Skolems solution for integer-linear-recurrences, with commensurable arguments for characteristic-roots of the same modulus

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Abstract: - For a homogeneous linear-recurrence f_n with integer coefficients and integer starting points, we derive a deterministic algorithm that finds the upper bound of the last non-periodic position n where $f_n=0$, for a large family of special cases. First, when θ is a given irrational constant, then we show that, the eventual lower bound of minimum(absolute(cos(mπθ)), over positive integers m less than n), for large positive integers n, is (2θ / (sqrt(5) n)). Our deterministic algorithm is based on the key concept that this lower bound decreases at a lower rate than the n^{th} power of the ratio of root moduli since the ratio is lesser than 1. Our deterministic algorithm is developed for the special cases where G(x), the characteristic polynomial of f_n, has either equal absolute values of arguments or commensurate arguments of those complex roots, whose moduli are equal. In an attempt to extend this algorithm as a general solution to Skolems problem, we obtain the lower bound of the distance between a zero and the next $(2^{(m+1)})^{th}$ zero, in the weighted sum of m continuous cosine functions, where the weights are given real-algebraic constants.

1. Introduction

Given integer constants $\{b_1, b_2, \ldots b_L, a_0, a_1, a_2 \ldots a_{L-1}\}$, we denote our homogenous integer linear recurrence sequence f_n as follows:

1. $f_n = 0$, for integers $n < 0$
2. $f_n = a_n$, for integers $0 \le n \le (L-1)$
3. $f_n = b_1 f_{n-1} + b_2 f_{n-2} + \ldots + b_L f_{n-L}$, for integers $n \ge L$

It is known from the Skolem-Mahler-Lech (SML) theorem that the set of zeros of f_n is the union of an infinite periodic set and a finite set. In other words, the set Z of n for which $f_n = 0$, is such that Z = (P union Q), where P is an infinite set of positive integers that is a finite set of infinite arithmetic progressions, and where Q is a finite set of positive integers. It is also known how to describe P and to determine whether or not P is empty, but an upper bound on the largest integer in Q is still an open problem. Consequently, it remains an open problem to determine whether or not Q is empty within finite time, and this open problem may be simply reworded as determining whether or not $f_n = 0$ for some integer n>0 [1][2][3][5][7].

In this paper, we discuss what constitutes the zeros in sets P and Q. We then derive a deterministic algorithm to describe the set Z for a given f_n, for certain special cases. We finally present a result that attempts to generalize the algorithm.

2. Approach

2.1 Notations

Further to the notation of f_n given in Section 1, we use the following additional notations:

1. A number x is said to be algebraic if it is the root of some polynomial with integer coefficients.
2. If x is a complex number = y + iz, where i=square_root(-1), then modulus(x) = square_root($y^2 + z^2$). Also, real(x)=y and imaginary(x)=z. Also, x = (modulus(x) $e^{i\theta}$) = modulus(x)(cos(θ) + i sin(θ)).
3. If x and y are two positive real numbers, then quotient(x/y)=floor(x/y) = the largest non-negative integer below x/y. Also, fraction(x/y) = ((x/y) - quotient(x/y)).
4. If x is a real number, then absolute(x) = x if $x \ge 0$, and absolute(x) = -x if $x < 0$.
5. BODMAS (Brackets, Order of Powers or Roots, Division, Multiplication, Addition, Subtraction) order is followed in all expressions where the order is not explicitly mentioned by brackets. For example, the expression $ax + b^{0.5}y - z/d^2 = ((ax) + ((b^{0.5})y) - (z/(d^2)))$.
6. Given a function with integer domain, f_n, where $f_n = 0$ for integers $n < 0$, and where f_n is a real for integers $n \ge 0$, then $F(z) = SUM((f_n z^n)$, over integers $n \ge 0) = (f_0 z^0 + f_1 z^1 + f_2 z^2 + \ldots)$ is the generating function for f_n.

7. $ax = a*x$ both denote the product of a with x, and either notation is used wherever convenient.
8. $a^x = a\text{^}x$ both denote a to the power of x.
9. log(x) is the natural logarithm of x, i.e., log of x to the base of e.
10. remainder(x,y) = x - (y floor(x/y)) = (y fraction(x/y)), where x and y are both real positive numbers.
11. floor(x) is the largest integer $\leq$ x, where x is a real positive number.
12. g is the golden ratio = (1+sqrt(5))/2, which is irrational, and approximately 1.618034.
13. π or PI is the irrational ratio between the lengths of the circumference and diameter of a circle, which is approximately 3.14159.
14. sqrt(x) is the square root of x.
15. G(x) is the characteristic polynomial of f_n. Basically, $G(x) = (1 - b_1 x - b_2 x^2 - b_3 x^3 - ... - b_L x^L)$.
16. (Zero of f_n) = (n, such that f_n=0, where integer n > 0). Similarly, zero of a function f(x), where x is a continuous real variable, is x such that f(x)=0.
17. Two non-zero real numbers x and y are said to be commensurable, if and only if, x/y is rational, i.e. iff (x/y)=(p/q), where p and q are both integers. If (x/y) is irrational, then x and y are said to be incommensurable.
18. If A and B are two boolean statements, then:
    $A \rightarrow B$ denotes "If A is true, then B is true"
    $A \leftrightarrow B$ denotes "A is true, if and only if, B is true".
19. A variable indexed by an integer, is represented in either or both of two ways:- with underscore or with subscript. For example, γ_i is used interchangeably with γ_i.

Theorem 1 below is a reworded form of a known result, on the stateless expression of f_n [6].

