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ARTICLE TYPE

Embedding of Octonion Fourier Transform in Geometric Algebra of $\mathbb{R}^3$ and Polar Representations of Octonion Analytic Signals in Detail[†]

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Summary

We show how the octonion Fourier transform can be embedded and studied in Clifford geometric algebra of three-dimensional Euclidean space $Cl(3,0)$. We apply a new form of dimensionally minimal embedding of octonions in geometric algebra, that expresses octonion multiplication non-associativity with a sum of up to four (individually associative) geometric algebra product terms. This approach leads to new polar representations of octonion analytic signals and signal reconstruction formulas.

KEYWORDS:
Clifford geometric algebra, octonions, Fourier transform, analytic signal, polar representation, signal reconstruction

1 | INTRODUCTION

This paper is an extension of the conference proceedings[14]. Hypercomplex Fourier transforms experienced rapid development during the last 30 years. A historical overview of this field can be found in[3], a variety of approaches is included in[8], and a recent comprehensive textbook is[10]. For an up-to-date survey of signal and image processing in Clifford geometric algebra, see Section 6 of[12]. In Definition 9 of[4] a Clifford algebra based hypercomplex Fourier transform producing a multidimensional analytic signal was defined. In the book[6] this approach is applied for the non-associative and non-commutative hypercomplex algebra of octonions. Apart from its non-associativity, octonions have many outstanding algebraic properties (e.g. the highest dimensional normed division algebra). Octonion Fourier transforms (OFT) have already found a wide range of applications (for more details see Chapters 5.6 and 9.4 of[6], and the references cited therein) to modulation theory, including the modulation of amplitude, frequency, single-sidebands, compatible single-sidebands and single-quadrant modulation, Hilbert filters and signal power analysis. Further applications are to electromagnetic fields, field theory, physics, relativistic quantum mechanics, holomorphicity, analytic signal entropy, medicine (e.g., medical image processing), noise analysis and image processing.

It is therefore of great interest for us in this work to use a recently invented minimal embedding[13,15] of octonions in the Clifford geometric algebra of three-dimensional space $Cl(3,0)$ and consequently embed the OFT in $Cl(3,0)$. This embedding allows to break down non-associative octonion multiplication into sums of associative geometric products, and therefore to easily apply existing geometric algebra computing software[1,2,18]. And it allows to establish new polar representations for octonion analytic signals, based on the polar decomposition (exponential factorization)[11,19] of geometric algebra multivectors.

We first review in Section 2 fundamental properties of octonions[16] and in Section 3 the new embedding of octonions in Clifford geometric algebra $Cl(3,0)$. Then we present in Section 4 the OFT of[6], as well as octonion analytic signals, and in Section 5 embed the OFT in $Cl(3,0)$. Finally, in Section 6 we utilize the polar decomposition of[11,19] for complex biquaternions

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[0]**Abbreviations:** GA, geometric algebra; OFT, octonion Fourier transform;

and multivectors in $Cl(3,0)$ to introduce new polar representations for octonion analytic signals and the reconstruction formulas of the original real signal. The paper concludes with Section 7, references and two appendixes on new simplified formulas for the polar decomposition of multivectors in $Cl(3,0)$ and with example computations.

2 | OCTONIONS

Here we first briefly summarize important octonion algebra properties (see [16], pp. 300–302, [13,5]), assuming $a, b, c, x, y \in \mathbb{O}$.

- Octonions $\mathbb{O}$ form an eight-dimensional bilinear algebra over the reals $\mathbb{R}$ with basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$.
- The multiplication table[1] is given by $(1 \leq i, j \leq 7)$

$$\mathbf{e}_i \star \mathbf{e}_i = -1, \quad \mathbf{e}_i \star \mathbf{e}_j = -\mathbf{e}_j \star \mathbf{e}_i \text{ for } i \neq j, \qquad \mathbf{e}_i \star \mathbf{e}_{i+1} = \mathbf{e}_{i+3}, \tag{1}$$

  where $(i, i+1, i+3)$ can be permuted cyclically and translated modulo 7.
- Via the Cayley-Dickson doubling process, octonions can directly be defined from pairs of quaternions $p_1, p_2, q_1, q_2 \in \mathbb{H}$ (note the order of factors, qc(…) is quaternion conjugation):

$$(p_1, q_1) \star (p_2, q_2) = \big(p_1 p_2 - \mathrm{qc}(q_2) q_1,\; q_2 p_1 + q_1 \mathrm{qc}(p_2)\big). \tag{2}$$

- $\mathbb{O}$ has no zero divisors, i.e., $ab = 0$ implies $a = 0$ or $b = 0$.
- $\mathbb{O}$ is a division algebra, i.e., $ax = b$ and $ya = b$ have unique solutions x, y for non-zero a.
- $\mathbb{O}$ admits unique inverses.
- $\mathbb{O}$ is non-associative, i.e., in general $a(bc) \neq (ab)c$.
- $\mathbb{O}$ is alternative, i.e., $a(ab) = a^2 b$ and $(ab)b = ab^2$.
- $\mathbb{O}$ is one of only four alternative division algebras over $\mathbb{R}$: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.
- $\mathbb{O}$ is flexible, i.e., $a(ba) = (ab)a$.
- $\mathbb{O}$ has a (positive-definite quadratic form) norm $\| \dots \| : \mathbb{O} \to \mathbb{R}$, the norm is preserved (i.e. admits composition), such that $\|ab\| = \|a\|\|b\|$.
- $\mathbb{O}$ is one of only four unital norm-preserving division algebras over $\mathbb{R}$: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.
- $\mathbb{O}$ is essential for treating *triality*, an automorphism of the universal covering spin group Spin(8) of the rotation group SO(8) or $\mathbb{R}^8$. Triality is not an inner automorphism, nor an orthogonal matrix similarity, nor a linear transformation $Cl(8,0) \to Cl(8,0)$, nor a linear automorphism of SO(8). Triality permutes three elements in the center of $Cl(8,0)$, namely $\{-1, e_{12345678}, -e_{12345678}\}$, with basis vectors e_i, $(1 \leq i \leq 8)$, of $\mathbb{R}^8$. Triality is a restriction of a polynomial mapping $Cl(8,0) \to Cl(8,0)$ of degree two.

Furthermore, like for complex numbers, quaternions and biquaternions, there is a *polar decomposition* for octonions[19].

3 | EMBEDDING OF OCTONIONS IN CLIFFORD GEOMETRIC ALGEBRA OF THREE-DIMENSIONAL EUCLIDEAN SPACE

For readers not familiar with Clifford geometric algebra we refer to the excellent textbook[16], and to the tutorial introduction[7]. The current section summarizes the results needed from[13].

The Clifford geometric algebra $Cl(3,0)$ of Euclidean space $\mathbb{R}^3$ has eight basis elements

$$\{1, \sigma_1, \sigma_2, \sigma_3, I\sigma_1 = \sigma_{23}, I\sigma_2 = \sigma_{31}, I\sigma_3 = \sigma_{12}, I = \sigma_{123}\}, \tag{3}$$

[1] This depends obviously on deliberate ordering and sign choices for the basis elements.

Table 1 GA $Cl(3,0)$ multiplication table, $Cl(3,0) \cong$ Pauli algebra.

Left factors	Right factors: 1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
1	1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
$I\sigma_1$	$I\sigma_1$	-1	$-I\sigma_3$	$I\sigma_2$	I	$-\sigma_3$	σ_2	$-\sigma_1$
$I\sigma_2$	$I\sigma_2$	$I\sigma_3$	-1	$-I\sigma_1$	σ_3	I	$-\sigma_1$	$-\sigma_2$
$I\sigma_3$	$I\sigma_3$	$-I\sigma_2$	$I\sigma_1$	-1	$-\sigma_2$	σ_1	I	$-\sigma_3$
σ_1	σ_1	I	$-\sigma_3$	σ_2	1	$I\sigma_3$	$-I\sigma_2$	$I\sigma_1$
σ_2	σ_2	σ_3	I	$-\sigma_1$	$-I\sigma_3$	1	$I\sigma_1$	$I\sigma_2$
σ_3	σ_3	$-\sigma_2$	σ_1	I	$I\sigma_2$	$-I\sigma_1$	1	$I\sigma_3$
I	I	$-\sigma_1$	$-\sigma_2$	$-\sigma_3$	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	-1

where $\{\sigma_1, \sigma_2, \sigma_3\}$ forms an orthonormal vector basis of $\mathbb{R}^3$. Its multiplication table is given in Table 1. The eight components of a general multivector $M \in Cl(3,0)$ can be grouped by grade into the scalar part $\langle M \rangle = \langle M \rangle_0$, the three-dimensional vector part $\langle M \rangle_1 \in \mathbb{R}^3$ (where usually $\mathbb{R}^3$ is identified with the grade one vector subspace $Cl_1(3,0)$), the three-dimensional bivector part $\langle M \rangle_2 \in Cl_2(3,0)$ spanned by $\{\sigma_{23}, \sigma_{31}, \sigma_{12}\}$, and the trivector (pseudoscalar) part $\langle M \rangle_3$.

