \;+\; \frac{\varepsilon}{2} \;=\; \sum_{k=1}^{\infty} m_N\Big(I_k \cap \Big(\bigcup_{k'=1}^{\infty} J'_{k'} \Big) \Big) \;+\; \frac{\varepsilon}{2} \\
&\leq\; \sum_{k=1}^{\infty} m_N(I_k) \;+\; \frac{\varepsilon}{2} \;<\; m_N^*(X) \;+\; \varepsilon.
\end{aligned}$$

$\square$

Theorem 5 *We assume that $N \in \{1,2,3,\ldots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ is an open set, and that $f : U \to \mathbb{R}^N$ is a mapping that is differentiable in its domain. We assume that there exists a constant $C \in \mathbb{R}$ such that*

$$\|\nabla f_n(x)\| \;\leq\; C \qquad \forall\, n \in \{1,2,\ldots,N\},\; x \in U.$$

Then for all sets $X \subset U$

$$m_N^*(f(X)) \;\le\; (NC)^N m_N^*(X).$$

If $C = 0$ and $m_N^*(X) = \infty$, we use the convention $0 \cdot \infty = 0$.

Proof If $C = 0$ or $m_N^*(X) = \infty$, the claim is obvious, so we can assume that $C > 0$ and $m_N^*(X) < \infty$. Let's fix some $\varepsilon > 0$. Then according to Theorem 4 there exists a sequence $C_1, C_2, C_3, \ldots \subset U$ of cubes such that

$$X \;\subset\; \bigcup_{k=1}^{\infty} C_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} m_N(C_k) \;<\; m_N^*(X) \;+\; \varepsilon.$$

Let's denote that $x_1, x_2, x_3, \ldots \in U$ are the center points of the cubes, and that $\ell_1, \ell_2, \ell_3, \ldots \in \mathbb{R}$ are the widths of the cubes. Let's define a new sequence of cubes $D_1, D_2, D_3, \ldots \subset \mathbb{R}^N$ by setting them to be closed, their center points to be $f(x_1), f(x_2), f(x_3), \ldots$, and their widths to be $NC\ell_1, NC\ell_2, NC\ell_3, \ldots$. Then the measures of the new cubes can be written as

$$m_N(D_k) \;=\; (NC)^N m_N(C_k) \qquad \forall\, k \in \{1, 2, 3, \ldots\}.$$

Let's set our objective to be to prove the relation

$$f(C_k) \;\subset\; D_k \qquad \forall\, k \in \{1, 2, 3, \ldots\}.$$

We fix some $k \in \{1, 2, 3, \ldots\}$ and $y \in f(C_k)$. There exists $x \in C_k$ such that $y = f(x)$. Since C_k is a convex set that belongs to U, also the line between x_k and x belongs to U. According to Mean Value Theorem of differentation, for all $n \in \{1, 2, \ldots, N\}$ there exists a point ξ_n along the line between x_k and x such that

$$f_n(x) - f_n(x_k) \;=\; (x - x_k) \cdot \nabla f_n(\xi_n).$$

Then

$$\begin{aligned}
\big\|y - f(x_k)\big\|^2 \;&=\; \sum_{n=1}^{N} \big(y_n - f_n(x_k)\big)^2 \;=\; \sum_{n=1}^{N} \big((x - x_k) \cdot \nabla f_n(\xi_n)\big)^2 \\
&\le\; \sum_{n=1}^{N} \|x - x_k\|^2 \big\|\nabla f_n(\xi_n)\big\|^2 \;\le\; \sum_{n=1}^{N} \Big(\frac{\sqrt{N}\ell_k}{2}\Big)^2 C^2 \\
&=\; \frac{1}{4} N^2 \ell_k^2 C^2.
\end{aligned}$$

This means

$$\big\|y - f(x_k)\big\| \;\le\; \frac{1}{2} NC\ell_k,$$

and $y \in D_k$, so we succeeded in proving the relation $f(C_k) \subset D_k$. Then

$$f(X) \ \subset \ f\Big(\bigcup_{k=1}^{\infty} C_k\Big) \ = \ \bigcup_{k=1}^{\infty} f(C_k) \ \subset \ \bigcup_{k=1}^{\infty} D_k,$$

and

$$m_N^*(f(X)) \ \leq \ m_N^*\Big(\bigcup_{k=1}^{\infty} D_k\Big) \ \leq \ \sum_{k=1}^{\infty} m_N(D_k) \ = \ \sum_{k=1}^{\infty} (NC)^N m_N(C_k)$$
$$< \ (NC)^N \big(m_N^*(X) \ + \ \varepsilon\big).$$

We get the claim of the theorem by taking the limit $\varepsilon \to 0$. $\square$

We can assume it to be known that a measure of a parallelogram is given by an absolute value of a determinant, and that a linear mapping $A \in \mathbb{R}^{N\times N}$ transforms the outer measure of a set $X \subset \mathbb{R}^N$ according to the formula $m_N^*(AX) = |\det(A)| m_N^*(X)$. Then you might guess that if a function is approximately $\varphi(x) \approx Ax$, it should also have the approximation $m_N^*(\varphi(X)) \approx |\det(A)| m_N^*(X)$. Theorem 6 below is one way of turning this approximation into a rigor form.