<u>Theorem 1</u>: For each integer n$\geq$0, f_n = SUM(SUM(($n^j\, r_k^n\, s_{n,j,k}$), over all integers k in [1,L]), over all integers j in [0,L-1]), where:

1. **(r_k e^(iθ_k)) is a constant algebraic root of G(x) and (r_k e^(-iθ_k)) is its conjugate.**
2. **$s_{n,j,k}$ = SUM(($d_{j,k}$ cos(nθ_k)) , over all integers k in [1,L]).**
3. **Each of {$d_{j,k}$, r_k, cos(θ_k), sin(θ_k)} is a constant real algebraic number, and r_k>0**

<u>Proof</u>: It is well-known that f_n = C_1(n) (r_1^n (e^(inθ_1) + e^(-inθ_1))) + C_2(n) (r_2^n (e^(inθ_2) + e^(-inθ_2))) + ... + C_L(n) (r_L^n (e^(inθ_L) + e^(-inθ_L))) , where for each integer k in [1, L]:

1. C_k(n) is a constant univariate polynomial in n with degree = L-1, with constant algebraic coefficients. So C_k(n) = ($c_{0,k}$ e^(i$\beta_{0,k}$)) + ($c_{1,k}$ e^(i$\beta_{1,k}$)) n + ($c_{2,k}$ e^(i$\beta_{2,k}$)) n^2 + ... + ($c_{L-1,k}$ e^(i$\beta_{L-1,k}$)) n^{L-1}, where each of {$c_{j,k}$, cos($\beta_{j,k}$), sin($\beta_{j,k}$)} is a constant real algebraic number for each integer j in [0,L-1].
2. (r_k e^(iθ_k)) is a constant algebraic root of G(x) and (r_k e^(-iθ_k)) is its conjugate, where each of {r_k , cos(θ_k), sin(θ_k)} is a constant real algebraic number.

Since the complex roots with imaginary components appear in conjugates, the imaginary parts of f_n have to disappear. Also, since two or more roots may be of equal modulus, and of same or different arguments, we can further group the expression of f_n as a sum of product terms, where each product term is a product between a trigonometric function and the term ($n^j\, r_k^n$). Thus, f_n can be expressed as follows:
f_n = SUM(SUM(($n^j\, r_k^n\, s_{n,j,k}$), over all integers k in [1,L]), over all integers j in [0,L-1]), where $s_{n,j,k}$ = SUM(($d_{j,k}$ cos(nθ_k)) , over all integers k in [1,L]) and where each $d_{j,k}$ is a real algebraic constant.
<u>Hence Proved Theorem 1.</u>

Before proceeding further, let us remember that our aim is to describe the set Z of the zeros of f_n, which we earlier mentioned is the union of sets P and Q. Theorem 2 describes what constitutes the set of zeros in P and Q.

<u>Theorem 2</u>: The finite set of non-periodic zeros Q of f_n is the set of zeros of the expression f_n = SUM(SUM(($n^j\, r_k^n\, s_{n,j,k}$), over all integers k in [1,L]), over all integers j in [0,L-1]), after removing cosines of rational multiples of π, which common factors from every $s_{n,j,k}$

<u>Proof</u>: If we take out the common cosine factors from all $s_{n,j,k}$, then we may rewrite:
f_n = (cos(π(n$\varphi_{rat,1}$+$\gamma_{rat,1}$)) cos(π(n$\varphi_{rat,2}$+$\gamma_{rat,2}$)) ... cos(π(n$\varphi_{rat,U}$+$\gamma_{rat,U}$))) (cos(π(n$\varphi_{real,1}$+$\gamma_{real,1}$)) cos(π(n$\varphi_{real,2}$+$\gamma_{real,2}$)) ... cos(π(n$\varphi_{real,V}$+$\gamma_{real,V}$))) SUM(SUM(($n^j\, r_k^n\, s'_{n,j,k}$), over all integers k in [1,L]), over all integers j in [0,L-1]), where U and V are some positive integer constants, each of {$\varphi_{rat,i}$,$\gamma_{rat,i}$} is a rational constant, and each {$\varphi_{real,i}$,$\gamma_{real,i}$} is a real constant. It becomes clear that the set P of zeros of f_n is due to the first product (cos(π(n$\varphi_{rat,1}$+$\gamma_{rat,1}$)) cos(π(n$\varphi_{rat,1}$+$\gamma_{rat,1}$)) ... cos(π(n$\varphi_{rat,U}$+$\gamma_{rat,U}$))) resulting in a finite union of infinite-sized arithmetic progressions. The second product (cos(π(n$\varphi_{real,1}$+$\gamma_{real,1}$))

$\cos(\pi(n\varphi_{real,2}+\gamma_{real,2}))$ ... $\cos(\pi(n\varphi_{real,V}+\gamma_{real,V})))$ might contribute to a maximum of V zeros to Q that are also easy to find. But it is the third product SUM(SUM((n^j r_k^n $s'_{n,j,k}$), over all integers k in [1,L]), over all integers j in [0,L-1]) that makes the most interesting contribution to the finite set of zeros of Q.

Hence Proved Theorem 2.

Since it is already known how to define the set P of zeros of f_n, we shall assume without loss of generality, that when we write f_n = SUM(SUM((n^j r_k^n $s_{n,j,k}$), over all integers k in [1,L]), over all integers j in [0,L-1]), in the remaining part of this paper, all common cosine factors have already been factored out and removed, and that the zeros of this expression of f_n is what forms the set of non-periodic zeros Q that we are trying to find..

We now make the next important Theorem 3, which serves as the foundation for the solution to many special cases in Skolems problem to the upper bound of the non-periodic zeros of f_n.

Theorem 3: **Let λ_A and λ_B be two positive incommensurable constants, where $\lambda_A > \lambda_B$. Consider the following algorithm:**

Algorithm description starts

$i = 0$;

$n_A = 0$;

$n_B = 0$;

$p_0 = \lambda_A$;

REPEAT:

If ($n_A\lambda_A < n_B\lambda_B$)

{ $n_A = n_A+1$;}

Else

{ $n_B = n_B+1$;}

If (absolute($n_A\lambda_A$ - $n_B\lambda_B$) < p_i)

{ $i = i+1$;

p_i = absolute($n_A\lambda_A$ - $n_B\lambda_B$);}

Goto REPEAT;

Algorithm description ends

Then the following statements are true:

1. **The following recurrence sequence can be used to find p_i :**
   $p_0 = \lambda_A$
   $p_1 = \lambda_B$
   p_{i+2} = remainder(p_i , p_{i+1}), for integers $i \geq 0$.
2. **When $\lambda_B=(\lambda_A/g)$, then $p_i = (\lambda_A/g^i)$ for all integers $i \geq 0$.**
3. **$p_i \geq (\lambda_A/g^i)$, for all values of λ_A and λ_B.**
4. **The i^{th} term in the Fibonacci sequence, is the lower bound for both n_A and n_B. That is, everytime (absolute($n_A\lambda_A$ - $n_B\lambda_B$) < p_i) in the algorithm, each of $\{n_A, n_B\} \geq t_i$, where $t_{i+2} = t_{i+1} + t_i$, for integers $i \geq 0$, and $t_0=0$ and $t_1=1$.**

Proof: The algorithm described in this Theorem can be viewed as finding how close a zero of $\sin(\theta_A x)$ comes to a zero of $\sin(\theta_B x)$, where $\theta_A=2\pi/(2\lambda_A)=\pi/\lambda_A$ and $\theta_B=2\pi/(2\lambda_B)=\pi/\lambda_B$, as the real variable x is increased continuously from 0, and where the counter i is incremented by 1 each time the present absolute distance becomes lesser than the previous absolute distance, between these zeros. Consider the recurrence sequence:

$p_0 = \lambda_A$

$p_1 = \lambda_B$

p_{i+2} = remainder(p_i , p_{i+1}), for integers $i \geq 0$.

Writing out the first few terms, we get:

$p_0 = \lambda_A$

$p_1 = \lambda_B$

p_2 = remainder(p_0 , p_1)

p_3 = remainder(p_1 , p_2)

p_4 = remainder(p_2 , p_3)

p_5 = remainder(p_3 , p_4)

... and so on.

Denoting floor(p_i / p_{i+1}) = M_i , we can write:
$p_0 = \lambda_A - 0\ \lambda_B$
$p_1 = \lambda_B - 0\ \lambda_A$
$p_2 = p_0 - M_0\ p_1 = \lambda_A - M_0\ \lambda_B$
$p_3 = p_1 - M_1\ p_2 = \lambda_B(1 + M_0M_1) - \lambda_A(M_1)$
$p_4 = p_2 - M_2\ p_3 = \lambda_A(1 + M_1M_2) - \lambda_B(M_0 + M_2 + M_0M_1M_2)$
$p_5 = p_3 - M_3\ p_4 = \lambda_B(1 + M_0M_1 + M_0M_3 + M_2M_3 + M_0M_1M_2M_3) - \lambda_A(M_1 + M_3 + M_1M_2M_3)$
...
and so on.

It follows that if p_i has $x_{A,i}$ product terms in the "sum of products" coefficient of λ_A and $x_{B,i}$ product terms in the "sum of products" coefficient of λ_B, then $x_{A,i+2} = x_{A,i+1} + x_{A,i}$, and $x_{B,i+2} = x_{B,i+1} + x_{B,i}$. The following patterns are straightforward by induction:

1. the number of product terms in the coefficient of λ_B follows the well-known Fibonacci sequence t_i : 0, 1, 1, 2, 3, 5, 8, 13, etc, where $t_{i+2} = t_{i+1} + t_i$, starting from p_0.
2. the number of product terms in the coefficient of λ_A follows the Fibonacci sequence : 0, 1, 1, 2, 3, 5, 8, 13, etc, starting from p_1.
3. the coefficient of λ_B in p_i has the product term $M_0M_1M_2...M_{i-2}$.
4. the coefficient of λ_A in p_i has the product term $M_1M_2...M_{i-2}$.

The exact formula for the n^{th} term of the Fibonacci sequence is well-known to be $t_n = (1/\text{sqrt}(5))(((1+\text{sqrt}(5))/2)^n - ((1-\text{sqrt}(5))/2)^n)$. A well-known approximation formula for $t_n = g^n/\text{sqrt}(5)$, and the exact formula $t_n = \text{round}(g^n/\text{sqrt}(5))$ to the nearest integer, is true for all integers $n \geq 2$.

Plugging in the least possible values $M_i = 1$, for all positive integers i, one possible solution that allows this would be that $\lambda_B=(\lambda_A/r)$, and $p_{i+1} = (p_i / r)$ for all integers $i \geq 0$,where r is a real > 1. We are now in a position to find the value of r that allows this by writing these 3 equations:
$p_{i+2} = (p_{i+1} / r)$
$p_{i+1} = (p_i / r)$
$p_{i+2} + p_{i+1} = p_i$

From the first two equations, we get $p_{i+2} = (p_i / r^2)$. From the second equation, we get $p_{i+1} = (p_i / r)$. Plugging these into the third equation, and eliminating p_i, we get $(1/r)(1+(1/r))=1$, which means $r = (1+(1/r))$, and the only solution for r that satisfies this is the golden ratio g. Hence, $p_i = \lambda_A/g^i$ when $\lambda_B=(\lambda_A/r)$, for integers $i \geq 0$. Since $t_i = t_{i-1} + t_{i-2}$, we know that $t_{i-2} = t_i - t_{i-1}$, implying that after every t_{i-2} consecutive values of n_A in the algorithm, $p_{i+1} = p_i/g$.