We can construct in $Cl(3,0)$ an octonionic product [13], after splitting it in its even subalgebra $Cl^+(3,0)$ with basis

$$\{1, \sigma_{23}, \sigma_{31}, \sigma_{12}\}, \tag{4}$$

and the set $Cl^-(3,0)$ of odd grade (w.r.t. grades in $Cl(3,0)$) elements

$$\{\sigma_1, \sigma_2, \sigma_3, I = \sigma_{123}\}. \tag{5}$$

We will use the Clifford conjugation[2] (indicated by an overbar $\overline{M}$), i.e. the composition of (main) grade involution[3] ($\widehat{M}$) and reversion[4] ($\widetilde{M}$). Clifford conjugation preserves grades zero and three, but changes the signs of grades one and two in $Cl(3,0)$. A realization of the octonionic product of M, N in $Cl(3,0)$ is given by four (individually associative) geometric algebra product terms

$$\begin{aligned} M &= M_+ + M_-, \quad N = N_+ + N_-, \\ M \star N &= M_+N_+ + \overline{N_-}M_- + N_-M_+ + M_-\overline{N_+}, \end{aligned} \tag{6}$$

with even grade parts $M_+, N_+ \in Cl^+(3,0)$ and odd grade parts $M_-, N_- \in Cl^-(3,0)$. The multiplication table is Table 2, with octonionic product illustration in Fano plane diagram form in Fig. 1.

The octonion conjugate (anti-involution) in $Cl(3,0)$ is given by

$$M^\star = \widetilde{M}_+ - M_- = \overline{M}_+ - M_-, \quad (M \star N)^\star = N^\star \star M^\star. \tag{7}$$

Computing the octonion norm yields (including norm-preservation):

$$\|M\| = M \star M^\star = \langle M\widetilde{M} \rangle = M * \widetilde{M} = \sum_{i=1}^{8} M_i^2, \qquad \|M \star N\| = \|M\|\|N\|. \tag{8}$$

where $M_i \in \mathbb{R}, 1 \le i \le 8$, are the coefficients of M in the $Cl(3,0)$ basis (3), and $A * B = \langle AB \rangle$ means to compute the scalar product of $A, B \in Cl(3,0)$, i.e. the scalar part of the geometric product.

The above reviewed (dimensionally) minimal embedding is very flexible. It even allows to reversely embed Clifford geometric algebra $Cl(3,0)$ in octonions by defining the geometric product in terms of the octonionic product[5] (see [13], Section 3.3 for

[2] Note that by construction $\overline{M_\pm} = (\overline{M})_\pm$.
[3] Grade involution $\widehat{M}$ changes the sign of all odd grade parts, i.e., of grades one and three.
[4] Reversion $\widetilde{M}$ reverts the order of all products and thus changes the sign of grades two and three.
[5] Note that the factors $\mathbf{e}_7$ (later identified with the pseudoscalar $I \in Cl(3,0)$) in the last equation of (9) are essential.

Table 2 Multiplication table for octonion embedding in $Cl(3,0)$. The upper left 4×4-block corresponds to M_+N_+, the upper right 4×4-block to N_-M_+, the lower left 4×4-block to $M_-\overline{N}_+$, and the lower right 4×4-block to $\overline{M}_-N_-$ of (6).

Left factors \ Right factors	1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
1	1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
$I\sigma_1$	$I\sigma_1$	-1	$-I\sigma_3$	$I\sigma_2$	I	σ_3	$-\sigma_2$	$-\sigma_1$
$I\sigma_2$	$I\sigma_2$	$I\sigma_3$	-1	$-I\sigma_1$	$-\sigma_3$	I	σ_1	$-\sigma_2$
$I\sigma_3$	$I\sigma_3$	$-I\sigma_2$	$I\sigma_1$	-1	σ_2	$-\sigma_1$	I	$-\sigma_3$
σ_1	σ_1	$-I$	σ_3	$-\sigma_2$	-1	$I\sigma_3$	$-I\sigma_2$	$I\sigma_1$
σ_2	σ_2	$-\sigma_3$	$-I$	σ_1	$-I\sigma_3$	-1	$I\sigma_1$	$I\sigma_2$
σ_3	σ_3	σ_2	$-\sigma_1$	$-I$	$I\sigma_2$	$-I\sigma_1$	-1	$I\sigma_3$
I	I	σ_1	σ_2	σ_3	$-I\sigma_1$	$-I\sigma_2$	$-I\sigma_3$	-1

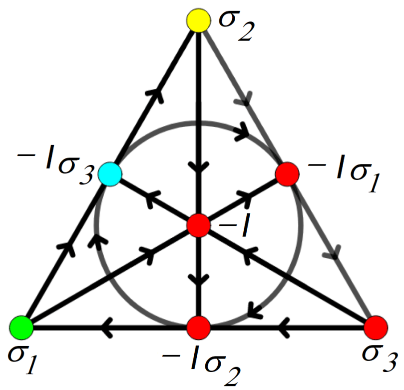

Figure 1 Illustration of $Cl(3,0)$ basis elements under the octonionic product (6) in Table 2, see [13] for details.

details):

$$M_+N_+ \overset{(6)}{=} M_+ \star N_+, \quad M_-N_- \overset{(6)}{=} N_- \star \overline{M}_-,$$
$$M_-N_+ \overset{(6)}{=} N_+ \star M_-, \quad M_+N_- = -(N_- \star \mathbf{e}_7) \star (M_+ \star \mathbf{e}_7), \tag{9}$$

with $M_+ \in \text{span}[1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ and $M_- \in \text{span}[\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7]$. Note that on the right side of the second equality in (9), the conjugation operation $\overline{M}_-$ is defined on the relevant octonion components as $\overline{\mathbf{e}_4} = -\mathbf{e}_4$, $\overline{\mathbf{e}_5} = -\mathbf{e}_5$, $\overline{\mathbf{e}_6} = -\mathbf{e}_6$, and $\overline{\mathbf{e}_7} = \mathbf{e}_7$.

4 | OCTONION FOURIER TRANSFORM

From now on, if no brackets are given, the order of multiplication is assumed to be from left to right, e.g., $ABC = (AB)C$, etc. According to Section 4.2.1 of[6], the OFT of an integrable three-dimensional[6] real signal $f \in L^1(\mathbb{R}^3, \mathbb{R})$ can be defined as

$$\mathcal{F}\{f\}(\mathbf{u}) = \int_{\mathbb{R}^3} f(\mathbf{x}) e^{-\mathbf{e}_1 2\pi u_1 x_1} e^{-\mathbf{e}_2 2\pi u_2 x_2} e^{-\mathbf{e}_4 2\pi u_3 x_3} d^3x, \tag{10}$$

with three-dimensional position vectors and frequency vectors, and volume element

$$\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3, \quad d^3x = dx_1 dx_2 dx_3, \tag{11}$$

[6]These signals can, e.g., be temperature data in a space volume, a density distribution, local chemical concentrations, pressure data, etc.

respectively, and octonion units $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$ in the exponents. As pointed out in[6], any triplet of octonion units could be used in the octonionic kernel of (10), as long as the three do not form a quaternionic subalgebra, by that reason, e.g., the triplet $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is excluded, compare the multiplication table Table 2.3 and its Fano plane visualization Fig. 2.2 in[6]. In the latter the triplet $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ clearly lies on a straight line.