Theorem 6 *We assume that $N \in \{1, 2, 3, \ldots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $X \subset \mathbb{R}^N$ is some set, that $A \in \mathbb{R}^{N\times N}$ is an invertible matrix, and that a function $\varphi : X \to \mathbb{R}^N$ has been defined by a formula*

$$\varphi(x) \ = \ \psi(x) \ + \ Ax,$$

where the function $\psi : X \to \mathbb{R}^N$ has the property that it is differentiable in $\mathrm{int}(X)$*, and*

$$\|\nabla \psi_n(x)\| \ \leq \ \epsilon \qquad \forall\, n \in \{1, 2, \ldots, N\},\ x \in \mathrm{int}(X)$$

with some constant $\epsilon \geq 0$. Then

$$m_N^*\big(\varphi(X)\big) \ \leq \ \big(1{+}2\epsilon N \|A^{-1}\|_{\mathrm{F}}\big)^N |\det(A)| m_N\big(\mathrm{int}(X)\big) \ + \ m_N^*\big(\varphi(X{\cap}\partial X)\big).$$

Here $\mathrm{int}(X)$ means the interior of X, and $\|\bullet\|_{\mathrm{F}}$ means the Frobenius norm.

Proof Relations

$$\varphi(X) \ = \ \varphi\big(\mathrm{int}(X) \cup (X \cap \partial X)\big) \ = \ \varphi(\mathrm{int}(X)) \ \cup \ \varphi(X \cap \partial X)$$

and

$$m_N^*\big(\varphi(X)\big) \ \leq \ m_N^*\big(\varphi(\mathrm{int}(X))\big) \ + \ m_N^*\big(\varphi(X \cap \partial X)\big)$$

are clear, so it is sufficient to prove

$$m_N^*\big(\varphi(\mathrm{int}(X))\big) \;\le\; \big(1+2\epsilon N\|A^{-1}\|_{\mathrm{F}}\big)^N|\det(A)|m_N\big(\mathrm{int}(X)\big).$$

If $m_N(\mathrm{int}(X)) = \infty$, the claim is obvious, so we can assume $m_N(\mathrm{int}(X)) < \infty$. Then also

$$m_N\big(A\mathrm{int}(X)\big) \;=\; |\det(A)|m_N\big(\mathrm{int}(X)\big) \;<\; \infty.$$

Since A is invertible, $A\mathrm{int}(X)$ is open. Let's fix some $\varepsilon > 0$. Then according to Theorem 3 there exist a sequence $C_1, C_2, C_3, \ldots \subset \mathbb{R}^N$ of cubes such that

$$A\mathrm{int}(X) \;=\; \bigcup_{k=1}^{\infty} C_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} m_N(C_k) \;<\; |\det(A)|m_N(\mathrm{int}(X)) \;+\; \varepsilon.$$

Let's denote that $\ell_1, \ell_2, \ell_3, \ldots \in \mathbb{R}$ are the widths of the cubes $C_1, C_2, C_3, \ldots$. Let's define new cubes $C_1(\epsilon), C_2(\epsilon), C_3(\epsilon), \ldots \subset \mathbb{R}^N$ by setting $C_k(\epsilon)$ to be that closed cube which is obtained from C_k by extending it to both directions of all axes by amount $\epsilon N\|A^{-1}\|_{\mathrm{F}}\ell_k$. This means that

$$m_N\big(C_k(\epsilon)\big) \;=\; \big(1+2\epsilon N\|A^{-1}\|_{\mathrm{F}}\big)^N m_N(C_k) \qquad \forall\, k \in \{1,2,3,\ldots\}.$$

Let's choose some $x_k \in A^{-1}C_k$ for all $k \in \{1,2,3,\ldots\}$. Then also $x_1, x_2, x_3, \ldots \in X$. Let's set our objective to be to prove the relation

$$\varphi\big(\mathrm{int}(X)\big) \;\subset\; \bigcup_{k=1}^{\infty}\big(\psi(x_k) \;+\; C_k(\epsilon)\big). \tag{1}$$

If we succeed in this, then

$$\begin{aligned} m_N^*\big(\varphi\big(\mathrm{int}(X)\big)\big) \;&\le\; \sum_{k=1}^{\infty} m_N^*\big(\psi(x_k) \;+\; C_k(\epsilon)\big) \\ &=\; \big(1+2\epsilon N\|A^{-1}\|_{\mathrm{F}}\big)^N \sum_{k=1}^{\infty} m_N(C_k) \\ &<\; \big(1+2\epsilon N\|A^{-1}\|_{\mathrm{F}}\big)^N\big(|\det(A)|m_N\big(\mathrm{int}(X)\big) \;+\; \varepsilon\big), \end{aligned}$$

and we can complete the proof by taking the limit $\varepsilon \to 0$. So now everything depends on proving (1).

Let's fix some $y \in \varphi(\mathrm{int}(X))$. Then there exists $x \in \mathrm{int}(X)$ such that $y = \varphi(x)$. There also exists $k \in \{1,2,3,\ldots\}$ such that $Ax \in C_k$. If we succeed in proving that

$$\varphi(x) \;\in\; \psi(x_k) \;+\; C_k(\epsilon),$$

we succeed in proving (1). Relation

$$\varphi(x) \;=\; \psi(x_k) \;+\; \big(\psi(x) - \psi(x_k)\big) \;+\; Ax \;\in\; \psi(x_k) \;+\; \big(\psi(x) - \psi(x_k)\big) \;+\; C_k$$

is clear, so (1) follows, if we prove

$$\big(\psi(x) - \psi(x_k)\big) \;+\; C_k \;\subset\; C_k(\epsilon). \tag{2}$$

Now everything depends on proving (2).