Another way of looking at why ensuring $p_{i+1} = p_i/g$ for all integers $i \geq 2$ leads to the fastest possible reduction in p_i, is to consider these cases of the value of K, where $p_{i+1} = (p_i/K)$ and $p_{i+1} < p_i$:

<u>Case 1</u>: K>=2. In this case, the coefficient of λ_B in p_i would be atleast $(t_i - 1 + M_0M_1M_2...M_{i-2})$ that is $(t_i -1 + K^{(i-1)})$ by putting each M_i=floor(K). Since our final aim is to look for n beyond which $(\text{ratio})^\wedge(t_n -1 + K^{(n-2)}) > (K^\wedge n)$, where "ratio" in the LHS represents the ratio (>1) between the largest root modulus to the next largest root modulus. As the LHS grows at a faster rate wrt n, the inequality is satisfied after a much smaller value of n, when K>=2. Note that the growth of the RHS represents the rate at which p_i decreases with respect to i.

<u>Case 2</u>: 1<K<2. In this case, $p_{i+2} = p_i - p_{i+1} = p_{i+1}(K-1)$. We can further consider three sub-cases:

<u>Subcase 2.1</u>: g<K<2. In this subcase, (g-1)<(K-1)<1, so $p_{i+1}(g-1) < p_{i+1}(K-1) < p_{i+1}$, so $p_{i+1}/g < p_{i+2} < p_{i+1}$, showing that p_{i+2} has not reduced much compared to p_{i+1}, and it also leads to Subcase 2.2.

<u>Subcase 2.2</u>: 1<K<g. In this subcase, 0<(K-1)<(g-1), so $0 < p_{i+2} < p_{i+1}(g-1)$, which means $0 < p_{i+2} < p_{i+1}/g$. If $0 < p_{i+2} <= p_{i+1}/2$, it would lead to Case 1. If $p_{i+1}/2 < p_{i+2} < p_{i+1}/g$, it would lead to Subcase 2.1.

<u>Subcase 2.3</u>: K=g. In this subcase, $p_{i+2} = p_{i+1}(g-1) = p_{i+1}/g$, since it is known that 1/g = (g-1), and M_i=1 for every integer $i \geq 2$, and the value of p_i has the fastest reduction for the RHS.

In summary, Case 1, subcase 2.1 and subcase 2.2 each have slower reduction of p_i than subcase 2.3, with respect to i.

<u>Hence Proved Theorem 3</u>.

<u>Theorem 4</u>: Let λ be a given irrational constant. For every integer $n \geq 0$, let $L_B(n)$ be a lower bound on the values of both:

1. **minimum(absolute(cos(2mπ/λ)), over positive integers $m \leq n$).**
2. **minimum(absolute(sin(2mπ/λ)), over positive integers $m \leq n$).**

Then one choice for $L_B(n)$ is given by $L_B(t_n) = 4/(\lambda g^n)$, where $t_{n+2} = t_{n+1} + t_n$, and t_0=0 and t_1=1. This lower bound may be approximated by the formula $L_B(n) = (4/(\text{sqrt}(5)\ \lambda\ n))$ for large n.

Proof: It is clear that the distance of the zeros of $\sin(4\pi t/\lambda)$ from the nearest integer, decreases at a rate equal to or faster than that of both $\sin(2\pi t/\lambda)$ or $\cos(2\pi t/\lambda)$, as t is continuously increased from a small positive quantity. This is because the zeros of $\sin(2\pi t/\lambda)$ and the zeros of $\cos(2\pi t/\lambda)$, are each separately subsets of the zeros of $\sin(4\pi t/\lambda)$. The lower bound of the distance of the $t_n{}^{th}$ zero of $\sin(4\pi t/\lambda)$ from the nearest integer, is directly available from Theorem 3, by substituting $\lambda_A=1$ and λ_B=fraction(λ), which gives a lower bound of $1/g^n$. A good choice for $L_B(t_n)$ would therefore be the product of $1/g^n$ and $4/\lambda$, the slope of the line joining the origin (0,0) to the point $(\lambda/4,1)$. So $L_B(t_n) = 4/(\lambda g^n)$, where $t_{n+2} = t_{n+1} + t_n$, and $t_0=0$ and $t_1=1$. Using the well-known result that t_n=round(g^n/sqrt(5)), or that t_n = approximately (g^n/sqrt(5)) for large n, we can say that $L_B(t_n)=4/(\lambda g^n)$=approximately (4/(sqrt(5) λ t_n)) for large n. So $L_B(n)$=approximately (4/(sqrt(5) λ n)) for large n.
Hence Proved Theorem 4.