Given suitable integrability conditions, the inverse OFT can be computed as

$$f(\mathbf{x}) = \mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(\mathbf{x}) = \int_{\mathbb{R}^3} \mathcal{F}\{f\}(\mathbf{u}) e^{\mathbf{e}_4 2\pi u_3 x_3} e^{\mathbf{e}_2 2\pi u_2 x_2} e^{\mathbf{e}_1 2\pi u_1 x_1} d^3u, \qquad d^3u = du_1 du_2 du_3. \tag{12}$$

Abbreviating $s_k = \sin(2\pi u_k x_k)$, $c_k = \cos(2\pi u_k x_k)$, $k = 1, 2, 3$, we can express the kernel of (10), using multiplication table Table 2.3 of[6], as

$$\begin{aligned} & e^{-\mathbf{e}_1 2\pi u_1 x_1} e^{-\mathbf{e}_2 2\pi u_2 x_2} e^{-\mathbf{e}_4 2\pi u_3 x_3} = (c_1 - s_1\mathbf{e}_1)(c_2 - s_2\mathbf{e}_2)(c_3 - s_3\mathbf{e}_4) \\ & \quad = c_1c_2c_3 - s_1c_2c_3\mathbf{e}_1 - c_1s_2c_3\mathbf{e}_2 - c_1c_2s_3\mathbf{e}_4 + s_1s_2c_3\mathbf{e}_3 + s_1c_2s_3\mathbf{e}_5 + c_1s_2s_3\mathbf{e}_6 - s_1s_2s_3\mathbf{e}_7. \end{aligned} \tag{13}$$

The significance of this decomposition is, that therefore a real signal $f \in L^1(\mathbb{R}^3, \mathbb{R})$ is decomposed by the OFT (10) into eight spectral components of distinct even-odd symmetries: {eee,oee,eoe,eeo,ooe,oeo,eoo,ooo}, where e=even, o=odd. Following the multiplication table Table 2.3 of[6], and using the alternative octonion multiplication property of Section 2, we find the following conjugations $(i, j = 2, \dots, 7)$

$$\alpha_i(\mathbf{e}_j) = \mathbf{e}_i\mathbf{e}_j\mathbf{e}_i = \begin{cases} \mathbf{e}_j, & i \neq j \\ -\mathbf{e}_j, & i = j \end{cases}. \tag{14}$$

This allows to express all $\mathcal{F}\{f\}(\pm u_1, \pm u_2, \pm u_3)$ in terms of $\mathcal{F}\{f\}(\mathbf{u})$ each time using four suitable α_i conjugations. For example,

$$\mathcal{F}\{f\}(-u_1, u_2, u_3) = \alpha_1(\alpha_3(\alpha_5(\alpha_7(\mathcal{F}\{f\}(\mathbf{u}))))). \tag{15}$$

As a consequence the OFT in all eight octants of the three-dimensional frequency space can be obtained from the OFT only applied to the first octant, where all three frequency components are positive (i.e. $\{u_1 \geq 0, u_2 \geq 0, u_3 \geq 0\}$).

4.1 | Hypercomplex Analytic Signal

A real signal $f \in L^1(\mathbb{R}, \mathbb{R})$ can be extended to a complex analytic signal with *positive* frequency by multiplying its Fourier transform $\mathcal{F}_{\mathbb{R}}\{f\}(u)$ with $(1 + \operatorname{sgn} u)$, $u \in \mathbb{R}$ being the frequency, and back transforming

$$\psi(x) = \mathcal{F}_{\mathbb{R}}^{-1}\big\{(1 + \operatorname{sgn} u)\mathcal{F}_{\mathbb{R}}\{f\}(u)\big\}(x), \tag{16}$$

equivalent to direct application of the Hilbert transform, where $\circledast$ means convolution,

$$H[f(x)] = (\frac{1}{\pi x}) \circledast f(x), \qquad \psi(x) = f(x) + iH[f(x)] = [\delta(x) + i\frac{1}{\pi x}] \circledast f(x). \tag{17}$$

We can recover the original signal as the real part of $\psi(x)$, i.e.,

$$f(x) = \frac{1}{2}\big(\psi(x) + \operatorname{cc}(\psi(x))\big), \tag{18}$$

where cc(…) refers to complex conjugation.

Analogously, we can construct for real three-dimensional signals $f \in L^1(\mathbb{R}^3, \mathbb{R})$ an analytic hypercomplex signal with triple convolution by (see Section 5.2.3 of[6] for details)

$$\begin{aligned} \psi(x_1, x_2, x_3)_1 &= [\delta(x_1) + \mathbf{e}_1\frac{1}{\pi x_1}] \times [\delta(x_2) + \mathbf{e}_2\frac{1}{\pi x_2}] \times [\delta(x_3) + \mathbf{e}_4\frac{1}{\pi x_3}] \circledast \circledast \circledast f(x_1, x_2, x_3) \\ &= f + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_{12}\mathbf{e}_3 + v_3\mathbf{e}_4 + v_{13}\mathbf{e}_5 + v_{23}\mathbf{e}_6 + v\mathbf{e}_7, \end{aligned} \tag{19}$$

which has only three-dimensional frequency values $\mathbf{u} = (u_1, u_2, u_3)$ in the first octant of frequency space, where all three frequency components are positive $(+++)$. The original signal $f \in L^1(\mathbb{R}^3, \mathbb{R})$ is the scalar real component of $\psi(x_1, x_2, x_3)$. The corresponding analytic signals $\psi(x_1, x_2, x_3)_k$, $k = 2, \dots, 8$ in the other seven octants are obtained by simply changing the three plus signs in the first line of (19) to $(-++), (+-+), (--+), (++-), (-+-), (+--), (---)$, respectively. And we can recover the original signal simply by

$$f(x_1, x_2, x_3) = \frac{1}{8}\sum_{k=1}^{8} \psi(x_1, x_2, x_3)_k. \tag{20}$$

Instead of computing $\psi(x_1, x_2, x_3)_k$, $k = 2, \dots, 8$, one by one, we can obviously also obtain them from $\psi(x_1, x_2, x_3)_1$ by applying to it compositions of octonionic conjugations (14) as, e.g., in (15). We note that[6], p. 167, states for $\psi(x_1, x_2, x_3)_1$ of (19): *The exact polar representation of this signal is unknown.*

This outline of the OFT and its corresponding analytic first octant frequency spectrum signal may suffice here to be able to somewhat appreciate its uniquely interesting properties, due to its octonionic kernel. For more details we refer to[6]. Polar reconstruction will be discussed in Section 6.

5 | EMBEDDING THE OFT IN CLIFFORD GEOMETRIC ALGEBRA OF THREE-DIMENSIONAL EUCLIDEAN SPACE

Now we reach the main purpose of this work to extend the embedding of octonions in Clifford geometric algebra $Cl(3,0)$ of Section 3 to a full embedding of the OFT. An essential first step is the question on how to identify the three unit octonions $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_4$ with corresponding non-scalar basis elements of $Cl(3,0)$. In[6], page 70, when defining the OFT, it is emphasized that the choice of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_4$, for constructing the transformation kernel is not unique, but other triplets suggested always include $\mathbf{e}_2$, located at the center of the Fano diagram Fig. 2.2 in[6]. Comparing this situation with our Fano diagram Fig. 1, we conveniently choose the three basis blades $\sigma_1, -I, -I\sigma_3$. We observe that $\sigma_1, -I \in Cl^-(3,0)$ are both odd-, and $-I\sigma_3 \in Cl^+(3,0)$ is even graded, respectively.

We therefore define the *embedding* in the geometric algebra $Cl(3,0)$ of the OFT of a real signal $f \in L^1(\mathbb{R}^3, \mathbb{R})$ as

$$\mathcal{F}\{f\}(\mathbf{u}) = \int_{\mathbb{R}^3} f(\mathbf{x})\, e^{-\sigma_1 2\pi u_1 x_1} \star e^{I 2\pi u_2 x_2} \star e^{I\sigma_3 2\pi u_3 x_3} d^3x. \tag{21}$$

