# Algorithmic computation of multivector inverses and characteristic polynomials in non-degenerate Clifford algebras

Dimiter Prodanov[1,2][0000−0001−8694−0535]

[1] Neuroelectronics Research Flanders, IMEC, Leuven, Belgium
[2] ITSDP, IICT, Bulgarian Academy of Science, Sofia, Bulgaria;
dimiter.prodanov@imec.be

**Abstract.** Clifford algebras provide the natural generalizations of complex, dual numbers and quaternions into the concept of non-commutative Clifford numbers. The paper demonstrates an algorithm for the computation of inverses of such numbers in a non-degenerate Clifford algebra of an arbitrary dimension. The algorithm is a variation of the Faddeev–LeVerrier–Souriau algorithm and is implemented in the open-source Computer Algebra System Maxima. Symbolic and numerical examples in different Clifford algebras are presented.

**Keywords:** multivector · Clifford algebra · computer algebra

## 1 Introduction

Clifford algebras provide the natural generalizations of complex, dual and split-complex (or hyperbolic) numbers into the concept of *Clifford numbers*. The development of Clifford algebras is based on the insights of Hamilton, Grassmann, and Clifford from the $19^{\text{th}}$ century. After a hiatus lasting many decades, the Clifford geometric algebra experienced a renaissance with the advent of contemporary computer algebra systems. Clifford algebras can be implemented in a variety of general-purpose computer languages and computational platforms. Recent years have seen renewed interest in Clifford algebra platforms: notably, for Maple, Matlab, Mathematica, Maxima, *Ganja.js* for JavaScript, *GaLua* for Lua, *Galgebra* for Python, *Grassmann* for Julia.

Computation of Clifford inverses has drawn attention in the literature [1, 4, 6]. The present contribution demonstrates an algorithm for Clifford number inversion, which involves only multiplications and subtractions and has a variable number of steps, depending on the maximal grade of the Clifford number. The algorithm is implemented using the Clifford Maxima package [5]. The algorithm is a direct translation of the Faddeev–LeVerrier–Souriau (FVS) algorithm for matrix inverse computation. The algorithm is in fact a proof certificate for the existence of an inverse. As a side product, the algorithm can compute the characteristic polynomial of the Clifford number and its determinant also without any resort to a matrix representation.

## 2 Notation and Preliminaries

$C\ell_n$ will denote a Clifford algebra of order $n$ but with unspecified signature. Clifford multiplication is denoted by simple juxtaposition of symbols. Algebra generators will be indexed by Latin letters. Multi-indices will be considered as index **lists** and not as sets and will be denoted with capital letters. The operation of taking k-grade part of an expression will be denoted by $\langle . \rangle_k$ and in particular the scalar part will be denoted by $\langle . \rangle_0$. Set difference is denoted by $\triangle$. Matrices will be indicated with bold capital letters, while matrix entries will be indicated by lowercase letters. The *scalar product* of the blades will be denoted by $*$.

**Definition 1.** *The generators of the Clifford algebra will be denoted by indexed symbol $e$. It will be assumed that there is an ordering relation $\prec$, such that for two natural numbers $i < j \implies e_i \prec e_j$. The **extended basis** set of the algebra will be defined as the ordered power set $\mathbf{B} := \{P(E), \prec\}$ of all generators $E = \{e_1, \ldots, e_n\}$ and their irreducible products.*

**Definition 2.** *Define the diagonal scalar product matrix as $\mathbf{G} := \{\sigma_{IJ} = e_I * e_J |\ e_I, e_J \in \mathbf{B}, I \prec J\}$.*

A Clifford number will be written as $A = a_1 + \sum_{k=1}^{r} \langle A \rangle_k = a_1 + \sum\limits_J a_J e_J$. The maximal grade of $A$ will be denoted by $\mathrm{gr}[A]$.

## 3 Clifford algebra real matrix representation map

In the present article we will focus on non-degenerate Clifford algebras, therefore the non-zero elements of $\mathbf{G}$ are valued in $\{-1, 1\}$.

**Definition 3 (Clifford coefficient map).** *Define the linear map acting element-wise $C_a : C\ell_n \mapsto \mathbb{R}$ by the action $C_a(ax + b) = x$ for $x \in \mathbb{R}, a, b \in \mathbf{B}$.*

*Define the Clifford **coefficient map** indexed by $e_S$ as $C_S(\mathbf{M}) := \mathbf{A}_S$, where $\mathbf{M}$ is the multiplication table of the extended basis $\mathbf{M} = \{\mathcal{R}(e_M e_N) \mid e_M, e_N \in \mathbf{B}\}$.*

**Definition 4 (Canonical matrix map).** *Define the map $\pi : \mathbf{B} \mapsto \mathbf{Mat}_{\mathbb{R}}(2^n \times 2^n)$, $n = p + q + r$ as $\pi : e_S \mapsto \mathbf{E}_s := \mathbf{G}\mathbf{A}_s$ where $s$ is the ordinal of $e_S \in \mathbf{B}$ and $\mathbf{A}_S$ is computed as in Def. 3.*

**Proposition 1.** *The $\pi$-map is linear.*

The proposition follows from the linearity of the coefficient map and matrix multiplication with a scalar.

**Theorem 1 (Semigroup property).** *Let $e_s$ and $e_t$ be generators of $C\ell_{p,q}$ . Then the following statements hold*

1. *The map $\pi$ is a homomorphism with regard to the Clifford product (i.e. $\pi$ distributes over the Clifford products): $\pi(e_s e_t) = \pi(e_s)\pi(e_t)$.*
2. *The set of all matrices $\mathbf{E}_s$ forms a multiplicative semigroup.*

*Proof.* Let $\mathbf{E}_s = \pi(e_s), \mathbf{E}_t = \pi(e_t), \mathbf{E}_{st} = \pi(e_s e_t)$. We specialize the result of Lemma 2 for $S = \{s\}$ and $T = \{t\}$ and observe that $m_{\lambda\lambda'}\, e_{st} = m_{\lambda\mu}\sigma_\mu m_{\mu\lambda'}\, e_{st}$ for $\lambda, \lambda', \mu \leq n$ and $\sigma_\lambda m_{\lambda\lambda'} = \sigma_\lambda m_{\lambda\mu}\sigma_\mu m_{\mu\lambda'}$. In summary, the map $\pi$ acts on $C\ell_{p,q}$ according to the following diagram:

$$\begin{tikzcd}
e_s \arrow[r, "\pi"] \arrow[d, "e_t"] & \mathbf{E}_s \arrow[d, "\mathbf{E}_t"] \\
e_s e_t \equiv e_{st} \arrow[r, "\pi"] & \mathbf{E}_{st} \equiv \mathbf{E}_s \mathbf{E}_t,\ st = s \cup t
\end{tikzcd}$$

Therefore, $\mathbf{E}_{st} = \mathbf{E}_s \mathbf{E}_t$. Moreover, we observe that $\pi(e_s e_t) = \mathbf{E}_{st} = \mathbf{E}_s \mathbf{E}_t = \pi(e_s)\pi(e_t)$.

For the semi-group property observe that since $\pi$ is linear it is invertible. Since $\pi$ distributes over Clifford product its inverse $\pi^{-1}$ distributes over matrix multiplication:

$$\pi^{-1}(\mathbf{E}_s \mathbf{E}_t) \equiv \pi^{-1}(\mathbf{E}_{st}) = e_{st} \equiv e_s e_t = \pi^{-1}(\mathbf{E}_s)\ \pi^{-1}(\mathbf{E}_t)$$

However, $C\ell_{p,q}$ is closed by construction, therefore, the set $\{\mathbf{E}\}_s$ is closed under matrix multiplication.

**Proposition 2.** *Let* $\mathbf{L} := \{l_i |\ l_i \in \mathbf{B}\}$ *be a column vector and* $\mathbf{R}_s$ *be the first row of* $\mathbf{E}_s$*. Then* $\pi^{-1} : \mathbf{E}_s \mapsto \mathbf{R}_s \mathbf{L}$*.*

*Proof.* We observe that by the Prop. 4 the only non-zero element in the first row of $\mathbf{E}_s$ is $\sigma_1 m_{1s} = 1$. Therefore, $\mathbf{R}_s \mathbf{L} = e_s$.

**Theorem 2 (Complete Real Matrix Representation).** *Define the map* $g : \mathbf{A} \mapsto \mathbf{GA}$ *as matrix multiplication with* $\mathbf{G}$*. Then for a fixed multiindex* $s$ $\pi = C_s \circ g = g \circ C_s$*. Further,* $\pi$ *is an isomorphism inducing a Clifford algebra representation in the real matrix algebra:*

$$C\ell_{p,q}(\mathbb{R}) \underset{\pi^{-1}}{\overset{\pi}{\rightleftarrows}} \mathbf{Mat}_{\mathbb{R}}\left(2^n \times 2^n\right)$$

*Proof.* The $\pi$-map is a linear isomorphism. The set $\{\mathbf{E}_s\}$ forms a multiplicative group, which is a subset of the matrix algebra $\mathbf{Mat}_{\mathbb{R}}(N \times N), N = 2^n$. Let $\pi(e_s) = \mathbf{E}_s$ and $\pi(e_t) = \mathbf{E}_t$. It is claimed that

1. $\mathbf{E}_s \mathbf{E}_t \neq \mathbf{0}$ by the Sparsity Lemma 1.
2. $\mathbf{E}_s \mathbf{E}_t = -\mathbf{E}_t \mathbf{E}_s$ by Prop. 5.
3. $\mathbf{E}_s \mathbf{E}_s = \sigma_s \mathbf{I}$ by Prop. 6.