Theorem 5: Let λ be a given irrational constant. Let β be a real constant. For every integer n$\geq$0, let $L_C(n)$ be a lower bound on the non-zero absolute value of minimum(absolute($\cos(2m\pi/\lambda)+\cos(2\pi/\beta)$)), over positive integers m $\leq$ n). Then one choice for $L_C(n)$ is given by $L_C(t_n) = 4/(\lambda g^{n+c})$, where $t_{n+2} = t_{n+1} + t_n$, and t_0=0 and t_1=1, and where c is a non-negative constant integer dependent on β and λ. This lower bound may be approximated by the formula $L_C(n)$ = (4/(sqrt(5) λ g^c n)) for large n.
Proof: This Theorem is a more generalized version of Theorem 4. We can write $(\cos(2m\pi/\lambda)+\cos(2\pi/\beta))$ = -2 $\sin(\pi((m/\lambda)+(1/\beta)))$ $\sin(\pi((m/\lambda)-(1/\beta)))$. So $L_C(n)$ for $(\cos(2m\pi/\lambda)+\cos(2\pi/\beta))$ = 2 minimum($L_C(n)$ for $\sin(\pi((m/\lambda)+(1/\beta)))$, $L_C(n)$ for $\sin(\pi((m/\lambda)-(1/\beta)))$). $L_C(n)$ for $\sin(\pi((m/\lambda)\pm(1/\beta)))$, can be $\leq L_B(n)$ for $\sin(\pi m/\lambda)$. This is because $\sin(\pi m/\lambda)=0$ when m=0, but then is situated at a distance of remainder($1/\lambda$) from the subsequent zero. Contrast this with the fact that absolute($\sin(\pi((m/\lambda)\pm(1/\beta)))$)>0 at m=0, but is situated at a distance of remainder($(1+\beta),\lambda$) or remainder($(1-\beta),\lambda$) from the next closest 0, either of which can be lesser than remainder($1/\lambda$). Note that in some cases, it can be greater too, however, we are interested only in a worst-case scenario. So absolute($\sin(2\pi m/\lambda)+\cos(2\pi/\beta)$) can get an initial constant advantage in being closer to a zero, over absolute($\sin(2\pi m/\lambda)$) for the same value of m>0. This has an effect of $L_C(t_n) = 4/(\lambda g^{n+c})$, where integer constant c>0 depends on the values of β and λ. Using t_n = approximately (g^n/sqrt(5)) for large n, we can say that $L_C(t_n)=4/(\lambda g^{n+c})$=approximately (4/(sqrt(5) λ g^c t_n)) for large n. So $L_C(n)$=approximately (4/(sqrt(5) λ g^c n)) for large n.
Hence Proved Theorem 5.

Having established the lower bounds of the absolute non-zero values of the cosine functions in $s_{n,j,k}$, we are now in a position to use them to develop our deterministic algorithms.

Theorem 6: Let G(x) be such that, for every pair of complex roots of G(x), (r_k e^($i\theta_k$)) and ($r_{k'}$ e^($i\theta_{k'}$)), this statement is true: ($r_k=r_{k'}$)$\rightarrow$(absolute(θ_k)=absolute($\theta_{k'}$)). Then there exists a deterministic algorithm to find the largest non-periodic zero of f_n.
Proof: This Theorem basically states that if the absolute values of arguments of every pair of complex roots of G(x) of equal moduli, are equal, then there exists a deterministic algorithm for finding an upper bound of the largest non-periodic zero of f_n. The algorithm is described below:

1. From Theorem 1, we can write f_n =
($r_1{}^n \cos(n\theta_1)$) ($d_{0,1} + d_{1,1}$ n + $d_{2,1}$ n^2 + ... + $d_{L-1,1}$ n^{L-1})) +
($r_2{}^n \cos(n\theta_2)$) ($d_{0,2} + d_{1,2}$ n + $d_{2,2}$ n^2 + ... + $d_{L-1,2}$ n^{L-1})) +
... +
($r_L{}^n \cos(n\theta_L)$) ($d_{0,L} + d_{1,L}$ n + $d_{2,L}$ n^2 + ... + $d_{L-1,L}$ n^{L-1})),
such that $r_1 > r_2 > ... > r_L$.
2. Set k=1.
3. Set w=0.
4. While ((θ_k is a rational multiple of π) and (k<L)).
5. {
6. k = k+1.
7. }
8. If (θ_k is an irrational multiple of π)
9. {
10. Set $L_{B,k,n}$ = (4/(sqrt(5) λn)) = ($2\theta_k$/(sqrt(5) πn)), as described in Theorem 4, which is the lower bound of $\cos(n\theta_k)$.
11. Set w = value of n, beyond which ($r_k{}^n$ $L_{B,k,n}$ absolute($d_{0,k}$ + $d_{1,k}$ n + $d_{2,k}$ n^2 + ... + $d_{L-1,k}$ n^{L-1})) $\geq$ SUM(($r_{k'}{}^n$ (absolute($d_{0,k'}$) + absolute($d_{1,k'}$)n + absolute($d_{2,k'}$)n^2 + ... + absolute($d_{L-1,k'}$)n^{L-1}) , over integers k' in [k+1, L]).
12. } OUTPUT w

The existence of w in step 11 of the above algorithm, is proved by dividing both sides of the inequality by (r_k^n). On doing so, the LHS becomes $L_{B,k,n}$ absolute$(d_{0,k} + d_{1,k}\, n + d_{2,k}\, n^2 + ... + d_{L-1,k}\, n^{L-1})$, and the RHS eventually tends to 0 faster due to the presence of n^{th} powers of positive reals less than 1.
<u>Hence Proved Theorem 6</u>.

