Correcting Rudin's Perfect Sets of $\mathbb{R}^k$ Are Uncountable Proof

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Abstract

It is fascinating fact that the reals are uncountably infinite. Usually Cantor's diagonal method is used to show this. Rudin gives a second proof that promises to be more rigorous than this method. But his proof is a little confusing, if not incorrect. His proof does not stipulate that the perfect set be bounded, but its proof hinges on a local, bounded phenomenon. We duplicate Rudin's proof and argue using two examples that assuming any indexing scheme for the presumed countable set can't work. We then give two proofs: one re-indexes points and the other indexes in the course of the proof.

Rudin's Proof

Rudin in his *Principles of Mathematical Analysis* gives a proof that non-empty perfect subsets of $\mathbb{R}^k$ are uncountable. We reproduce the proof.

Theorem. *Let P be a non-empty perfect set in $\mathbb{R}^k$. Then P is uncountable.*

Proof. Since P has limit points, P must be infinite. Suppose P is countable, and denote the points of P by $\mathbf{x}_1, x_2, x_3, \ldots$. We shall construct a sequence $\{V_n\}$ of neighborhoods, as follows.

Let V_1 be any neighborhood of x_1. If V_1 consists of all $y \in \mathbb{R}^k$ such that $|y - x_1| < r$, the closure of $\overline{V_1}$ of V_1 is the set of all $y \in \mathbb{R}^k$ such that $|y - x_1| \leq r$.

Suppose V_n has been constructed so that $V_n \cap P$ is not empty. Since every point of P is a limit point of P there is a neighborhood V_{n+1} such that (i) $\overline{V}_{n+1} \subset V_n$,

(ii) $x_n \notin \overline{V}_{n+1}$, and (iii) $V_{n+1} \cap P$ is not empty. By (iii) V_{n+1} satisfies our induction hypothesis, and the construction can proceed.

Put $K_n = \overline{V}_n \cap P$. Since K_n is closed and bounded, $\overline{V}_n$ is compact. Since $x_{n+1} \notin K_n$, no point of P lies in $\bigcap_1^\infty K_n$. Since $K_n \subset P$ this implies that $\bigcap_1^\infty K_n$ is empty. But each K_n is not empty, by (iii), and $K_n \supset K_{n+1}$ by (i); this contradicts the corollary to Theorem 2.36. $\square$

Problematic Examples

Example 1. Prove $[0, 1]$ is uncountable. Per Rudin's proof we can be given any countable set of points in any order. If we are to nest x_1, x_2, x_3, given that is their order, there is no problem: symbolically,

$$\boxed{1\boxed{2\boxed{3}}}, \tag{1}$$

where boxes indicate the sets. Concretely: $1.1 \in (1, 3.1) = V_1$, $2.1 \in (2, 3) = V_2$, and $3 \in (2.9, 3.1) = V_3$ with $1.1 \notin V_2$, $3 \notin V_2$. They nest

$$V_3 \subset V_2 \subset V_1$$

and systematically exclude earlier numbers from later set. This results in an empty intersection relative to $\{1.1, 2.1, 3\}$. But suppose the points are not in order. The points are ordered x_3, x_1, x_2:

$$\boxed{3\boxed{1\boxed{2}}}.$$

One can't get a intervals, neighborhoods such that x_1 is not in the neighborhood for x_2. You are forced to re-index to (1).

Example 2. Prove $\mathbb{R}^+$ is uncountable. This is an unbounded set. Assume the indexed set are the positive integers. The radius needed to fit $x_1 = 1$ into each neighborhood for the nesting to work must grow to infinity, but, per Rudin's proof this radius can be arbitrary, not infinity.

Here are two ways to correct these problems.

The Swap Method

Theorem. *Non-empty perfect subsets of* $\mathbb{R}^k$ *are uncountable.*

Proof. The well-ordering principle states that there is a smallest element in a set of positive integers. It is taken as obviously true.

Armed with the well-ordering principle, suppose P is countable and its elements are $\{x_1, x_2, \dots\}$. Using x_1, form the neighborhood $N_{r_1}(x_1)$ for any $r_1 > 0$. Let

$$C_1 = \{d(x_1, x_j) < r_1 : j \in \mathbb{N} \text{ and } 2 \leq j\}$$

and suppose $d(x_1, x_{m_1}) = \min C_1$; the element closest to x_1 is x_{m_1}. There will be infinitely many x_{m_1} such that $d(x_1, x_{m_1}) < r_1$, but there will be a smallest index for exactly one, even if the distance is the same for more than one. We know that such an index will exist as the set of indices consists of a set of positive integers.

Notice this is where the idea of the proof becomes clear. A perfect set of reals will not have an element closest to another element; making the set countable forces this to become true.

Now re-index by exchanging 2 for m_1. That is set x_2 to the value of x_{m_1} and the value of x_{m_1} to the old value of x_2. Form $N_{r_2}(x_2)$ with r_2 small enough to reside in $N_{r_1}(x_1)$ with $x_1 \notin N_{r_2}(x_2)$ and repeat. Eventually you have proved that the assumption of countable P has forced all elements into an any arbitrarily small neighborhood, a contradiction.

Rudin's argument also now applies. □

Index as you go

Theorem. *Non-empty perfect subsets of* $\mathbb{R}^k$ *are uncountable.*

Proof. A perfect set is defined as a closed set with all elements limit points. Let P be non-empty perfect subset of $\mathbb{R}^k$. Every neighborhood of every point of P has infinitely many points of P in it. We know, then, that P must be countably or uncountably infinite.

Suppose, to get a contradiction, that P is countably infinite.

Take any element of P and label it x_1; give it any radius r_1. We know by the definition of a limit point that there exists $x_2 \in N_{r_1}(x_1)$, $x_2 \neq x_1$. There may be other elements of P in this neighborhood, in fact there are infinitely many, but they have not been indexed and can be indexed as we like. Suppose this has been done for k elements. We then have a nesting sequence of neighborhoods $N_{r_j}(x_j)$, $1 \leq j \leq k$. There will exist x_{k+1} and r_{k+1} such that $x_{k+1} \in N_{r_{k+1}}$, but $x_k \notin N_{r_{k+1}}$. We choose using the Axiom of Choice any remaining non-indexed element. We notice that if $k = 1$ and we are indexing on the real line, we can always choose a greater or lesser number than the max or min of our indexed set and avoid the problem given in Example 1.

We can take r_js such that $\lim_{j\to\infty} r_j = 0$.

Next

$$\bigcap_{j=1}^{\infty} N_{r_j}(x_j) = \emptyset,$$

as all indexed points x_j are excluded by not being in $N_{r_{j+1}}(x_{r_{j+1}})$ and points not indexed will eventually be excluded from a neighborhood of an indexed element of P when the radii are small enough.

Finally, define the closed and bounded, hence compact sets

$$K_j = P \cap \overline{N_{r_j}(x_j)}$$

and their intersection will also be be the empty set, a contradiction, per Rudin's argument. $\square$

Conclusion

Reading Rudin, one gets the impression that you can start with any indexing you like, but this is confusing, especially as he does not stipulate that you can re-index any countable set without changing its cardinality.

References

[1] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill.