THE TOTAL MECHANICAL ENERGY (Δ): THEORY AND TYPICAL APPLICATIONS

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Abstract. The principles of science, whose history can be traced back to centuries of presence in scientific writings and whose fidelity appears to have been verified through a variety of experiments and applications, could still need some corrections or even supplementary proposals so that their content becomes whole. The principle of energy conservation is at least two hundred years old and is one of the most undeniable principles, with universal application in classical and quantum physics. However, as we expand in this paper, dynamic energy could be defined more precisely by including cases that deviate from it. In other words, we highlight and fill a scientific gap relevant to the principle of energy conservation. Of course, this historical scientific principle cannot be restated. For scientific as well as pedagogical reasons, however, we highlight our findings and note the fields, both traditional and modern, of classical and quantum physics, that researchers must take into account to fully understand an object under study. We also introduce and apply the conservation of total mechanical energy in the cases of terminal velocity (falling object), and forced oscillation in classical mechanics as well as in the case of quantum mechanics.

Keywords:*Energy conservation; Potential Energy; Kinetic Energy;Total Mechanical Energy;Total Potential Energy*

1. Introduction

The concept of energy is one of the oldest in the history of Physics [1-2]. The total energy of a system is consisted of its dynamic and kinetic energy. Kinetic energy is determined by the motion of a body - or the composite motion of smaller bodies that make it up - and the dynamic energy expresses the body's ability to change its motion. It generally depends on the position of the body within a field and can be stored inside that field [3]. The word “energy”originates from ancient Greek and means "activity, function" [4] which also appears in the work of Aristotle in the 4th century BC. In the late 17th century, Gottfried Leibniz proposed the idea of the Latin: vis viva, or living force, defined as the squared product of an object's mass with its velocity. Leibniz believed that the total vis viva is a conserved quantity. In the early 18th century, Émilie du Châtelet proposed the concept of conservation of energy in the margins of her French translation of Newton's Principia Mathematica, which represented the first formulation of a conserved measurable quantity different from momentum and which later was called "energy". In 1807, Thomas Young was probably the first to use the term "energy" instead of vis viva, in the modern sense. These developments led to the theory of energy conservation, largely formalized by William Thomson (Lord Kelvin) as the field of thermodynamics. Since 1918, according to Noether's theorem, theorists have understood that the conservation of energy is a consequence of the fact that the laws of physics do not change over time and that the law of energy conservation is the direct mathematical consequence of symmetry of a quantity coupled with energy, i.e. time. In quantum mechanics, the term energy is defined by the energy operator (Hamiltonian), and the Schrödinger equation correlates the energy operator to the total energy of a particle or system. Its results can be the definition of energy measurement in quantum mechanics. The Schrödinger equation describes the space and time dependence of slowly varying (non-relativistic) wave function of quantum systems. The solution of this equation for a bound system is discrete (a set of permitted states, each characterized by an energy level) and leads to the concept of quanta. By solving the Schrödinger equation for any oscillator (vibrator), and also for electromagnetic waves in vacuum, the resulting energy levels are related to the frequency by the Planck relation of $E = h\nu$, (where h is the Planck constant and ν the frequency). In the case of an electromagnetic wave, these energy levels are called quanta of light or photons.

The structure of this paper consists of the Methods (second sector), where new physical quantities are introduced, while the third sector provides the physical meaning of the proposed quantities. Then, two examples include the terminal (limiting) velocity and forced oscillation. Afterward, in the fifth sector, we mention the forced oscillation in Quantum Mechanics while we introduce the term Total Dynamic Energy into it. Finally, in the sixth section, the conclusions are given.

