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Calculation of the mathematical expectancy of Cauchy's law and extension to other improper integrals

Abstract: It is well known that the calculation of the mathematical expectancy of Cauchy's law in probability generates an indeterminate form. We show here that this indeterminacy can be lifted and the calculation leads to a fixed value. Moreover, we show that other improper integrals with an indeterminate result can be computed.

The normalized Cauchy's law probability density defined for a continuous random variable on $\mathbb{R}$ is:

$$\rho_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

Its mathematical expectancy is given by the integral:

$$E[X] = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x}{1+x^2}dx = \frac{1}{\pi}\left\{\int_{-\infty}^{0}\frac{x}{1+x^2}dx + \int_{0}^{+\infty}\frac{x}{1+x^2}dx\right\}$$

For the first integral we set: $x' = -x$ so $dx' = -dx$ and the lower bound becomes $+\infty$

$$E[X] = \frac{1}{\pi}\left\{\int_{+\infty}^{0}\frac{x'}{1+x'^2}dx' + \int_{0}^{+\infty}\frac{x}{1+x^2}dx\right\}$$

As the primitive exists, we get:

$$E[X] = \frac{1}{\pi}\left\{\left[\frac{1}{2}ln\left(1+x'^2\right)\right]_{+\infty}^{0} + \left[\frac{1}{2}ln\left(1+x^2\right)\right]_{0}^{+\infty}\right\}$$

At this stage, we can identify x' by x and:

$$E[X] = \frac{1}{2\pi}\left\{\ln(1) - \lim_{x\to\infty}\ln\left(1+x^2\right) + \lim_{x\to\infty}\ln\left(1+x^2\right) - \ln(1)\right\}$$

In this form, the calculation brings back the indeterminate value - ∞ + ∞. But the fact of having cut the integral into two parts makes it possible to use the property of the natural logarithm, so:

$$E[X]=\frac{1}{2\pi}\left\{\lim_{x\to\infty}\ \ln\left(\frac{1+x^2}{1+x^2}\right)\right\}$$

The argument of the logarithm is then a fraction which *authorizes* the analytical calculation. So, dividing numerator and denominator by x^2 we get:

$$E[X]=\frac{1}{2\pi}\left\{\lim_{x\to\infty}\ \ln\left(\frac{\left(1+x^2\right)\frac{1}{x^2}}{\left(1+x^2\right)\frac{1}{x^2}}\right)\right\}=\frac{1}{2\pi}\left\{\lim_{x\to\infty}\ \ln\left(\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+1}\right)\right\}=\frac{1}{2\pi}\ln(1)=0$$

It is then immediate to establish that the general form of Cauchy's law:

$$\rho_X(x)=\frac{a}{\pi}\frac{1}{\left(x-x_0\right)^2+a^2}$$ with $a > 0$ has its mathematical expectancy: x_0

Corollary: other improper integrals with an indeterminate result can thus be calculated if, through one or more appropriate changes of variable, a division into two equivalent parts and by showing the natural logarithm, they allow a grouping of two terms authorizing an analytical calculation as defined above.

Thus, without being exhaustive, the following functions are then integrable because they can be reduced to the Cauchy function with the indicated changes of variable.

- $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} tg(x)\,dx=0$ change of variable: $t= tg(x)$
- $\int_{-\infty}^{+\infty}\frac{1}{x}\,dx=0$ change of variable: $t=\sqrt{x-1}$
- $\int_{0^-}^{0^+}\frac{1}{x^2\left(1+x^2\right)}\,dx=0$ change of variable:: $t = 1/x$
- $\int_{-1}^{+1}\frac{x}{1-x^2}\,dx=0$ change of variable: $t = \sin(x)$, then $y= tg(t)$
- $\int_{-\infty}^{+\infty}\frac{x}{\ln\left[\frac{1}{2x^2}\left(1\pm\sqrt{1-4x^2}\right)\right]}\,dx=0$ change of variable: $t=\frac{x}{1+x^2}$

There are certainly other improper integrals whose indeterminacy is then removed.