Relations $x \in A^{-1}C_k$ and $x_k \in A^{-1}C_k$ are true, and since $A^{-1}C_k$ is convex, the line between x and x_k also belongs to $A^{-1}C_k \subset \mathrm{int}(X)$. According to Mean Value Theorem of differentiation, for all $n \in \{1, 2, \dots, N\}$, there exists ξ_n on the line between x and x_k such that

$$\psi_n(x) - \psi_n(x_k) \;=\; (x - x_k) \cdot \nabla\psi_n(\xi_n).$$

Then

$$\begin{aligned}
\big\|\psi(x) - \psi(x_k)\big\|^2 \;&=\; \sum_{n=1}^{N} \big(\psi_n(x) - \psi_n(x_k)\big)^2 \;=\; \sum_{n=1}^{N} \big((x - x_k) \cdot \nabla\psi_n(\xi_n)\big)^2 \\
&\leq\; \sum_{n=1}^{N} \|x - x_k\|^2 \big\|\nabla\psi_n(\xi_n)\big\|^2 \;\leq\; N\|x - x_k\|^2\epsilon^2,
\end{aligned}$$

and

$$\begin{aligned}
\big\|\psi(x) - \psi(x_k)\big\| \;&\leq\; \sqrt{N}\|x - x_k\|\epsilon \;=\; \sqrt{N}\|A^{-1}A(x - x_k)\|\epsilon \\
&\leq\; \sqrt{N}\|A^{-1}\|_{\mathrm{F}}\|A(x - x_k)\|\epsilon \;\leq\; \sqrt{N}\|A^{-1}\|_{\mathrm{F}}\big(\sqrt{N}\ell_k\big)\epsilon \\
&=\; \epsilon N\|A^{-1}\|_{\mathrm{F}}\ell_k.
\end{aligned}$$

Inequality $\|Ax - Ax_k\| \leq \sqrt{N}\ell_k$ comes from the fact that since the points Ax and Ax_k both belong to the cube C_k, whose side length is ℓ_k, the distance between the points cannot be larger than $\sqrt{N}\ell_k$.

The cube $C_k(\epsilon)$ was defined so that C_k was extended to all axis directions by the amount $\epsilon N\|A^{-1}\|_{\mathrm{F}}\ell_k$, and now the length of $\psi(x) - \psi(x_k)$ is equal or less than this amount, so we can conclude that (2) is true. □

Theorem 7 *We assume that $N \in \{1, 2, 3, \dots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ is an open set, that $X \subset U$ is a compact set, and that $V \subset X$ is an open set. We assume that $\varphi : U \to \mathbb{R}^N$ is a mapping that is continuously differentiable in its domain, and whose Jacobian $D\varphi(x)$ is non-singular for all $x \in U$. Then*

$$m_N^*\big(\varphi(V)\big) \;\leq\; \int_V \big|\det(D\varphi(x))\big| dm_N(x).$$

Most people who see Theorem 7 probably wonder that why do we use the relation $V \subset X \subset U$ in a such complicated way, and wouldn't it be simpler to just estimate $m_N^*(\varphi(U))$? The answer is that estimating $m_N^*(\varphi(U))$ directly would be more difficult, and the relation $V \subset X \subset U$ makes the proof easier. You can criticize Theorem 7 for being a weak result due to its formulation, but we will use Theorem 7 as a tool to prove a similar stronger result Theorem 8 later below.

Proof According to the assumptions $(D\varphi(x))^{-1}$ exists for all $x \in U$. Let's justify that the mapping $x \mapsto (D\varphi(x))^{-1}$ is continuous for all $x \in U$. There exists a formula for inverse matrices where each element of the inverse matrix depends on finite amount of additions, subtractions, multiplications and divisions of the elements of the matrix, so in domain where the determinant is non-zero, inverse matrix depends continuously on the matrix. We have assumed that $x \mapsto D\varphi(x)$ is continuous, so $x \mapsto (D\varphi(x))^{-1}$ is a composite of two continuous functions. Also the Frobenius norm $\| \bullet \|_{\mathrm{F}}$ is continuous. Real valued continuous functions reach their maximal values on compact sets, so we can define

$$M \;:=\; \max_{x \in X} \|(D\varphi(x))^{-1}\|_{\mathrm{F}},$$

and then $M < \infty$.

Let's fix some $\varepsilon > 0$. Continuous mappings on compact sets are always uniformly continuous, so the restriction $X \to \mathbb{R}^{N \times N}$, $x \mapsto D\varphi(x)$ is uniformly continuous. We can choose $\delta_1 > 0$ such that the relation

$$\begin{array}{l} \|x - x'\| < \delta_1 \qquad \Longrightarrow \qquad \|\nabla\varphi_n(x) - \nabla\varphi_n(x')\| < \varepsilon \\ \forall\, n \in \{1, 2, \ldots, N\},\; x, x' \in X \end{array}$$

is true. Also $X \to \mathbb{R}$, $x \mapsto \det(D\varphi(x))$ is uniformly continuous, so we can choose $\delta_2 > 0$ such that the relation

$$\|x - x'\| < \delta_2 \qquad \Longrightarrow \qquad \big|\det(D\varphi(x)) - \det(D\varphi(x'))\big| < \varepsilon \qquad \forall\, x, x' \in X$$

is true. Let's denote $\delta := \min\{\delta_1, \delta_2\}$.