2. Methods

In this paragraph, we will methodologically develop the introduction and presentation of the new Physical Quantities by successively following the following steps:

- Definition of Total Dynamic(Potential) Energy (Λ) (newly introduced quantity)
- Formulation of the principle of energy conservation for the case of Total Dynamic (Potential) Energy (newly introduced quantity)
- Physical Meaning of Total Dynamic Energy
- Definition of Total Mechanical Energy (Δ) (newly introduced quantity)
- New theorem of the Total Mechanical Energy
- Typical examples
- Proof of the new theorem
- Extension of physical concepts in Quantum Mechanics

Let's have a body of mass m on which n forces F_1, $F_2, \dots, F_\nu$. are applied. For these forces and displacement of the body from initial point (A) to final point (B) the work-energy theorem is written as [5-6]:

$$T_B - T_A = W_{F_1} + W_{F_2} + \cdots + W_{F_\nu} \quad (1)$$
$$= \int_A^B \boldsymbol{F_1} \cdot d\boldsymbol{r} + \int_1^2 \boldsymbol{F_2} \cdot d\boldsymbol{r} + \cdots + \int_1^2 \boldsymbol{F_\nu} \cdot d\boldsymbol{r}$$
$$= \int_A^B (\boldsymbol{F_1} + \boldsymbol{F_2} + \cdots + \boldsymbol{F_\nu}) \cdot d\boldsymbol{r}$$

$$T_B - T_A = \int_A^B \boldsymbol{F_{tot}} \cdot d\boldsymbol{r} \tag{2}$$

where: $\boldsymbol{F_{tot}} = \boldsymbol{F_1} + \boldsymbol{F_2} + \cdots + \boldsymbol{F_v}$

The conservative or otherwise known as the preservative force relies on the independence of the path from the work production (e.g. movement of a particle from point A to point B regardless of the intermediate path). Any conservative force can be related to the gradient of a scalar function [6-7]. A consequence of this fact is the equivalent proposition that (a) the work of a conservative force is independent of the path it follows, and that (b) along a closed path, the work of a conservative force is zero. As a scalar function or monometer physical quantity, is called that physical quantity, which is adequately described by a single number (i.e. its measure):

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r}) \tag{3}$$

If the constitutive force($\boldsymbol{F_{tot}}$) is conservative, it can be expressed as the negative slope of a scalar function of body position:

$$\boldsymbol{F_{tot}} = -\boldsymbol{\nabla} \Lambda(\boldsymbol{r}) \tag{4}$$

or else known as $W = \oint_c \overrightarrow{F_{tot}} \cdot d\vec{r} = 0$where Λ is called total potential or total potential energy (total dynamic energy), and then relation (1) is written:

$$T_B - T_A = \int_A^B -\boldsymbol{\nabla} \Lambda(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_A^B -d\Lambda = \Lambda_A - \Lambda_B \tag{5}$$

3. Physical Meaning

The function Λ expresses the total energy that can potentially be the kinetic energy of the body of mass m.By definition, Λ has a value of zero at the point in space where the body acquires maximum kinetic energy, which does not have the possibility (i.e. dynamics) to further increase its kinetic energy (equation 6):

$$T_2 - T_1 = \Lambda_1 - \Lambda_2$$

$$T_1 + \Lambda_1 = T_2 + \Lambda_2$$

$$\Delta_1 = \Delta_2 \tag{6}$$

where Δ = T + Λ is called the Total Mechanical Energy of a body.

3.1 Theorem

The introduced theorem mentions that "*if the combined acting force on a body is conservative, regardless of whether the individual ones of which it is composed are conservative or not, then the total mechanical energy of the body (Δ) is a conserved quantity*".

It is notable though that perhaps, at first reading, one might assume that the existence of these new quantities, total potential energy (Λ), and total mechanical energy (Δ) do not represent anything new. The following explanations can clarify any emerging questions:

- When all the acting forces on a body are conservative, or cumulatively conservative for any path, then the work of the non-conservative ones equals zero. In turn, indeed, the total potential energy (Λ) is identical to the potential energy (U), and the total mechanical energy (Δ) is identical to mechanical energy (E).
- However, in cases where the constitutive (resultant) force is conservative and the cumulative work of the non-conservative ones for any path is different from zero, then the mechanical energy (E) is not a conservative quantity in contrast to the total mechanical energy (Δ). In this attribute lies precisely the focal point of this study, which adequately interprets the total energy of motion concerning the possibility of changing the state of a quantity of energy.

4. Typical application examples

In this sector, we show the application of the aforementioned inside two systems, thus we give an extension of one of these cases of classical mechanics to quantum mechanics (in the next section).