The kernel of the embedded OFT (21) can be expressed in geometric algebra, using multiplication table Table 2, as

$$\begin{aligned}
K(x_1, x_2, x_3) &= [e^{-\sigma_1 2\pi u_1 x_1} \star e^{I2\pi u_2 x_2}] \star e^{I\sigma_3 2\pi u_3 x_3} \\
&= [(c_1 - \sigma_1 s_1) \star (c_2 + I s_2)] \star (c_3 + I\sigma_3 s_3) \\
&= c_1c_2c_3 - s_1c_2c_3\sigma_1 + c_1s_2c_3 I + c_1c_2s_3 I\sigma_3 - s_1s_2c_3\sigma_1 \star I - s_1c_2s_3\sigma_1 \star (I\sigma_3) \\
&\quad + c_1s_2s_3 I \star (I\sigma_3) - s_1s_2s_3[\sigma_1 \star I] \star (I\sigma_3) \\
&= c_1c_2c_3 - s_1c_2c_3\sigma_1 + c_1s_2c_3 I + c_1c_2s_3 I\sigma_3 - s_1s_2c_3 I\sigma_1 + s_1c_2s_3\sigma_2 + c_1s_2s_3\sigma_3 - s_1s_2s_3 I\sigma_2 \\
&= c_1c_3(c_2 + s_2 I) - s_1c_3(c_2 + s_2 I)\sigma_1 + s_1s_3(c_2 - s_2 I)\sigma_2 + c_1s_3(c_2 - s_2 I) I\sigma_3 \\
&= c_3(c_1 - s_1\sigma_1)(c_2 + s_2 I) + s_3(s_1\sigma_2 + c_1\sigma_1\sigma_2)(c_2 - s_2 I) \\
&= c_3(c_1 - s_1\sigma_1)(c_2 + s_2 I) + s_3\sigma_1\sigma_2(c_1 - s_1\sigma_1)(c_2 - s_2 I) \\
&= [c_3(c_2 + s_2 I) + s_3\sigma_1\sigma_2(c_2 - s_2 I)](c_1 - s_1\sigma_1) \\
&= [c_3 e^{I2\pi u_2 x_2} + s_3 I\sigma_3 e^{-I2\pi u_2 x_2}](c_1 - s_1\sigma_1)
\end{aligned} \tag{22}$$

Now we observe that to change the sign of any of the three frequency components in the result, GA has seven very simple involutions

$$\begin{aligned}
K(-u_1, u_2, u_3) &= \sigma_3 K(u_1, u_2, u_3)\sigma_3, \quad & K(u_1, -u_2, u_3) &= \sigma_3 \widehat{K}(u_1, u_2, u_3)\sigma_3, \\
K(u_1, u_2, -u_3) &= \sigma_1 K(u_1, u_2, u_3)\sigma_1, \quad & K(-u_1, -u_2, u_3) &= \widehat{K}(u_1, u_2, u_3), \\
K(-u_1, u_2, -u_3) &= \sigma_2 K(u_1, u_2, u_3)\sigma_2, \quad & K(u_1, -u_2, -u_3) &= \sigma_2 \widehat{K}(u_1, u_2, u_3)\sigma_2, \\
K(-u_1, -u_2, -u_3) &= \sigma_1 \widehat{K}(u_1, u_2, u_3)\sigma_1, & &
\end{aligned} \tag{23}$$

Note that the frequency sign change only operating in octonion algebra always requires a composition of *four* conjugations (as e.g. in (15)). For later use, we tabulate the action of these involutions on all basis elements of $Cl(3,0)$ in Table 3. Note that each involution reproduces the respective basis element up to a sign factor listed in the table, e.g., $\sigma_3\sigma_1\sigma_3 = -\sigma_1$, $\sigma_1\widehat{\sigma_2}\sigma_1 = +\sigma_2$, etc.

5.1 | Embedding of Octonion Analytic Signal in Geometric Algebra $Cl(3,0)$

We now ask how the octonion analytic signal, defined in (19), can be embedded in the geometric algebra $Cl(3,0)$ of three-dimensional Euclidean space $\mathbb{R}^3$? Similar to our study of the kernel of the embedding of the OFT, we therefore need to apply

Table 3 Action (sign changes) of all involutions in (23) on all basis elements A of $Cl(3,0)$.

Basis blade A	Involution							
	identity	$\widehat{A}$	$\sigma_3 A\sigma_3$	$\sigma_3\widehat{A}\sigma_3$	$\sigma_1 A\sigma_1$	$\sigma_2 A\sigma_2$	$\sigma_2\widehat{A}\sigma_2$	$\sigma_1\widehat{A}\sigma_1$
1	+	+	+	+	+	+	+	+
σ_1	+	−	−	+	+	−	+	−
σ_2	+	−	−	+	−	+	−	+
σ_3	+	−	+	−	−	−	+	+
$I\sigma_1$	+	+	−	−	+	−	−	+
$I\sigma_2$	+	+	−	−	−	+	+	−
$I\sigma_3$	+	+	+	+	−	−	−	−
I	+	−	+	−	+	+	−	−

the embedding of octonion multiplication in geometric algebra to the convolution factor product that appears in the definition of the octonion analytic signal in the first line of (19). We again replace $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_4$, by the three $Cl(3,0)$ basis blades σ_1, $-I$, and $-I\sigma_3$, respectively, and obtain[7]

$$\begin{aligned}&\left\{[\delta(x_1)+\sigma_1\frac{1}{\pi x_1}]\star[\delta(x_2)-I\frac{1}{\pi x_2}]\right\}\star[\delta(x_3)-I\sigma_3\frac{1}{\pi x_3}]\\&=\left[\delta(x_3)\big(\delta(x_2)-I\frac{1}{\pi x_2}\big)-I\sigma_3\frac{1}{\pi x_3}\big(\delta(x_2)+I\frac{1}{\pi x_2}\big)\right]\big(\delta(x_1)+\sigma_1\frac{1}{\pi x_1}\big).\end{aligned}\tag{24}$$

The following threefold convolution, carried out algebraically in the geometric algebra $Cl(3,0)$, will therefore give the embedding of the octonion analytic signal of (19) in $Cl(3,0)$

$$\begin{aligned}\psi(x_1,x_2,x_3)_1&=\left[\delta(x_3)\big(\delta(x_2)-I\frac{1}{\pi x_2}\big)-I\sigma_3\frac{1}{\pi x_3}\big(\delta(x_2)+I\frac{1}{\pi x_2}\big)\right]\big(\delta(x_1)+\sigma_1\frac{1}{\pi x_1}\big)\circledast\circledast\circledast f(x_1,x_2,x_3)\\&=f+v_1\sigma_1-v_2I-v_3I\sigma_3-v_{12}I\sigma_1+v_{13}\sigma_2+v_{23}\sigma_3+vI\sigma_2.\end{aligned}\tag{25}$$

Furthermore, the seven simple GA involutions of (23) will also analogously yield the embedded version of the octonion analytic signal for the other seven octants, which corresponds to changing one, two or all three signs of σ_1, $-I$, and $-I\sigma_3$, in (25):

$$\begin{aligned}&\psi(x_1,x_2,x_3)_2=\sigma_3\psi(x_1,x_2,x_3)_1\sigma_3,\quad \psi(x_1,x_2,x_3)_3=\sigma_3\widehat{\psi}(x_1,x_2,x_3)_1\sigma_3,\\&\psi(x_1,x_2,x_3)_4=\widehat{\psi}(x_1,x_2,x_3)_1,\quad \psi(x_1,x_2,x_3)_5=\sigma_1\psi(x_1,x_2,x_3)_1\sigma_1,\\&\psi(x_1,x_2,x_3)_6=\sigma_2\psi(x_1,x_2,x_3)_1\sigma_2,\quad \psi(x_1,x_2,x_3)_7=\sigma_2\widehat{\psi}(x_1,x_2,x_3)_1\sigma_2,\\&\psi(x_1,x_2,x_3)_8=\sigma_1\widehat{\psi}(x_1,x_2,x_3)_1\sigma_1,\end{aligned}\tag{26}$$

where in number ordering of the octants we simply follow Fig. 4.10 and Table 5.4 of[6]. The original scalar signal can always be reconstructed from the eight octant specific signals of (25) and (26), and therefore from the purely positive frequency (in the first octant of the three-dimensional frequency space) signal $\psi(x_1,x_2,x_3)_1$, as

$$f(x_1,x_2,x_3)=\frac{1}{8}\sum_{k=1}^{8}\psi(x)_k,\tag{27}$$

which is the consequent octant generalization of the reconstruction (18) of a real one-dimensional signal from its complex analytic signal. The single complex conjugation in (18) is replaced by the seven geometric algebra involutions of (26). With the help of Table 3 that has eight positive signs in the first row of scalars 1, and precisely four positive and four negative signs[8] in each of the other seven rows, it is obvious that the sum of the eight involutions (including the identity) in (27) will give eight times the scalar part of $\psi(x_1,x_2,x_3)_1$ and zero for all the non-scalar parts.

[7]Note the close algebraic analogy to the computation in (22), associating c_k and $\delta(x_k)$, as well as s_k and $-1/(\pi x_k)$, for $k=1,2,3$.
[8]Corresponding to a set of four elementary octonion conjugations (14).