Let's write V in form $V = V_1 \cup V_2 \cup \cdots \cup V_K$, where $V_1, V_2, \ldots, V_K$ are almost disjoint, and where each V_k has been defined so that it is an intersection of V and some cube that is of the form

$$\Big[j_1 \frac{\delta}{2\sqrt{N}}, (j_1{+}1)\frac{\delta}{2\sqrt{N}}\Big] \times \Big[j_2 \frac{\delta}{2\sqrt{N}}, (j_2{+}1)\frac{\delta}{2\sqrt{N}}\Big] \times \cdots \times \Big[j_N \frac{\delta}{2\sqrt{N}}, (j_N{+}1)\frac{\delta}{2\sqrt{N}}\Big],$$

with some indices $j_1, j_2, \ldots, j_N \in \mathbb{Z}$. We can assume that all $V_1, V_2, \ldots, V_K$ are non-empty. Then for all $k \in \{1, 2, \ldots K\}$ the set V_k has the property

$$x, x' \in V_k \qquad \Longrightarrow \qquad \|x - x'\| \;\leq\; \frac{\delta}{2} \;<\; \delta.$$

Let's fix some $k \in \{1, 2, \ldots, K\}$, and investigate what kind of upper bounds we can find for $m_N^*(\varphi(V_k))$. Let's fix some $x_k \in V_k$, and define a mapping

$$\psi : V_k \to \mathbb{R}^N, \qquad \psi(x) \;=\; \varphi(x) \;-\; (D\varphi(x_k))x.$$

Now ψ has the property that it is differentiable in $\mathrm{int}(V_k)$, and

$$\begin{aligned} &\big\|\nabla\psi_n(x)\big\| \;=\; \big\|\nabla\varphi_n(x) - \nabla\varphi_n(x_k)\big\| \;<\; \varepsilon \\ &\forall\, n \in \{1, 2, \ldots, N\},\; x \in \mathrm{int}(V_k) \end{aligned}$$

If we write φ in form

$$\varphi(x) \;=\; \psi(x) \;+\; (D\varphi(x_k))x,$$

we see that according to Theorem 6

$$\begin{aligned} m_N^*(\varphi(V_k)) \;\le\; & \big(1 + 2\varepsilon N \|(D\varphi(x_k))^{-1}\|_{\mathrm{F}}\big)^N \big|\det(D\varphi(x_k))\big| m_N(\mathrm{int}(V_k)) \\ & +\; m_N^*\big(\varphi(V_k \cap \partial V_k)\big). \end{aligned}$$

Let's justify that $m_N^*(V_k \cap \partial V_k) = 0$. In general boundaries don't necessarily have measure zero, but boundaries of cubes do, and $V_k \cap \partial V_k$ is a subset of boundary of the cube (that was defined with some indices $j_1, j_2, \ldots, j_N \in \mathbb{Z}$). Assume as an antithesis that some $x \in V_k \cap \partial V_k$ would not belong to a boundary of the cube. Then x belongs to the interior of the cube, so some $B(x, r_1)$ is a subset of the cube. Also x belongs to V that is an open set, so some $B(x, r_2)$ is a subset of V. Now $B(x, \min\{r_1, r_2\})$ is a subset of V_k, meaning that $x \in \mathrm{int}(V_k)$, and not $x \in V_k \cap \partial V_k$.

Then according to Theorem 5

$$m_N^*\big(\varphi(V_k \cap \partial V_k)\big) \;\le\; \Big(N \max_{\substack{x \in X \\ n \in \{1,2,\ldots,N\}}} \|\nabla\varphi_n(x)\|\Big)^N \underbrace{m_N^*(V_k \cap \partial V_k)}_{=0} \;=\; 0.$$

By using inequalities $\|(D\varphi(x_k))^{-1}\|_{\mathrm{F}} \le M$ and $m_N(\mathrm{int}(V_k)) \le m_N(V_k)$ we get an upper bound

$$m_N^*(\varphi(V_k)) \;\le\; (1 + 2\varepsilon N M)^N \big|\det(D\varphi(x_k))\big| m_N(V_k).$$

A relation

$$\big|\det(D\varphi(x_k))\big| \;\le\; \big|\det(D\varphi(x))\big| \;+\; \varepsilon \qquad \forall\, x \in V_k$$

is true, so we can write the upper bound as

$$\begin{aligned} m_N^*(\varphi(V_k)) \;&\le\; (1 + 2\varepsilon N M)^N \int\limits_{V_k} \big|\det(D\varphi(x_k))\big| dm_N(x) \\ &\le\; (1 + 2\varepsilon N M)^N \Big(\int\limits_{V_k} \big|\det(D\varphi(x))\big| dm_N(x) \;+\; \varepsilon m_N(V_k)\Big). \end{aligned}$$

Since $V_1, V_2, \ldots, V_K$ are almost disjoint, we can write an integral over V as a sum of integrals over $V_1, V_2, \ldots, V_K$.

$$\begin{aligned}
m_N^*(\varphi(V)) &= m_N^*\Big(\bigcup_{k=1}^{K} \varphi(V_k)\Big) \;\leq\; \sum_{k=1}^{K} m_N^*(\varphi(V_k)) \\
&\leq (1+2\varepsilon NM)^N \sum_{k=1}^{K} \Big(\int\limits_{V_k} \big|\det(D\varphi(x))\big| dm_N(x) \;+\; \varepsilon m_N(V_k)\Big) \\
&= (1+2\varepsilon NM)^N \Big(\int\limits_{V} \big|\det(D\varphi(x))\big| dm_N(x) \;+\; \varepsilon m_N(V)\Big)
\end{aligned}$$

Since X is compact, it is bounded and has a finite measure, so $m_N(V) \leq m_N(X) < \infty$. We get the claim of the theorem by taking the limit $\varepsilon \to 0$. $\square$

There is a theorem that states that if a continuous mapping maps sets of zero measure to sets of zero measure, then it maps Lebesgue measurable sets to Lebesgue measurable sets. This is based on a result that if A is Lebesgue measurable, then it is possible to write it in a form $A = E \cup X_1 \cup X_2 \cup X_3 \cup \cdots$, where E has measure zero, and $X_1, X_2, X_3, \ldots$ are compact. A continuous mapping will map $X_1, X_2, X_3, \ldots$ into compact sets, which are Lebesgue measurable, so essentially the image of A will be Lebesgue measurable, if the image of E is Lebesgue measurable. If the image of E has measure zero, then it is Lebesgue measurable.