4.1 Terminal velocity

Let's take the case of a body (mass) falling under the effect of its weight and accept air resistance to be proportional to its speed and opposite to its motion ($F = -bv$), where b is the airresistanceconstantwhichdepends on the geometriccharacteristics of the movingbody and fromairdensity (Figure 1). At some point, it reaches the terminal velocity and continues to fall with it. The mechanical energy of the mass given by equation (7) constantly decreases [5, 8].

$$E = T + U = \frac{1}{2}mv^2 + mgh \qquad (7)$$

As mechanical energy of the body in this case, we mean the sum of its kinetic and dynamic gravitational energy, because by using a stricter description of mechanical energy, in the condition of even one non-conservative force, this definition of energy is not valid. Moreover, the total mechanical energy of the body is equal to $\Delta = 0$ since the constitutive force exerted on it is given by $F_{tot} = 0$. Since $\Delta = 0$, this value expresses that the body does not have the ability (i.e. the dynamics) to increase its kinetic energy. Nevertheless, the principle of energy conservation dictates something different; the "change" in mechanical energy is delivered in the form of heat to the environment (i.e. $\Delta E = Q$, where E is the mechanical energy of the system and Q is the heat delivered to the environment).

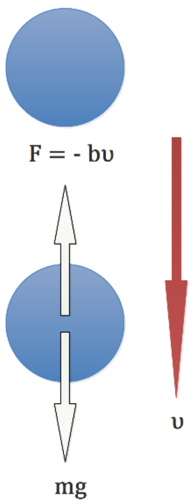

Fig. 1. Body fall

4.2 Forced oscillation

In this case, after the transient condition, the body performs simple harmonic oscillation. Consequently, let's take a horizontal spring performing forced oscillation (Figure 2). This choice simplifies the operations without affecting the generalization of the conclusions. We assume that on a body of m mass are applied the exciting force is $F_\delta = F_0 cos\omega t$, the decelerating (retarding) force$R = -bv = -b\frac{dx}{dt}$, and the spring force $F_{spr.} = -kx$. Then the 2nd Newton's law is found in (8):

$$F_0 cos\omega t - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2} \qquad (8)$$

The solution of this differential equation is provided by equation (9) [9-14]which is the equation of the oscillating body's motion and is an equation of motion of simple harmonic oscillation.:

$$x = Acos(\omega t + \delta) \qquad (9)$$

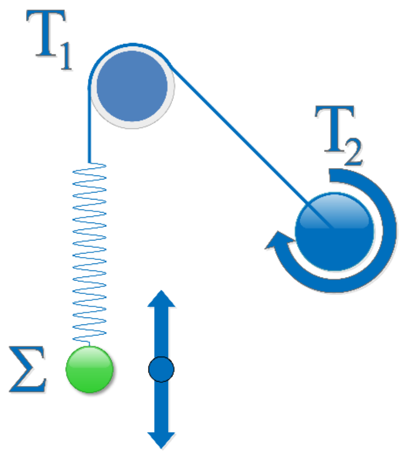

Fig. 2. Forced oscillation

By getting the derivative with respect to time we get the velocity of the body (equation 10):

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \delta) where v_{max} = \omega A \qquad (10)$$

If we again get the new derivative with respect to time we get the body acceleration:

$$\alpha = \frac{dv}{dt} = -\omega^2 Acos(\omega t + \delta) \; where \; \alpha_{max} = \omega^2 A \qquad (11)$$

Consequently, the total force acting on the body is given by:

$$F_{tot} = -m\omega^2 A\sigma\upsilon\nu(\omega t + \delta) = -m\omega^2 x \qquad (12)$$

We set$\Delta = D = m\omega^2$, and the formula of the constitutiveforceexerted on the bodygives:

$$F_{tot} = -Dx \qquad (13)$$

This constitutive force is conservative, i.e. there is a function Λ (total potential energy) such that:

$$F_{tot} = -\frac{d\Lambda}{dx} \qquad (14)$$

Λ is easily calculated (with the help of the previous two equations) as it is the derivative function of F_{tot} with respect to x and constitutes the equation of the total potential energy of the forced oscillation (equation 15):

$$\Lambda = \frac{1}{2} Dx^2 \qquad (15)$$

Moreover, for the oscillation the following applies:

$$K_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}DA^2 = \Lambda_{max} \quad (16)$$

So the total mechanical energy (Δ) in the forced oscillation is conserved, while the system's mechanical energy cannot be defined. This fact applies here because not all forces acting on the body are conservative.