6 | POLAR REPRESENTATION OF EMBEDDED OCTONION ANALYTIC SIGNAL

First we review in Section 6.1 two proposals in[6] for polar representations of octonion analytic signals. Then in Section 6.2 we look at new candidates for polar representations of octonion analytic signals, after embedding them in the Clifford geometric algebra $Cl(3,0)$.

6.1 | Previous Candidates for Polar Representations of Octonion Analytic Signals

For octonion signals with spectrum in the first octant (19), Hahn and Snopek first propose in Section 7.5.2 of[6] a polar form with one amplitude function $A_0(x_1,x_2,x_3)$ (the octonion norm of (19)) and seven phase angle functions[9] $\Phi_k(x_1,x_2,x_3)$, $1 \leq k \leq 7$

$$\psi_1^{HS}(x_1,x_2,x_3) = A_0 e^{\mathbf{e}_1\Phi_1} e^{\mathbf{e}_2\Phi_2} e^{\mathbf{e}_3\Phi_3} e^{\mathbf{e}_7\Phi_7} e^{\mathbf{e}_4\Phi_4} e^{\mathbf{e}_5\Phi_5} e^{\mathbf{e}_6\Phi_6},$$
$$A_0 = \sqrt{f^2 + v_1^2 + v_2^2 + v_3^2 + v_{12}^2 + v_{13}^2 + v_{23}^2 + v^2}, \tag{28}$$

where we have omitted for brevity the arguments (x_1,x_2,x_3) of all seven phase angles Φ_k and all functions $f, v_1, v_2, v_3, v_{12}, v_{13}, v_{23}, v$. After defining

$$c_k = \cos\Phi_k(x_1,x_2,x_3), \quad s_k = \sin\Phi_k(x_1,x_2,x_3), \quad 1 \leq k \leq 7, \tag{29}$$

Hahn and Snopek provide the 16 term *reconstruction formula* for the scalar real signal as

$$\begin{aligned} f_{rec}(x_1,x_2,x_3) = A_0[&c_1c_2c_3c_4c_5c_6c_7 + s_1s_2s_3c_4c_5c_6c_7 - s_1c_2c_3s_4s_5c_6c_7 + c_1s_2s_3s_4s_5c_6c_7 \\ &- s_1s_2c_3s_4c_5s_6c_7 + s_1c_2s_3c_4s_5s_6c_7 - c_1c_2s_3c_4s_5s_6c_7 - s_1s_2c_3c_4s_5s_6c_7 \\ &+ c_1c_2s_3s_4c_5c_6s_7 + s_1s_2c_3s_4c_5c_6s_7 + c_1s_2c_3c_4s_5c_6s_7 - s_1c_2s_3c_4s_5c_6s_7 \\ &- s_1c_2c_3c_4c_5s_6s_7 + c_1s_2s_3c_4c_5s_6s_7 - c_1c_2c_3s_4s_5s_6s_7 - s_1s_2s_3s_4s_5s_6s_7]. \end{aligned} \tag{30}$$

On the other hand, for the simpler case of three-dimensional separable real signals

$$f'(x_1,x_2,x_3) = g_1(x_1)g_2(x_2)g_3(x_3), \quad g_k \in L^1(\mathbb{R}^1,\mathbb{R}^1), \quad k = 1,2,3, \tag{31}$$

the proposed polar representation and its reconstruction (see Section 7.5.2.1 of[6]) look much easier

$$\psi_1^{HS'}(x_1,x_2,x_3) = A_0' e^{\mathbf{e}_1\Phi_1} e^{\mathbf{e}_2\Phi_2} e^{\mathbf{e}_4\Phi_4}, \quad f'_{rec} = A_0' \cos\Phi_1 \cos\Phi_2 \cos\Phi_4, \tag{32}$$

again omitting for brevity the arguments (x_1,x_2,x_3) of A_0', Φ_1, Φ_2 and Φ_3.

6.2 | New Polar Representations of Embedded Octonion Analytic Signals

6.2.1 | Polar Representation Based on Polar Decomposition in $Cl(3,0)$

As shown in[19], Theorem 1, there exists an elegant and very compact polar decomposition for complex biquaternions. Due to the isomorphism between complex biquaternions and the Clifford algebra $Cl(3,0)$, this can be carried over to multivectors in $Cl(3,0)$ as well, see[11], Section 4.3, equation (49). In the following we will first summarize the polar decomposition of $Cl(3,0)$ multivectors provided in[19,11], then provide a set of direct (computationally) simplified formulas for its computation, followed by an explicit example. The simplified formulas are derived in appendix A, and the example is fully computed in appendix B.

A summary of the polar decomposition of $Cl(3,0)$ multivectors in[19,11] can be given as follows. As for notation, all unit vectors u (two degrees of freedom (DOF)), all unit bivectors i_2 (two DOF), and the central unit pseudoscalar $I = \sigma_{123}$ in $Cl(3,0)$ square to

$$u^2 = +1, \qquad i_2^2 = -1, \qquad I^2 = -1. \tag{33}$$

The even subalgebra of $Cl(3,0)$ is *isomorphic to quaternions* $\mathbb{H}$: $Cl^+(3,0) \cong \mathbb{H}$. That means general multivectors M in $Cl(3,0)$ can always be represented as complex ($I^2 = -1$) (bi)quaternions:

$$M = M_+ + M_- = p + Iq, \tag{34}$$

[9] Hahn and Snopek[6], p. 168, state that the factor $e^{\mathbf{e}_7\Phi_7}$ is placed arbitrarily at the center.

where p and q are (isomorphic to) quaternions

$$p = M_+ = a_p e^{\alpha_p i_p}, \quad q = I^{-1} M_- = a_q e^{\alpha_q i_q}, \quad a_p, a_q \in \mathbb{R}_0^+, \quad i_p^2 = i_q^2 = -1, \tag{35}$$

with unit bivectors $i_p, i_q \in Cl_2(3,0)$.

The polar decomposition of $M \in Cl(3,0)$ is

$$M = p + Iq = \begin{cases} e^{\alpha_0} e^{\alpha_2 i_2} & \text{for} \quad q = 0, \\ I e^{\alpha_0} e^{\alpha_2 i_2} & \text{for} \quad p = 0, \\ e^{\alpha_0} e^{\alpha_2 i_2} \frac{1+I\mathbf{f}}{2} & \text{for} \quad q = p\mathbf{f}, \\ e^{\alpha_0} e^{\alpha_1 u'} e^{\alpha_2 i_2} e^{\alpha_3 I} & \text{otherwise.} \end{cases} \tag{36}$$

where in line three (compare (26) in[11]) we have the special case that the quotient $p^{-1}q$ results in a unit bivector $\mathbf{f} = p^{-1}q$. The value of $i_2 = i_p$ in lines one (compare (19) in[11]) and three, $i_2 = i_P$ in line four, while in line two we have $i_2 = i_q$. We note that line one is a special case of line four for $\alpha_1 = \alpha_3 = 0$. Line two (compare (19) in[11]) is a special case of line four for $\alpha_1 = 0$ and $\alpha_3 = \pi/2$. So essentially only lines three and four of (36) matter, and we have one special (line three) case with idempotent factor $(\frac{1+I\mathbf{f}}{2})$, signaling that M is not invertible, and one general case (line four: see Section 4.2 of[11] for all computational details) with full exponential factorization. The latter has the necessary eight DOF: four DOF are given by the phase angles α_k, $k = 0,1,2,3$, two DOF by unit vector u' and two by unit bivector i_2.

Here we present a *computationally simplified* set of formulas for computing the polar decomposition of a general multivector $M \in Cl(3,0)$. First we compute the central number $M\overline{M} \in \mathbb{R} \oplus I\mathbb{R}$, i.e. a scalar plus a pseudoscalar (algebraically like a complex number).