Theorem 8 *We assume that $N \in \{1,2,3,\ldots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ is an open set, that $X \subset U$ is a compact set, and that $A \subset \mathrm{int}(X)$ is a Lebesgue measurable set. We assume that $\varphi : U \to \mathbb{R}^N$ is a mapping that is continuously differentiable in its domain, and whose Jacobian $D\varphi(x)$ is non-singular for all $x \in U$. Then $\varphi(A)$ is Lebesgue measurable, and*

$$m_N\big(\varphi(A)\big) \;\leq\; \int\limits_{A} \big|\det(D\varphi(x))\big| dm_N(x).$$

Proof To prove that $\varphi(A)$ is Lebesgue measurable, it is sufficient to prove that φ maps any subset of $\mathrm{int}(X)$, that has a measure zero, into a set of measure zero. Let $E \subset \mathrm{int}(X)$ be some set such that $m_N^*(E) = 0$. Then according to Theorem 5

$$m_N^*(\varphi(E)) \;\leq\; \underbrace{\Big(N \max_{\substack{x\in X \\ n\in\{1,2,\ldots,N\}}} \|\nabla\varphi_n(x)\|\Big)^N}_{<\infty} \underbrace{m_N^*(E)}_{=0} \;=\; 0.$$

Let's fix some $\varepsilon > 0$. Since $m_N(A) \le m_N(X) < \infty$, there exists an open set $V \subset \mathbb{R}^N$ such that

$$A \subset V \qquad \text{and} \qquad m_N(V \setminus A) < \varepsilon.$$

Then $V \cap \mathrm{int}(X)$ is an open set that satisfies the relation

$$V \cap \mathrm{int}(X) \subset X \subset U,$$

so according to Theorem 7

$$m_N^*\big(\varphi(V \cap \mathrm{int}(X))\big) \;\le\; \int\limits_{V\cap \mathrm{int}(X)} \big|\det(D\varphi(x))\big| dm_N(x).$$

Also

$$A \subset V \cap \mathrm{int}(X) \qquad \text{and} \qquad m_N\big((V \cap \mathrm{int}(X)) \setminus A\big) < \varepsilon,$$

so

$$\begin{aligned} m_N(\varphi(A)) \;\le\; m_N^*\big(\varphi(V\cap \mathrm{int}(X))\big) \;&\le\; \int\limits_{V\cap \mathrm{int}(X)} \big|\det(D\varphi(x))\big| dm_N(x) \\ &=\; \int\limits_A \big|\det(D\varphi(x))\big| dm_N(x) \;+\; \int\limits_{(V\cap \mathrm{int}(X))\setminus A} \big|\det(D\varphi(x))\big| dm_N(x) \\ &\le\; \int\limits_A \big|\det(D\varphi(x))\big| dm_N(x) \;+\; \varepsilon \max_{x\in X} \big|\det(D\varphi(x))\big|. \end{aligned}$$

We get the claim of the theorem by taking the limit $\varepsilon \to 0$. □

Theorem 9 *We assume that $N \in \{1, 2, 3, \dots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ and $V \subset \mathbb{R}^N$ are open sets. We assume that $\varphi : U \to V$ is a bijective mapping that is continuously differentiable in its domain, and whose Jacobian $D\varphi(x)$ is non-singular for all $x \in U$. We assume that $X \subset U$ is a compact set, and that $A \subset \mathrm{int}(X)$ is a Lebesgue measurable set. We denote $Y := \varphi(X)$ and $B := \varphi(A)$. Then Y is compact, B is Lebesgue measurable, and $B \subset \mathrm{int}(Y)$. If $f : B \to [0, \infty]$ is a Lebesgue measurable function, then*

$$\int\limits_B f(y) dm_N(y) \;=\; \int\limits_A (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x),$$

where the integrals are Lebesgue integrals. This means that either the both integrals are finite with the same value, or that they are both infinite.

Most people who see Theorem 9 probably wonder that why do we use the relation $A \subset \mathrm{int}(X) \subset X \subset U$ in a such complicated way, and wouldn't it be simpler to integrate over an arbitrary Lebesgue measurable set $A \subset U$? A challenge with an arbitrary Lebesgue measurable set $A \subset U$ is that the behaviour of φ can become bad in the limit where x approaches the boundary ∂U, and it is difficult to approximate how φ distorts subsets close to the boundary. By using the relation $A \subset \mathrm{int}(X) \subset X \subset U$, where X is compact, we isolate A from the boundary ∂U, which makes the proof easier. However, we will replace the relation $A \subset \mathrm{int}(X) \subset X \subset U$ with the simpler relation $A \subset U$ later in Theorem 11.

Proof Continuous mappings map compact sets into compact sets, so compactness of Y is clear. B is Lebesgue measurable according to Theorem 8. Let's prove the relation $B \subset \mathrm{int}(Y)$ next, and fix some $y \in B$. Then there exists $x \in A$ such that $y = \varphi(x)$. There also exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X$. According to Inverse Function Theorem φ^{-1} is continuous, so, also keeping in mind that V is open, there exists $\delta > 0$ such that $B(y, \delta) \subset V$ and $\varphi^{-1}(B(y, \delta)) \subset B(x, \varepsilon)$. Then also

$$B(y,\delta) \;\subset\; \varphi\big(B(x,\varepsilon)\big) \;\subset \varphi(X) \;=\; Y,$$

which means that $y \in \mathrm{int}(Y)$.