Here the conservative force is only that of the spring$F_{spr} = -kx$, where (k) is the constant of the spring and (x) its displacement from the equilibrium position. The potential energy of (F) is given by$U_{F_{spr}} = \frac{1}{2}kx^2$. It is valid that $k = m\omega_0^2$, where (ω_0)is the natural frequency of the spring. We have seen earlier that$D = m\omega^2$, where (ω) is the frequency of the exciter which it imposes on the excited system.

Let's make the following hypothesis: It is notable that only for the special case of resonance, where the frequency of the exciter is identical to the natural frequency of the spring ($\omega=\omega_0$) applies that $D=k$. In any other case, where ($\omega\neq\omega_0$) then $D\neq k$. The proof is shown below:

$$cos^2(\omega t + \delta) = \frac{x^2}{A^2}$$

$$sin^2(\omega t + \delta) = \frac{v^2}{\omega^2 A^2}$$

By taking into consideration that $cos^2\varphi + sin^2\varphi = 1$ then we get:

$$\frac{x^2}{A^2} + \frac{v^2}{\omega^2 A^2} = 1 \Leftrightarrow \omega^2 x^2 + v^2 = \omega^2 A^2 \Leftrightarrow$$

$$\frac{1}{2}m\omega^2 x^2 + \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \Leftrightarrow$$

$$\frac{1}{2}Dx^2 + \frac{1}{2}mv^2 = \frac{1}{2}DA^2$$

Consequently we take the result of Λ + T = Δ. So the theorem of conservation of total mechanical energy for forced oscillation is verified. We must highlight the fact that we verified the principle of conservation of the total energy, but we didn't use the principle of conservation of energy to prove our hypothesis.

5. Forced oscillation in Quantum Mechanics

As we know from the Newtonian expression of forced oscillation, this is given by the following equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t) \quad (17)$$

whose Hamiltonian equals to $H = \frac{p^2}{2m}e^{-\lambda t} + (\frac{1}{2}\mathrm{k}x^2 - xF)e^{\lambda t}$. In this case the Schrödinger equation becomes:

$$-\frac{\hbar^2}{2m}e^{-\lambda t}\frac{\partial^2\Psi}{\partial \mathrm{x}^2} + (\frac{1}{2}\mathrm{k}x^2 - xF)e^{\lambda t}\Psi = i\hbar\frac{\partial \Psi}{\partial t} \tag{18}$$

However, in the specific case that we present in this paper, where we consider the total potential energy (Λ) of the system, which corresponds to its constitutive force(F_{tot}) and which, as we saw earlier, is calculated by the formula $F_{tot} = -\nabla\Lambda(r)$, the Hamiltonian takes the following form:

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \Lambda = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \tag{19}$$

And therefore the time-independent Schrödinger equation becomes:

$$\frac{\partial^2\Psi}{\partial x^2} = -\left(\frac{2m\Delta}{\hbar^2} - \frac{m^2\omega^2 x^2}{\hbar^2}\right)\Psi \tag{19}$$

which gives solutions based on the following equation [15]:

$$\Delta = \hbar\omega\left(n + \frac{1}{2}\right) \tag{20}$$

where Δ is the total mechanical energy of the system, which takes quantized eigenvalues. The difference here is that ω is the frequency of the external exciter and not the natural frequency (ω_0) of the system which is governed by the same equation (20) with respect to Δ.

6. Conclusions

It is noticeable that in cases where the composite force exerted on a body is conservative, regardless of whether the individual forces (of which it is composed) are conservative or not. Then there is a quantity, which should be counted among the constants of the study, and this is none other than what we defined as total mechanical energy (Δ). This quantity is found both in applications of classical physics and quantum mechanics. The existence of this quantity is not only of theoretical value but also practical as new cutting-edge technologies can emerge if modern researchers take it into account.

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