<u>Theorem 7</u>: Let G(x) be such that, for every pair of complex roots of G(x), $(r_k\, e\text{^}(i\theta_k))$ and $(r_{k'}\, e\text{^}(i\theta_{k'}))$, this statement is true: $(r_k=r_{k'})\rightarrow$(There exist non-zero integers q_k and $q_{k'}$ such that $(q_k\theta_k=q_{k'}\theta_{k'})$). Then there exists a deterministic algorithm to find an upper bound of the largest non-periodic zero of f_n.
<u>Proof</u>: This Theorem basically states that if the absolute values of arguments of every pair of complex roots of G(x) of equal moduli, are commensurable, then there exists an algorithm for finding an upper bound of the largest non-periodic zero of f_n. In such a situation, from Theorem 1, we can write f_n = SUM($(n^j\, r_{V'}^n\, s_{n,j,V'}$), over all integers j in [0,L-1]), over all subsets $V'=\{k_1, k_2\ ...\ , k_v\}$ of integers belonging to [1,L] such that $r_{k\_1} = r_{k\_2} = ... = r_{k\_v} = r_{V'}$). Here, v denotes the cardinality of subset V'. Also, here, each $s_{n,j,V'} = (d_{j,k\_1} \cos(n\theta_{k\_1}) + d_{j,k\_2} \cos(n\theta_{k\_2}) + ... + d_{j,k\_v} \cos(n\theta_{k\_v}))$. Since there exist non-zero integers $\{q_1, q_2, ... , q_v\}$ such that $(q_1\theta_{k1}=q_2\theta_{k2}=...=q_v\theta_v)$, we can denote LCM$(q_1, q_2, ... , q_v)$=y. Then let $\theta_{V'}$ be such that $(q_1\theta_{k1}=q_2\theta_{k2}=...=q_v\theta_v=y\theta_{V'})$. Then write $\cos(n\theta_{k1})=\cos(ny\theta_{V'}/q_1)$, $\cos(n\theta_{k2})=\cos(ny\theta_{V'}/q_2)$, ... , $\cos(n\theta_{kv})=\cos(ny\theta_{V'}/q_v)$. Note that each (y/q_i) evaluates to an integer, for each integer i in [1,v]. Use the cosine expansion formula of expressing cos(mx) as a univariate polynomial of cos(x), where m is an integer, to express each $s_{n,j,V'}$ as a univariate polynomial in $\cos(n\theta_{V'})$ with real-algebraic coefficients. Factorize this polynomial to obtain all its roots. Assume that the factorized polynomial is $K((\cos(n\theta_{V'})-r_1)(\cos(n\theta_{V'})-r_2)...(\cos(n\theta_{V'})-r_{d\_V'})(\cos(n\theta_{V'})-c_1)(\cos(n\theta_{V'})-c_2)...(\cos(n\theta_{V'})-c_{e\_V'}))$, where each of $\{r_1, r_2, ... , r_{d\_V'}\}$ is a real-algebraic constant, and where each of $\{c_1, c_2, ... , c_{e\_V'}\}$ is a complex-algebraic constant, and K is real-algebraic. Note that $(\cos(n\theta_{V'})-c_1)(\cos(n\theta_{V'})-c_2)...(\cos(n\theta_{V'})-c_{e\_V'}))$ will itself evaluate to an $e_{V'}$ degree univariate polynomial in $\cos(n\theta_{V'})$ with real-algebraic coefficients, without a real root. The minimum absolute value of a rootless univariate polynomial with real-algebraic coefficients, will be a real-algebraic constant, hence we may denote the minimum absolute value of $(\cos(n\theta_{V'})-c_1)(\cos(n\theta_{V'})-c_2)...(\cos(n\theta_{V'})-c_{e\_V'}))$ as K_c, which is a real-algebraic constant. We can now say that LowerBound$(s_{n,j,V'}) \geq$ (K K_c PRODUCT(LowerBound(absolute($(\cos(n\theta_{V'})-r_i)$))), over integers i in $[1,d_{V'}]$). We know from Theorem 5, that a good choice for LowerBound(absolute($(\cos(n\theta_{V'})-r_i)$)) is $(4/(\mathrm{sqrt}(5)\ \lambda_{V'}\ g^{c\_i}\ n)) = (2\theta_{V'}/(\mathrm{sqrt}(5)\ \pi g^{c\_i} n))$, where $\gamma_{i,V'}$ is a constant positive integer, and where $\lambda_{V'}=2\pi/\theta_{V'}$. Therefore, LowerBound$(s_{n,j,V'}) \geq$ (K K_c PRODUCT($(2\theta_{V'}/(\mathrm{sqrt}(5)\ \pi g^{c\_i} n))$, over integers i in $[1,d_{V'}]$)), which can be written as $(K_{V'}(\theta_{V'}/\pi n)^{d\_V'})$, where $K_{V'}$ is a real-algebraic constant. Denote $P_{V'\_k,\ n,\ theta\_V'\_k}$ as a univariate real-algebraic polynomial in $\cos(n\theta_{V'\_k})$, for each integer k in [1,L]. Then our deterministic algorithm is described below:

1. From Theorem 1, we can write f_n =

$(r_{V'\_1}^n\, P_{V'\_1, n, theta\_V'\_1}(e_{0,V'\_1} + e_{1,V'\_1}\, n + e_{2,V'\_1}\, n^2 + ... + e_{L-1,V'\_1}\, n^{L-1}))$ +
$(r_{V'\_2}^n\, P_{V'\_2, n, theta\_V'\_2}(e_{0,V'\_2} + e_{1,V'\_2}\, n + e_{2,V'\_2}\, n^2 + ... + e_{L-1,V'\_2}\, n^{L-1}))$ +
... +
$(r_{V'\_L}^n\, P_{V'\_L, n, theta\_V'\_L}(e_{0,V'\_L} + e_{1,V'\_L}\, n + e_{2,V'\_L}\, n^2 + ... + e_{L-1,V'\_L}\, n^{L-1}))$,
such that $r_{V'\_1} > r_{V'\_2} > ... > r_{V'\_L}$.