Let's assume that

$$\int\limits_B f(y) dm_N(y) \;\leq\; \int\limits_A (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x).$$

is true under the assumptions of the theorem. What would this imply? According to Inverse Function Theorem $\varphi^{-1} : V \to U$ is continuously differentiable with a Jacobian $D\varphi^{-1}(y)$ that is non-singular for all $y \in V$. We can swap the places of the objects (x, A, X, U) and (y, B, Y, V), and apply the "$\leq$"-result by substituting functions $\varphi \leftarrow \varphi^{-1}$ and $f \leftarrow (f \circ \varphi)|\det(D\varphi)|$. We get

$$\begin{aligned}
&\int\limits_A (f \circ \varphi)(x) \big|\det(D\varphi)(x)\big| dm_N(x) \\
&\leq \int\limits_B \underbrace{\big((f \circ \varphi) \circ \varphi^{-1}\big)(y)}_{=f} \, \underbrace{\big|\det(D\varphi)(\varphi^{-1}(y))\big|\big|\det(D\varphi^{-1})(y)\big|}_{=1} \, dm_N(y) \\
&= \int\limits_B f(y) dm_N(y).
\end{aligned}$$

This is the "$\geq$"-direction of the claim of the theorem. So we see that it is sufficient to prove the "$\leq$"-direction of the claim of the theorem, since the "$\geq$"-direction follows on its own.

Let's assume that

$$\int\limits_B f(y) dm_N(y) \;<\; \infty$$

and fix some $\varepsilon > 0$. According to the definition of Lebesgue integral there exists disjoint Lebesgue measurable sets $B_1, B_2, \ldots, B_K \subset B$ and numbers $c_1, c_2, \ldots, c_K \in [0, \infty]$ such that for all $k \in \{1, 2, \ldots, K\}$ $c_k \le f(y)$ for all $y \in B_k$, and

$$\int\limits_B f(y) dm_N(y) \ < \ \sum_{k=1}^{K} c_k m_N(B_k) \ + \ \varepsilon.$$

Let's define sets $A_1, A_2, \ldots, A_K$ by setting $A_k := \varphi^{-1}(B_k)$ for all $k \in \{1, 2, \ldots, K\}$. Then $A_1, A_2, \ldots, A_K$ are disjoint, since φ is a bijection, and for all $k \in \{1, 2, \ldots, K\}$ $c_k \le (f \circ \varphi)(x)$ for all $x \in A_k$. According to Theorem 8 $A_1, A_2, \ldots, A_K$ are Lebesgue measurable, and

$$m_N(B_k) \ = \ m_N(\varphi(A_k)) \ \le \ \int\limits_{A_k} \big|\det(D\varphi(x))\big| dm_N(x).$$

Then

$$\begin{aligned}
\int\limits_B f(y) dm_N(y) \ &< \ \sum_{k=1}^{K} c_k m_N(B_k) \ + \ \varepsilon \\
&\le \ \sum_{k=1}^{K} c_k \int\limits_{A_k} \big|\det(D\varphi(x))\big| dm_N(x) \ + \ \varepsilon \\
&\le \ \sum_{k=1}^{K} \int\limits_{A_k} (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x) \ + \ \varepsilon \\
&= \ \int\limits_{A_1 \cup A_2 \cup \cdots \cup A_K} (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x) \ + \ \varepsilon \\
&\le \ \int\limits_{A} (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x) \ + \ \varepsilon.
\end{aligned}$$

We get the "$\le$"-direction of the claim of the theorem by taking the limit $\varepsilon \to 0$.

Let's assume that

$$\int\limits_B f(y) dm_N(y) \ = \ \infty$$

and fix some $R \in \mathbb{R}$. According to the definition of Lebesgue integral there exists disjoint Lebesgue measurable sets $B_1, B_2, \ldots, B_K \subset B$ and numbers $c_1, c_2, \ldots, c_K \in [0, \infty]$ such that for all $k \in \{1, 2, \ldots, K\}$ $c_k \le f(y)$ for all $y \in B_k$, and

$$R \ < \ \sum_{k=1}^{K} c_k m_N(B_k).$$

Let's again define sets $A_1, A_2, \ldots, A_K$ by setting $A_k := \varphi^{-1}(B_k)$ for all $k \in \{1, 2, \ldots, K\}$. Again $A_1, A_2, \ldots, A_K$ are disjoint and Lebesgue measurable, and

$$m_N(B_k) \;\leq\; \int\limits_{A_k} \big|\det(D\varphi(x))\big| dm_N(x).$$

Using similar steps as above we get

$$\begin{aligned} R \;&<\; \sum_{k=1}^{K} c_k m_N(B_k) \;\leq\; \sum_{k=1}^{K} c_k \int\limits_{A_k} \big|\det(D\varphi(x))\big| dm_N(x) \\ &\leq\; \cdots \leq\; \int\limits_{A} (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x). \end{aligned}$$

We get

$$\int\limits_{A} (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x) \;=\; \infty$$

by taking the limit $R \to \infty$. We conclude that the "$\leq$"-direction of the claim of the theorem is true whether the integral of $f(y)$ is finite or not. $\square$

Theorem 10 *We assume that $N \in \{1, 2, 3, \ldots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ is an open set. Then there exists a sequence $X_1 \subset X_2 \subset X_3 \subset \cdots \subset U$ of compact sets such that*

$$U \;=\; \bigcup_{k=1}^{\infty} \operatorname{int}(X_k).$$

Proof If $U = \mathbb{R}^N$, we can set $X_k = [-k, k]^N$, so we can next assume that $U \subsetneq \mathbb{R}^N$. Let's define a function $f : \mathbb{R}^N \to [0, \infty[$ by setting

$$f(x) \;=\; \operatorname{dist}(x, \mathbb{R}^N \setminus U) \;=\; \inf_{x' \in \mathbb{R}^N \setminus U} \|x - x'\|.$$