For $M\overline{M} = 0$ we have the special case of M being a *divisor of zero*, i.e., not invertible[10]. Then we can directly compute the entities of line 3 of (36) as

$$\alpha_0 = \ln 2 + \frac{1}{2}\ln(M_+\overline{M_+}), \quad \alpha_2 = \operatorname{atan2}(|\langle M\rangle_2|, \langle M\rangle_0), \quad i_2 = \frac{\langle M\rangle_2}{|\langle M\rangle_2|} \text{ for } \langle M\rangle_2 \neq 0, \quad \mathbf{f} = I^{-1}M_+^{-1}M_- = (M_+^{-1}M_-)^*, \tag{37}$$

where the upper star index of A^* applied to a multivector $A \in Cl(3,0)$ means geometric algebra *duality*, i.e., $A^* = AI^{-1} = -AI$. And we note that in this case we have

$$M_+ = 2e^{\alpha_0} e^{\alpha_2 i_2}, \qquad M_- = IM_+\mathbf{f}. \tag{38}$$

The above formulas also apply in the case of $\langle M\rangle_2 = 0$, i.e., $M_+ = \langle M\rangle_0$. Then $e^{\alpha_2 i_2}$ degenerates to ± 1, and it is simpler to express

$$M = 2\langle M\rangle_0 \frac{1+I\mathbf{f}}{2} = e^{\alpha_0}\operatorname{sgn}\langle M\rangle_0 \frac{1+I\mathbf{f}}{2}, \qquad \alpha_0 = \ln 2 + \ln|\langle M\rangle_0|, \qquad \mathbf{f} = I^{-1}\frac{M_-}{\langle M\rangle_0} = \frac{(M_-)^*}{\langle M\rangle_0}, \tag{39}$$

For $M\overline{M} \neq 0$, i.e., when M is not a divisor of zero (and thus invertible) we get with the normed multivector (compare (A11))

$$N = \frac{M}{\sqrt{M\overline{M}}} = N_+ + N_-, \qquad N_\pm = \Big(\frac{M}{\sqrt{M\overline{M}}}\Big)_\pm, \tag{40}$$

the general *simplified* decomposition formulas for invertible multivectors

$$\begin{aligned} \alpha_0 &= \frac{1}{4}\ln\big(\det(M)\big), \\ \alpha_1 &= \operatorname{atanh}\Big(-\frac{N_-\overline{N_-}}{N_+\overline{N_+}}\Big)^{\frac{1}{2}}, \qquad u' = \frac{N_-\overline{N_+}}{|N_-\overline{N_+}|}, \\ \alpha_2 &= \operatorname{atan2}(|\langle N\rangle_2|, \langle N\rangle_0), \qquad i_2 = \frac{\langle N\rangle_2}{|\langle N\rangle_2|} \text{ for } \langle N\rangle_2 \neq 0, \\ \alpha_3 &= \frac{1}{2}\operatorname{atan2}\big((\langle M\overline{M}\rangle_3)^*, \langle M\overline{M}\rangle_0\big). \end{aligned} \tag{41}$$

We note that for $\langle N\rangle_2 = 0$, i.e., $N_+ = \langle N\rangle_0$, the factor $e^{\alpha_2 i_2}$ degenerates to $\pm 1 = \operatorname{sgn}\langle N\rangle_0$, and thus α_2 and i_2 need not to be computed. The derivation of (37) to (41), based on the results of[19,11] can be found in appendix A.

[10] $M\overline{M}$ is a factor in the determinant of M, see (B22): $\det(M) = M\overline{M}\widetilde{M\overline{M}}$, showing that $\det(M) = 0 \Leftrightarrow M\overline{M} = 0$. The determinant is the same when computed in a matrix representation of $Cl(3,0)$.

To better understand how to compute the generic case decomposition of line four of (36), we present the following numerical example (see all computational details in Appendix B).

Example 6.1.

$$M = 1 + 2\sigma_1 + 3\sigma_2 + 4I\sigma_1 + 5I\sigma_3 + 6I = e^{1.0436}\, e^{1.5574\,u'}\, e^{0.66405\,i_2}\, e^{1.8304\,I},$$
$$u' = 0.9047\,\sigma_1 - 0.1544\,\sigma_2 + 0.3972\,\sigma_3, \qquad i_2 = -0.2959I\sigma_3 - 0.6685I\sigma_2 - 0.6823I\sigma_1. \tag{42}$$

We thus propose to use this new polar representation method (36) for the embedded octonion analytic signal (25), as *one way to answer* the open question for the exact polar representation of (19).

Now let us assume, we have a general embedded octonion analytic signal in this new form of polar decomposition

$$\psi_1(x_1, x_2, x_3) = e^{\alpha_0} e^{a} e^{B} e^{\alpha_3 I}, \tag{43}$$

with

$$a = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \alpha_1 u', \quad \alpha_1 = |a| = \sqrt{a^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}, \quad u' = \frac{a}{\alpha_1},$$
$$B = b_1\sigma_{23} + b_2\sigma_{31} + b_3\sigma_{12}, \quad \alpha_2 = |B| = \sqrt{-B^2} = \sqrt{b_1^2 + b_2^2 + b_3^2}, \quad i_2 = \frac{B}{\alpha_2},$$
$$b = i_2^* = i_2(-I) = \frac{b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3}{\alpha_2} \tag{44}$$

where α_0, $a_1, a_2, a_3, b_1, b_2, b_3$ and α_3 are scalar functions of (x_1, x_2, x_3), and consequently α_1, α_2, vector a and bivector B are also functions of (x_1, x_2, x_3). Note that the unit vector b is dual (and thus orthogonal) to the unit bivector i_2.

What does the reconstruction of the original real scalar signal f look like? In order to answer this question, we note that

$$e^{a} = e^{\alpha_1 u'} = \cosh\alpha_1 + u' \sinh\alpha_1, \quad e^{B} = e^{\alpha_2 i_2} = \cos\alpha_2 + i_2 \sin\alpha_2, \quad e^{\alpha_3 I} = \cos\alpha_3 + I\sin\alpha_3, \tag{45}$$

and abbreviate in this context

$$c_1 = \cosh\alpha_1, \quad s_1 = \sinh\alpha_1, \quad c_2 = \cos\alpha_2, \quad s_2 = \sin\alpha_2, \quad c_3 = \cos\alpha_3, \quad s_3 = \sin\alpha_3. \tag{46}$$

Now we can expand the above polar representation of the embedded octonion signal as

$$\psi_1(x_1, x_2, x_3) = e^{\alpha_0}(c_1 + u's_1)(c_2 + i_2 s_2)(c_3 + I s_3)$$
$$= e^{\alpha_0}(c_1c_2c_3 + c_1s_2c_3 i_2 + s_1c_2c_3u' + s_1s_2c_3u'i_2 + c_1c_2s_3I + c_1s_2s_3i_2I + s_1c_2s_3u'I + s_1s_2s_3u'i_2I). \tag{47}$$

We observe that only the first term and the scalar part of the last term do contribute to the scalar part of ψ_1

$$f_{rec} = \langle \psi_1(x_1, x_2, x_3)\rangle = e^{\alpha_0}(c_1c_2c_3 + s_1s_2s_3\langle u'i_2I\rangle) = e^{\alpha_0}(c_1c_2c_3 + s_1s_2s_3\, u'\cdot(-i_2^*)) = e^{\alpha_0}(c_1c_2c_3 - s_1s_2s_3\, u'\cdot b)$$
$$= e^{\alpha_0}(c_1c_2c_3 - s_1s_2s_3\cos\varphi_{ab}), \tag{48}$$

where in the last line we introduced the angle φ_{ab}, $\cos\varphi_{ab} = u' \cdot b$, between the vector a and the normal vector b of i_2. Comparing with the octonionic reconstruction result (30) in [6], we see that even without the assumption of separability, we obtain a considerably *simpler, more compact* and *geometrically intuitive* result in terms of the amplitude factor e^{α_0}, the (hyperbolic) cosines and sines of the parameters $\alpha_1, \alpha_2, \alpha_3$ and of the cosine of the angle φ_{ab} between the vectors a and b (normal to i_2).