2. Set k=1.
3. Set w=0.
4. While (($\theta_{V'\_k}$ is a rational multiple of π) and ($k<L$)).
5. {
6. k = k+1.
7. }
8. If ($\theta_{V'\_k}$ is an irrational multiple of π)
9. {
10. Set $L_{B,k,n} = (K_{V'\_k}(\theta_{V'\_k}/\pi n)^{d\_V'\_k})$, the lower bound of $P_{V'\_k, n, theta\_V'\_k}$, where d_V'_k is an integer constant.
11. Set w = value of n, beyond which $(r_{V'\_k}^n\, L_{B,k,n}$ absolute$(e_{0,k} + e_{1,k}\, n + e_{2,k}\, n^2 + ... + e_{L-1,k}\, n^{L-1})) \geq$ SUM($(r_{V'\_k'}^n$ (absolute$(e_{0,V'_k'})$ + absolute$(e_{1,V'_k'})n$ + absolute$(e_{2,V'_k'})n^2$ + ... + absolute$(e_{L-1,V'_k'})n^{L-1}$)) , over integers k' in [k+1, L]).
12. }
13. OUTPUT w

The existence of w in step 11 of the above algorithm, is proved by dividing both sides of the inequality by $(r_{V'\_k}^n)$. On doing so, the LHS becomes $L_{B,k,n}$ absolute$(e_{0,k} + e_{1,k}\, n + e_{2,k}\, n^2 + ... + e_{L-1,k}\, n^{L-1})$, and the RHS tends to 0 faster due to the presence of n^{th} powers of positive reals less than 1.
<u>Hence Proved Theorem 7</u>.

For our algorithms to be truly deterministic, one important criterion is that they can run on finite precision computers, and also be made to consume only finite-sized input data. So it is worth noting here that each of the cosines of arguments described so far are actually algebraic, i.e. each of $\{\cos(\theta_k), \cos(\pi\varphi_{rat,i}), \cos(\pi\gamma_{rat,i}), \cos(\pi\varphi_{real,i}), \cos(\pi\varphi_{real,i})\}$ is actually algebraic after factoring them from the characteristic polynomial G(x). However their respective arguments need not be algebraic, i.e., each of $\{\theta_k, \varphi_{rat,i}, \gamma_{rat,i}, \varphi_{real,i}, \varphi_{real,i}\}$ need not be algebraic and can be only described as being real. And the formulae described in this paper for the lower bounds involve the arguments and not the cosines of the arguments. Still, one can approximate the argument to some desired accuracy, since our aim is to finally obtain a lower bound, and not an exact value. For example, in the lower bound formula of step 10 in Theorem 6, we have $L_{B,k,n} = (4\theta_k/(\text{sqrt}(5)\ \pi n))$, where non-algebraic numerator terms need to be rounded down and denominator terms need to be rounded up. So if θ_k=1.56784... and π=3.14159..., simply choose a desired accuracy of 3 significant figures by approximating down θ_k=1.56 and approximating up π=3.15 to get a final lower bound figure that is algebraic. The only thing to keep in mind is the more the number of significant figures considered, one is likely to develop a lower bound that is as high as possible, so that the upper bound to the largest non-periodic zero is as low as possible.

Now that we have described deterministic algorithms for the special cases of G(x), our next task will be to attempt to generalize the same for all cases of G(x). As should be evident now, the key concept on which our paper is based to obtain the upper bound of the largest zero in set Q of f_n, is the determination of an effective lower bound for the absolute non-zero value of the summation of weighted cosines of arguments of roots, and then using the fact that the ratio (any ratio lesser than 1) of root moduli raised to the power of n eventually decreases at a much faster rate. In an attempt to get an effective lower bound of this weighted sum of m cosines (m>2), the next Theorem 8 finds the lower bound of the distance between every x^{th} zero.

<u>Theorem 8</u>: Let m be a given positive integer constant. Let d_k and θ_k each be a given real constant, for each integer k in [1,m]. Let s_t = SUM(($d_k \cos(t\theta_k)$) , over all integers k in [1,m]), for real variable t≥0. Let θ_{max} = maximum(θ_k, over all integers k in [1,m]). There exists a lower bound $L_{m,t} = (\pi / (\theta_{max}\ \text{sqrt}(5)\ t)$ between every $(2^{m+1})^{th}$ zero of s_t.

<u>Proof</u>: For each ($d_k \cos(t\theta_k)$), divide the domain of t≥0 into alternating convex and concave spaces with period of $t\theta_k$ equal to 2π, for integers n≥0:

1. concave spaces in $t\theta_k \in [2n\pi - \pi/2, 2n\pi + \pi/2[$.
2. convex spaces in $t\theta_k \in [2n\pi + \pi/2, 2n\pi + 3\pi/2[$.

Now divide the domain of t≥0 into blocks, not necessarily periodic, such that each block is bounded by a zero of $\cos(t\theta_k)$ for some integer k in [1,m]. So the lower bound of the width of a block is the lower bound of the distance of a zero of $\cos(t\theta_k)$ from a zero of $\cos(t\theta_{k'})$, where integers k and k' are not equal. Denote θ_{max}=maximum(θ_k, over all integers k in [1,m]), since $\cos(t\theta_{max})$ would have the least wavelength. Applying the result of Theorem 3, we get the lower bound of a block's width $L_{m,t}$ to approximately be $(2\pi / (2\theta_{max}\ \text{sqrt}(5)\ t)) = (\pi / (\theta_{max}\ \text{sqrt}(5)\ t))$.

It is obvious that each block will contain either a convex space or a concave space of ($d_k \cos(t\theta_k)$), but not both, for each integer k in [1,m]. This means that there are 2^m types of space combinations in each block.