Then f is continuous, because relation

$$f\Big(B\Big(x, \frac{\varepsilon}{2}\Big)\Big) \;\subset\; \big]f(x) - \varepsilon, f(x) + \varepsilon\big[$$

is true for all $x \in \mathbb{R}^N$ and $\varepsilon > 0$. Let's justify this. If we fix some arbitrary point from $f\big(B\big(x, \frac{\varepsilon}{2}\big)\big)$, we can denote it as $f(\overline{x})$ with some $\overline{x} \in \mathbb{R}^N$ such that $\|x - \overline{x}\| < \frac{\varepsilon}{2}$. There exists $x' \in \mathbb{R}^N \setminus U$ such that

$$\|x - x'\| \;<\; f(x) \;+\; \frac{\varepsilon}{2}.$$

Then

$$f(\overline{x}) \;\leq\; \|\overline{x} - x'\| \;\leq\; \|\overline{x} - x\| \;+\; \|x - x'\| \;<\; f(x) \;+\; \varepsilon.$$

There also exists $x'' \in \mathbb{R}^N \setminus U$ such that

$$\|\overline{x} - x''\| \;<\; f(\overline{x}) \;+\; \frac{\varepsilon}{2}.$$

Then

$$f(x) \;\le\; \|x - x''\| \;\le\; \|x - \overline{x}\| \;+\; \|\overline{x} - x''\| \;<\; f(\overline{x}) \;+\; \varepsilon.$$

We can summarize these results as

$$f(x) \;-\; \varepsilon \;<\; f(\overline{x}) \;<\; f(x) \;+\; \varepsilon.$$

Let's define sets $S_1, S_2, S_3, \ldots \subset \mathbb{R}^N$ by setting

$$S_k \;=\; f^{-1}\Big(\Big[\frac{1}{k}, \infty\Big[\Big) \qquad \forall\, k \in \{1, 2, 3, \ldots\}.$$

Then all $S_1, S_2, S_3, \ldots$ are closed, and satisfy relation $S_1 \subset S_2 \subset S_3 \subset \cdots$. Relation $S_k \subset U$ is true for all $k \in \{1, 2, 3, \ldots\}$, since for all $x \in S_k$ $\mathrm{dist}(x, \mathbb{R}^N \setminus U) \ge \frac{1}{k}$, meaning $x \notin \mathbb{R}^N \setminus U$.

Let's prove

$$f^{-1}\Big(\Big]\frac{1}{k}, \infty\Big[\Big) \;\subset\; \mathrm{int}(S_k) \qquad \forall\, k \in \{1, 2, 3, \ldots\}.$$

For this, let's fix arbitrary

$$x \;\in\; f^{-1}\Big(\Big]\frac{1}{k}, \infty\Big[\Big),$$

and define a radius

$$r \;:=\; \frac{1}{2}\Big(f(x) \;-\; \frac{1}{k}\Big).$$

Then $r > 0$. We want to prove that $B(x, r) \subset S_k$. This is the same as

$$f\big(B(x, r)\big) \;\subset\; \Big[\frac{1}{k}, \infty\Big[.$$

If we fix an arbitrary point from $f(B(x, r))$, we can denote it as $f(\overline{x})$ with some $\overline{x} \in \mathbb{R}^N$ such that $\|x - \overline{x}\| < r$. Then there exists $x' \in \mathbb{R}^N \setminus U$ such that

$$f(\overline{x}) \;>\; \|\overline{x} - x'\| \;-\; r \;\ge\; \underbrace{\|x - x'\|}_{\ge f(x)} \;-\; \underbrace{\|x - \overline{x}\|}_{< r} \;-\; r \;>\; f(x) \;-\; 2r \;=\; \frac{1}{k}.$$

Let's next prove that

$$U \;=\; \bigcup_{k=1}^{\infty} \mathrm{int}(S_k).$$

The "$\supset$"-direction is obvious, so it is sufficient to prove the "$\subset$"-direction. For this it is sufficient to prove that

$$U \subset \bigcup_{k=1}^{\infty} f^{-1}\Big(\Big]\frac{1}{k},\infty\Big[\Big).$$

Let's fix arbitrary $x \in U$. Since U is open, there exist a radius $r > 0$ such that $B(x,r) \subset U$. This means that $f(x) = \mathrm{dist}(x, \mathbb{R}^N \setminus U) \geq r$, and

$$f(x) \in \Big]\frac{1}{k},\infty\Big[\qquad \text{and} \qquad x \in f^{-1}\Big(\Big]\frac{1}{k},\infty\Big[\Big)$$

for k so large that $\frac{1}{k} < r$. Then also the relation

$$U = \bigcup_{k=1}^{\infty} \big(\mathrm{int}(S_k) \cap]-k,k[^N\big)$$

is obvious. If we define a sequence $X_1, X_2, X_3, \ldots \subset \mathbb{R}^N$ by setting

$$X_k := S_k \cap [-k,k]^N \qquad \forall\, k \in \{1,2,3,\ldots\},$$

then all $X_1, X_2, X_3, \ldots$ are compact and satisfy the relation $X_1 \subset X_2 \subset X_3 \subset \cdots \subset U$. If we prove that

$$\mathrm{int}(S_k) \cap]-k,k[^N \subset \mathrm{int}(X_k) \qquad \forall\, k \in \{1,2,3,\ldots\},$$

then

$$U = \bigcup_{k=1}^{\infty} \mathrm{int}(X_k)$$

follows, and the proof becomes complete. Let's fix arbitrary $x \in \mathrm{int}(S_k) \cap]-k,k[^N$. Then there exists $r_1 > 0$ such that $B(x,r_1) \subset S_k$ and $r_2 > 0$ such that $B(x,r_2) \subset [-k,k]^N$. If we set $r := \min\{r_1,r_2\}$, then $B(x,r) \subset S_k \cap [-k,k]^N = X_k$, meaning that $x \in \mathrm{int}(X_k)$. $\square$