6.2.2 | Polar Representation Based on Polar Decomposition in $Cl(3, 0)$ and Intuition from Separability

Another way to answer the above question for the polar decomposition of embedded octonion analytic signals can be proposed based on analysis of a separable three-dimensional signal (31) that leads to a decomposition of the form

$$\psi_1'(x_1, x_2, x_3) = A_1A_2A_3\left[\cos(\alpha_2)e^{-\alpha_3 I} - \sin(\alpha_2)I\sigma_3 e^{\alpha_3 I}\right]\left(\cos(\alpha_1) + \sin(\alpha_1)\sigma_1\right), \tag{49}$$

where the scalar amplitude- and angle parameters $A_1, A_2, A_3, \alpha_1, \alpha_2, \alpha_3$ are all functions of (x_1, x_2, x_3). More general, without assuming separability, we have

$$\psi_1'(x_1, x_2, x_3) = e^{\alpha_0}\left[\cos(\alpha_2)e^{-\alpha_3 I} + \sin(\alpha_2)i_2 e^{\alpha_3 I}\right]\left(\cos(\alpha_1) + \sin(\alpha_1)u\right), \tag{50}$$

with

$$a = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \alpha_1 u, \quad \alpha_1 = |a| = \sqrt{a^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}, \quad u = \frac{a}{\alpha_1},$$
$$B = b_1\sigma_{23} + b_2\sigma_{31} + b_3\sigma_{12}, \quad \alpha_2 = |B| = \sqrt{-B^2} = \sqrt{b_1^2 + b_2^2 + b_3^2}, \quad i_2 = \frac{B}{\alpha_2},$$
$$b = i_2^* = i_2(-I) = \frac{b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3}{\alpha_2}, \tag{51}$$

where α_0, $a_1, a_2, a_3, b_1, b_2, b_3$ and α_3 are scalar functions of (x_1, x_2, x_3), and consequently α_1, α_2, thus vector a and bivector B are also functions of (x_1, x_2, x_3).

The reconstruction formula for the real signal f' from the polar representation (50) amounts simply to compute its scalar part. In analogy to (48), we obtain with (note the different definitions of s_1 and c_1, compared to (46))

$$c_1 = \cos\alpha_1, \quad s_1 = \sin\alpha_1, \quad c_2 = \cos\alpha_2, \quad s_2 = \sin\alpha_2, \quad c_3 = \cos\alpha_3, \quad s_3 = \sin\alpha_3. \tag{52}$$

$$f'_{rec} = \langle \psi\varepsilon'_1(x_1, x_2, x_3)\rangle = e^{\alpha_0}(c_1c_2c_3 - s_1s_2s_3\, u\cdot b) = e^{\alpha_0}(c_1c_2c_3 - s_1s_2s_3\cos\varphi_{ab}), \tag{53}$$

where in the last line we again introduced the angle φ_{ab} between the vector a and the normal vector b of i_2. We note that the two geometric algebra embedding based reconstruction formulas (48) and (53) are formally identical, apart from the differences in using hyperbolic cosines and sines at the beginning of (46), while only trigonometric cosines and sines are used in (52).

Remark 1. We note that in the case of a truly separable signal, like in (49) with $u = \sigma_1$ and $i_2 = -I\sigma_3$, we have

$$-\cos\varphi_{ab} = -u\cdot b = \langle u i_2 I\rangle = \langle \sigma_1(-I\sigma_3)I\rangle = \langle \sigma_1\sigma_3\rangle = 0 \tag{54}$$

and hence the even simpler result

$$f'_{rec} = e^{\alpha_0}c_1c_2c_3, \tag{55}$$

formally identical to the above (32) also found in Section 7.5.2.1 of[6].

Further research has to show which of these two geometric algebra based polar representations of embedded octonion analytic signals may be preferable.

7 | CONCLUSIONS

We have briefly reviewed octonions and their new minimal embedding in the geometric algebra of three-dimensional space $Cl(3,0)$. We further reviewed the notion of OFT and octonion analytic signal, embedded both in $Cl(3,0)$, and finally suggested two interesting possibilities for polar decompositions of the embedded octonion analytic signal, together with the corresponding signal reconstruction formulas. In this context we have given for the polar decomposition of multivectors in $Cl(3,0)$ new simplified computation formulas. Further research, including concrete applications to non-separable signals, is desirable.

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Author contributions

The author is responsible for all parts of this work.

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APPENDIX

A DERIVATION OF SIMPLIFIED FORMULAS FOR EXPONENTIAL DECOMPOSITION IN $CL(3,0)$

Based on the general results for the exponential factorization of $Cl(3,0)$ multivectors M in[19,11], we now prove the simplified computation formulas (37) to (41).

For a general multivector $M \in Cl(3,0)$ we have the additive decomposition

$$M = M_+ + M_- = p + Iq, \qquad p, q \in Cl^+(3,0), \qquad p = M_+, \qquad q = I^{-1}M_- = (M_-)^*. \tag{A1}$$

For $M\overline{M} = 0$ the multivector M is a *divisor of zero* and will take on the factorized form

$$M = e^{\alpha_0}e^{\alpha_2 i_2}\frac{1+I\mathbf{f}}{2} = \frac{1}{2}e^{\alpha_0}e^{\alpha_2 i_2} + \frac{1}{2}e^{\alpha_0}e^{\alpha_2 i_2}I\mathbf{f}, \tag{A2}$$

with

$$M_+ = \langle M\rangle_0 + \langle M\rangle_2 = \frac{1}{2}e^{\alpha_0}e^{\alpha_2 i_2} = \frac{1}{2}e^{\alpha_0}\cos\alpha_2 + i_2\frac{1}{2}e^{\alpha_0}\sin\alpha_2, \qquad \langle M\rangle_0 = \frac{1}{2}e^{\alpha_0}\cos\alpha_2, \qquad \langle M\rangle_2 = i_2\frac{1}{2}e^{\alpha_0}\sin\alpha_2, \tag{A3}$$

and

$$M_- = \frac{1}{2}e^{\alpha_0}e^{\alpha_2 i_2}I\mathbf{f} = M_+I\mathbf{f} = M_+\mathbf{f}I. \tag{A4}$$

From (A3) we immediately find

$$M_+\overline{M_+} = \frac{1}{4}e^{2\alpha_0}e^{\alpha_2 i_2}e^{-\alpha_2 i_2} = \frac{1}{4}e^{2\alpha_0} \quad \Leftrightarrow \quad \alpha_0 = \frac{1}{2}\ln(4M_+\overline{M_+}) = 2 + \frac{1}{2}\ln(M_+\overline{M_+}), \tag{A5}$$

and

$$\tan\alpha_2 = \frac{|\langle M\rangle_2|}{\langle M\rangle_0} \quad \Leftrightarrow \quad \alpha_2 = \mathrm{atan2}(|\langle M\rangle_2|, \langle M\rangle_0). \tag{A6}$$

For the special case of $\langle M\rangle_2 = 0$, i.e., $M_+ = \langle M\rangle_0$, we have

$$\alpha_0 = 2 + \frac{1}{2}\ln(\langle M\rangle_0^2), \qquad e^{\alpha_2 i_2} \;\to\; \mathrm{sgn}\langle M\rangle_0 = \pm 1. \tag{A7}$$

For $\langle M\rangle_2 \neq 0$, we see from (A3) that

$$i_2 = \frac{\langle M\rangle_2}{|\langle M\rangle_2|}. \tag{A8}$$

Finally from (A4) we then conclude

$$\mathbf{f} = M_+^{-1}M_-I^{-1} = (M_+^{-1}M_-)^*, \tag{A9}$$

and if furthermore $\langle M\rangle_2 = 0$, i.e., $M_+ = \langle M\rangle_0$, this becomes simply

$$\mathbf{f} = \frac{(M_-)^*}{\langle M\rangle_0}. \tag{A10}$$

This completes the derivation of the simplified computation of the exponential factorization of a (non-invertible) multivector $M \in Cl(3,0)$ that is a divisor of zero.

Now we derive the simplified factorization expressions for a $M \in Cl(3,0)$ with $M\overline{M} \neq 0$, i.e. for M being invertible (not a divisor of zero). Division with $\sqrt{M\overline{M}}$ leads to a unit norm multivector

$$N = \frac{M}{\sqrt{M\overline{M}}}, \qquad N\overline{N} = 1, \tag{A11}$$

and hence the central (scalar and pseudoscalar) amplitude is given by

$$\begin{aligned}
\sqrt{M\overline{M}} &= e^{\alpha_0+\alpha_3 I} = e^{\alpha_0}e^{\alpha_3 I}, \qquad M\overline{M} = e^{2\alpha_0}e^{2\alpha_3 I} = e^{2\alpha_0}(\cos 2\alpha_3 + I\sin 2\alpha_3),\\
e^{2\alpha_0} &= \sqrt{|M\overline{M}|^2} = \left(M\overline{M}\,\widetilde{(M\overline{M})}\right)^{\frac{1}{2}} = \left(M\overline{M}\,\widehat{M}\widetilde{M}\right)^{\frac{1}{2}} = \sqrt{\det(M)}\\
\Leftrightarrow\quad \alpha_0 &= \frac{1}{4}\ln(M\overline{M}\,\widehat{M}\widetilde{M}) = \frac{1}{4}\ln\left(\det(M)\right),
\end{aligned} \tag{A12}$$

where we notice that $\det(M) = M\overline{M}\,\widehat{M}\widetilde{M}$, compare[21,9]. And

$$\tan(2\alpha_3) = \frac{\langle M\overline{M}\rangle_3 I^{-1}}{\langle M\overline{M}\rangle_0} = \frac{\langle M\overline{M}\rangle_3^*}{\langle M\overline{M}\rangle_0} \quad\Leftrightarrow\quad \alpha_3 = \frac{1}{2}\mathrm{atan}\Big((\langle M\overline{M}\rangle_3)^*, \langle M\overline{M}\rangle_0\Big). \tag{A13}$$

Interpreting $\sqrt{M\overline{M}}$ as a complex number, the computation of α_0 and α_3 simply means to obtain the logarithm of the magnitude, and the phase angle, respectively.