An inductive argument will prove that there can be no more than (2^{m+1}) zeros inside a block. Consider the SUM(($d_k \cos(t\theta_k)$) , over all integers k in [1,1]) within a block. It has a single piecewise function that is either concave or convex with not more than 2 zeros. Now consider the SUM(($d_k \cos(t\theta_k)$) , over all integers k in [1,2]) within a block. If both $\cos(t\theta_1)$ and $\cos(t\theta_2)$ are convex, then the resulting sum is also convex, and the maximum number of zeros remains 2. Similarly, if both $\cos(t\theta_1)$ and $\cos(t\theta_2)$ are concave, then the resulting sum is also concave, and the maximum number of zeros remains 2. But when one is convex and the other is concave, then the maximum number of piecewise concave functions in the sum is doubled, and so is the maximum number of concave functions and so is the maximum number of zeros in the sum. Now consider the SUM(($d_k \cos(t\theta_k)$) , over all integers k in [1,q]) where q<m within a block, and assume that it has a maximum of 2^{q+1} piecewise functions, that are alternatingly convex and concave, with a maximum of 2^{q+1} zeros. When we now add a convex or concave part of ($d_{q+1} \cos(t\theta_{q+1})$) to the sum, each of {maximum number of piecewise concave functions, maximum number of piecewise convex functions, maximum number of zeros} is doubled. Therefore, the maximum number of zeros in s_t within a block, becomes 2^{m+1}. Since we have not established a lower bound on the distance between each of these zeros within a block, we can only safely say that $L_{m,t} = (\pi / (\theta_{max}\ \text{sqrt}(5)\ t))$ is the lower bound of the distance between every $(2^{m+1})^{th}$ zero of s_t.

<u>Hence Proved Theorem 8</u>.

We now discuss some areas for future work.

3. Future Work

Though Theorem 8 establishes the lower bound of the horizontal distance between every x^{th} zero, where x is an integer constant, we are yet to find the lower bound of the non-zero value of $(d_1 \cos(n\theta_1) + d_2 \cos(n\theta_2) + ... + d_m \cos(n\theta_m))$, since:

1. We have not yet identified a lower bound of the distance between successive zeros of s_t.
2. We have not yet identified a lower bound to the absolute value of the slope of s_t near a zero, unlike in the case of m=1 where the slope at a point close to a zero of $(d_1 \cos(n\theta_1))$ tends to $(d_1\theta_1)$.

Let us conveniently assume that s_t is equivalent to a series of cosine waves, placed serially one after the other, each of progressively lower wavelength equal to the lower bound of Theorem 8. Though this assumption is obviously false, we can use this to make a conjecture. Applying the Theorem 3 to the current wavelength at n yields $(\pi / (2\theta_{max} g^n \text{sqrt}(5) n))$, which could be used as the lower bound of the absolute horizontal distance of s_t from the nearest 0. Another idea would be to evaluate the $\text{SUM}((\pi / (2\theta_{max} g^n \text{sqrt}(5) n))$, over integers n in [1,T]), which is approximately $((\pi/(2\theta_{max} \text{sqrt}(5))) \ln(T))$, where ln(T) denotes the natural logarithm of T. The fraction of this value could be the absolute lower bound of the horizontal distance of $(d_1 \cos(n\theta_1) + d_2 \cos(n\theta_2) + ... + d_m \cos(n\theta_m))$ from the nearest zero. That horizontal distance multiplied by the lower bound of the absolute non-zero slope would be a lower bound on the absolute non-zero value of $(d_1 \cos(n\theta_1) + d_2 \cos(n\theta_2) + ... + d_m \cos(n\theta_m))$, using which an algorithm can be then developed on the similar lines of Theorems 6 and 7, to find the largest non-periodic zero of the generic f_n.

The study of zeros of a weighted sum of three or more trigonometric functions, is already a well known topic [8] under the study of zeros of the Riemann Zeta Function, so ideas from there could be applied here.

4. Conclusion

We presented a deterministic algorithm to determine an upper bound for the largest non-periodic zero for an integer linear homogeneous recurrence f_n, when the absolute values of arguments of every pair of complex roots of its characteristic polynomial G(x) of equal moduli, are either equal or commensurable. Our algorithm is based on the concept that a ratio of reals raised to the power of n, as long as the ratio is lesser than 1, decreases at a faster rate than the rate at which the eventual lower bound decreases for the absolute non-zero value of $\cos(\pi a n)$ where a is an irrational constant. We then discussed the difficulty in extending this concept to the generic case of G(x) since one needs to calculate the lower bound of the absolute value of a weighted sum of m cosine terms, which is more challenging. We were able to only develop a lower bound between a zero and the x^{th} successive zero for the weighted sum of m cosines. We finally presented some directions for future work.

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I, Deepak Ponvel Chermakani, wrote this paper, which is original to the best of my knowledge, out of my own interest and initiative during my spare time. I completed a fulltime two-year Master of Science Degree in Electrical Engineering from University of Hawaii at Manoa USA (www.hawaii.edu) in Aug 2015, a fulltime one-year Master of Science Degree in Operations Research with Computational Optimization from University of Edinburgh UK (www.ed.ac.uk) in Sep 2010, a fulltime four-year Bachelor of Engineering Degree in Electrical and Electronic Engineering, from Nanyang Technological University Singapore (www.ntu.edu.sg) in Jul 2003, and fulltime high schooling from National Public School Indiranagar in Bangalore in India in Jul 1999. I am most grateful to my parents (my mother Mrs. Kanaga Rathinam Chermakani and my father Mr. T. Chermakani) for their sacrifices in educating me and bringing me up.