Theorem 11 *We assume that $N \in \{1,2,3,\ldots\}$ is some number, and equip $\mathbb{R}^N$ with Euclidean topology. We assume that $U \subset \mathbb{R}^N$ and $V \subset \mathbb{R}^N$ are open sets. We assume that $\varphi : U \to V$ is a bijective mapping that is continuously differentiable in its domain, and whose Jacobian $D\varphi(x)$ is non-singular for all $x \in U$. We assume that $A \subset U$ is a Lebesgue measurable set. Then $\varphi(A)$ is Lebesgue measurable. If $f : \varphi(A) \to [0,\infty]$ is a Lebesgue measurable function, then*

$$\int_{\varphi(A)} f(y)\,dm_N(y) = \int_A (f\circ\varphi)(x)\big|\det(D\varphi(x))\big|\,dm_N(x),$$

where the integrals are Lebesgue integrals. This means that either the both integrals are finite with the same value, or that they are both infinite.

Proof According to Theorem 10 there exists a sequence $X_1 \subset X_2 \subset X_3 \subset \cdots \subset U$ of compact sets such that

$$U = \bigcup_{k=1}^{\infty} \mathrm{int}(X_k).$$

Then relations

$$A = \bigcup_{k=1}^{\infty} \big(A \cap \mathrm{int}(X_k)\big)$$

and

$$\varphi(A) = \bigcup_{k=1}^{\infty} \varphi\big(A \cap \mathrm{int}(X_k)\big)$$

are true. According to Theorem 9 set $\varphi(A \cap \mathrm{int}(X_k))$ is Lebesgue measurable for all $k \in \{1, 2, 3, \ldots\}$. A countable union of Lebesgue measurable sets is Lebesgue measurable, so $\varphi(A)$ is Lebesgue measurable. According to Theorem 9

$$\int\limits_{\varphi(A \cap \mathrm{int}(X_k))} f(y) dm_N(y) = \int\limits_{A \cap \mathrm{int}(X_k)} (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x)$$

for all $k \in \{1, 2, 3, \ldots\}$. Also relations

$$A \cap \mathrm{int}(X_1) \subset A \cap \mathrm{int}(X_2) \subset A \cap \mathrm{int}(X_3) \subset \cdots$$

and

$$\varphi(A \cap \mathrm{int}(X_1)) \subset \varphi(A \cap \mathrm{int}(X_2)) \subset \varphi(A \cap \mathrm{int}(X_3)) \subset \cdots$$

are true, so according to Monotone Convergence Theorem

$$\begin{aligned} &\int\limits_A (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x) \\ &= \lim_{k \to \infty} \int\limits_{A \cap \mathrm{int}(X_k)} (f \circ \varphi)(x) \big|\det(D\varphi(x))\big| dm_N(x) \end{aligned}$$

and

$$\int\limits_{\varphi(A)} f(y) dm_N(y) = \lim_{k \to \infty} \int\limits_{\varphi(A \cap \mathrm{int}(X_k))} f(y) dm_N(y),$$

so

$$\int_{\varphi(A)} f(y)dm_N(y) \;=\; \int_A (f\circ\varphi)(x)\big|\det(D\varphi(x))\big|dm_N(x).$$

□

Proof of the main theorem Set $\varphi(A)$ is Lebesgue measurable according to Theorem 11. Let's denote $B = \varphi(A)$, and define new sets by setting

$$B_+ \;:=\; \big\{f(y) > 0 \;\big|\; y \in B\big\}, \quad B_- \;:=\; \big\{f(y) < 0 \;\big|\; y \in B\big\},$$

$$A_+ \;:=\; \big\{(f\circ\varphi)(x) > 0 \;\big|\; x \in A\big\} \quad \text{and} \quad A_- \;:=\; \big\{(f\circ\varphi)(x) < 0 \;\big|\; x \in A\big\}.$$

Sets $A_\pm$ and $B_\pm$ are Lebesgue measurable, since f is Lebesgue measurable and φ is continuous. Also the relation $B_\pm = \varphi(A_\pm)$ is true with the both sign choices. According to the definition of Lebesgue integral

$$\int_B f(y)dm_N(y) \;=\; \int_{B_+} f(y)dm_N(y) \;-\; \int_{B_-} \big(-f(y)\big)dm_N(y)$$

and

$$\begin{aligned}\int_A (f\circ\varphi)(x)\big|\det(D\varphi)(x)\big|dm_N(x) \;&=\; \int_{A_+} (f\circ\varphi)(x)\big|\det(D\varphi)(x)\big|dm_N(x)\\ &\quad - \int_{A_-} (-f\circ\varphi)(x)\big|\det(D\varphi)(x)\big|dm_N(x).\end{aligned}$$

Here the integrals on right are integrals of non-negative functions. According to Theorem 11

$$\int_{B_+} f(y)dm_N(y) \;=\; \int_{A_+} (f\circ\varphi)(x)\big|\det(D\varphi)(x)\big|dm_N(x)$$

and

$$\int_{B_-} \big(-f(y)\big)dm_N(y) \;=\; \int_{A_-} (-f\circ\varphi)(x)\big|\det(D\varphi)(x)\big|dm_N(x),$$

so we see that the claim of the main theorem is true. □