In[11] we find the definition of $P, Q \in Cl^+(3,0)$ as

$$P = N_+ = \langle M\sqrt{M\overline{M}}^{-1}\rangle_+, \qquad Q = N_- I^{-1} = \left(\langle M\sqrt{M\overline{M}}^{-1}\rangle_-\right)^*, \qquad \tanh\alpha_1 = \frac{a_Q}{a_P} = \sqrt{\frac{Q\overline{Q}}{P\overline{P}}}, \tag{A14}$$

hence

$$\alpha_1 = \mathrm{atanh}\sqrt{\frac{(N_-)^*\overline{(N_-)^*}}{N_+\overline{N_+}}} = \mathrm{atanh}\sqrt{-\frac{N_-\overline{N_-}}{N_+\overline{N_+}}}, \tag{A15}$$

because

$$Q\overline{Q} = N_- I^{-1}\overline{N_- I^{-1}} = N_- I^{-1} I^{-1}\overline{N_-} = -N_-\overline{N_-}. \tag{A16}$$

From we have

$$u' = \frac{\langle Ne^{-\alpha_P i_P}\rangle_1}{|\langle Ne^{-\alpha_P i_P}\rangle_1|} = \frac{\langle NP^{-1}\rangle_1}{|\langle NP^{-1}\rangle_1|} = \frac{IQP^{-1}}{|IQP^{-1}|} = \frac{N_- N_+^{-1}}{|N_- N_+^{-1}|} = \frac{N_-\overline{N_+}}{|N_-\overline{N_+}|}. \tag{A17}$$

According to[11], we have

$$e^{\alpha_2 i_2} = \cos\alpha_2 + i_2\sin\alpha_2 = e^{\alpha_P i_P} = \frac{P}{a_P} = \frac{N_+}{|N_+|} = \frac{1}{|N_+|}(\langle N\rangle_0 + \langle N\rangle_2), \tag{A18}$$

hence

$$\tan\alpha_2 = \frac{|\langle N\rangle_2|}{\langle N\rangle_0} \quad\Leftrightarrow\quad \alpha_2 = \mathrm{atan2}(|\langle N\rangle_2|, \langle N\rangle_0), \qquad i_2 = \frac{\langle N\rangle_2}{|\langle N\rangle_2|}. \tag{A19}$$

For the special case of $\langle N\rangle_2 = 0$ we have $e^{\alpha_2 i_2} \to \pm 1 = \mathrm{sgn}\langle N\rangle_0$, and do not need to compute α_2 and i_2.

B COMPUTATION OF EXAMPLE 6.1

We assume in $Cl(3,0)$ the multivector

$$M = 1 + 2\sigma_1 + 3\sigma_2 + 4I\sigma_1 + 5I\sigma_3 + 6I \tag{B20}$$

A first step is to norm M by division with the central square root of $M\overline{M}$.

$$\begin{aligned}
M\overline{M} &= (1 + 2\sigma_1 + 3\sigma_2 + 4I\sigma_1 + 5I\sigma_3 + 6I)(1 - 2\sigma_1 - 3\sigma_2 - 4I\sigma_1 - 5I\sigma_3 + 6I)\\
&= 1 - 4 - 9 + 16 + 25 - 36 + I(12 - 16) = -7 - 4I\\
&= \sqrt{65}\,\frac{-7-4I}{\sqrt{65}} = e^{2\times 1.0436}e^{2\times 1.8304 I}, \quad \langle M\overline{M}\rangle_0 = -7, \quad \langle M\overline{M}\rangle_3 = -4I,
\end{aligned} \tag{B21}$$

showing that $\alpha_0 = 1.0436$ and $\alpha_3 = 1.8304$. We can check the value of α_0 by computing the determinant

$$\det(M) = M\overline{M}\widehat{M}\widetilde{M} = 65, \qquad \alpha_0 = \frac{1}{4}\ln\det(M) = \frac{1}{4}\ln 65 = 1.0436, \tag{B22}$$

and similarly we can check (to make the angle positive, we add 2π)

$$\alpha_3 = \frac{1}{2}\mathrm{atan2}(-4I^*, -7) = \frac{1}{2}\mathrm{atan2}(-4, -7) = \frac{1}{2}(-2.6224) \cong \frac{1}{2}(-2.6224 + 2\pi) = 1.8304. \tag{B23}$$

We therefore have

$$\sqrt{M\overline{M}} = e^{1.0436}\, e^{1.8304 I}, \tag{B24}$$

and

$$\begin{aligned} N = M\sqrt{M\overline{M}}^{-1} &= Me^{-1.0436}e^{-1.8304I} \\ &= 1.9519 + 1.1807\sigma_1 - 0.2712\sigma_2 + 1.7019\sigma_3 \\ &\quad - 0.4520I\sigma_3 - 1.0212I\sigma_2 - 1.0424I\sigma_1 - 0.8828I \\ &= N_+ + N_-, \end{aligned} \tag{B25}$$

Therefore

$$\begin{aligned} N_+ &= 1.9519 - 0.4520I\sigma_3 - 1.0212I\sigma_2 - 1.0424I\sigma_1, \\ \langle N\rangle_0 &= 1.9519, \quad \langle N\rangle_2 = -0.4520I\sigma_3 - 1.0212I\sigma_2 - 1.0424I\sigma_1, \quad |\langle N\rangle_2| = 1.5276, \\ N_- &= -0.8828I + 1.1807\sigma_1 - 0.2712\sigma_2 + 1.7019\sigma_3. \end{aligned} \tag{B26}$$

And we represent N_+ as a rotor

$$N_+ = |N_+|e^{\alpha_2 i_2} = 2.4786\, e^{0.66405\times(-0.2959I\sigma_3 - 0.6685I\sigma_2 - 0.6823I\sigma_1)}, \qquad |N_+| = \sqrt{N_+\overline{N_+}} = 2.4786, \tag{B27}$$

that is

$$\begin{aligned} \alpha_2 &= \mathrm{atan2}(|\langle N\rangle_2|, \langle N\rangle_0) = 0.66405, \\ i_2 &= \frac{\langle N\rangle_2}{|\langle N\rangle_2|} = -0.2959I\sigma_3 - 0.6685I\sigma_2 - 0.6823I\sigma_1. \end{aligned} \tag{B28}$$

We finally have

$$e^{\alpha_1 u'} = Ne^{-\alpha_P i_P} = 2.4786 + 2.0517\sigma_1 - 0.3502\sigma_2 + 0.9008\sigma_3, \tag{B29}$$

with unit vector part

$$u' = \frac{N_-\overline{N_+}}{|N_-\overline{N_+}|} = 0.9047\,\sigma_1 - 0.1544\,\sigma_2 + 0.3972\,\sigma_3, \tag{B30}$$

and

$$\alpha_1 = \mathrm{atanh}\sqrt{-\frac{N_-\overline{N_-}}{N_+\overline{N_+}}} = \mathrm{atanh}\sqrt{\frac{5.1435}{6.1435}} = 1.5574. \tag{B31}$$

In summary the polar decomposition gives

$$\begin{aligned} M &= e^{1.0436}\, e^{1.5574\, u'}\, e^{0.66405\, i_2}\, e^{1.8304\, I}, \\ u' &= 0.9047\,\sigma_1 - 0.1544\,\sigma_2 + 0.3972\,\sigma_3, \qquad i_2 = -0.2959I\sigma_3 - 0.6685I\sigma_2 - 0.6823I\sigma_1. \end{aligned} \tag{B32}$$

All computations have been verified with The Clifford Multivector Toolbox for Matlab[18].

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