Therefore, the set $\{\mathbf{E}_S\}_{S=\{1\}}^{P(n)}$ is an image of the extended basis $\mathbf{B}$. Here $P(n)$ denotes the power set of the indices of the algebra generators.

What is special about the above representation is the relationship

$$\mathrm{tr}\mathbf{A} = 2^n \langle A \rangle_0 \tag{1}$$

for the image $\pi(A) = \mathbf{A}$ of a general multivector element $A$ and it will be used further in the proof of FVS algorithm.

*Remark 1.* The above construction works if instead of the entire algebra $C\ell_{p,q}$ we restrict a multivector to a sub-algebra of a smaller grade $\max \mathrm{gr} A = r$. In this case, we form grade-restricted multiplication matrices $\mathbf{G}_r$ and $\mathbf{M}_r$.

## 4 FVS multivector inversion algorithm

Multivector inverses can be computed using the matrix representation and the characteristic polynomial. The matrix inverse is $\mathbf{A}^{-1} = \hat{\mathbf{A}}/\det \mathbf{A}$, where ˆ denotes the matrix adjunct operation and $\det \mathbf{A}$ is the matrix determinant. The formula is not practical, because it requires the computation of $n^2+1$ determinants. With the help of the Cayley-Hamilton Theorem, the inverse of $\mathbf{A}$ can be expressed as a polynomial in $\mathbf{A}$. The inverse can be computed as the last step of the FVS algorithm [3]. The algorithm has a direct representation in terms of Clifford multiplications as follows:

**Theorem 3.** *Suppose that $A \in C\ell_{p,q}$ is a multivector of maximal grade $r \leq p + q$. The Clifford inverse, if it exists, can be computed by the algorithm in $k = 2^{\lceil r/2 \rceil}$ steps as*

$$\begin{array}{l|l} m_1 = A & c_1 = -kA * 1 \\ m_2 = Am_2 - t_1 & c_2 = -\frac{k}{2} A * m_1 \\ \ldots & \ldots \\ m_k = Am_{k-1} - t_k & c_k = -A * m_{k-1} \end{array}$$

*until the step where $m_k = 0$ so that*

$$A^{-1} = -\frac{m_{k-1}}{c_k} \tag{2}$$

*The inverse does not exist if $c_k = -\det A = 0$.*

*The (reduced) characteristic polynomial of the multivector $A$ of maximal grade $r$ is*

$$p_A(\lambda) = \lambda^k + c_1 \lambda^{k-1} + \ldots c_{k-1}\lambda + c_k \tag{3}$$

*Proof.* The proof follows from the homomorphism of the $\pi$ map. We recall the statement of FVS algorithm:

$$p_A(\lambda) = \det(\lambda \mathbf{I}_n - \mathbf{A}) = \lambda^n + c_1 \lambda^{n-1} + \ldots c_{n-1}\lambda + c_n, \quad n = \dim(\mathbf{A})$$

where

$$\begin{array}{l|ll} \mathbf{M}_1 = \mathbf{A} & t_1 = \mathrm{tr}[\mathbf{M}_1] & c_1 = -t_1 \\ \mathbf{M}_2 = \mathbf{A}\mathbf{M}_1 - t_1 \mathbf{I}_n & t_2 = \frac{1}{2}\mathrm{tr}[\mathbf{A}\mathbf{M}_1] & c_2 = -t_2 \\ \ldots & \ldots & \ldots \\ \mathbf{M}_n = \mathbf{A}\mathbf{M}_{n-1} - t_n \mathbf{I}_n & t_n = \frac{1}{n}\mathrm{tr}[\mathbf{A}\mathbf{M}_{n-1}] & c_n = -t_n \end{array}$$

The matrix inverse can be computed from the last step of the algorithm as $\mathbf{A}^{-1} = \mathbf{M}_{n-1}/t_n$ under the obvious restriction $t_n \neq 0$.

Therefore, the k$^{\text{th}}$ step of the algorithm is $\pi^{-1} : \mathbf{M}_k = \mathbf{A}\mathbf{M}_{k-1} - t_k\mathbf{I} \mapsto m_k = Am_{k-1} - t_k$. Furthermore, $\pi$ commutes with the trace operator giving $\pi^{-1}(\mathrm{tr}[\mathbf{M}_k]) = n\langle m_k\rangle_0$; hence, $t_k = n\langle m_k\rangle_0$. Moreover, the FVS algorithm terminates with $\mathbf{M}_n = 0$, which corresponds to the limiting case $n = 2^{p+q}$ wherever A contains all grades.

On the other hand, [2] make the observation that according to the Bott periodicity the number of steps can be reduced to $2^{\lceil n/2\rceil}$. This can be proven as follows. Consider the isomorphism $C\ell_{p,q} \supset C\ell^{+}_{p,q} \cong C\ell_{q-1,p-1}$. Therefore, if a property holds for an algebra of dimension $n$ it will hold also for the algebra of dimension $n-2$. Therefore, suppose that for $n$ even the characteristic polynomial is square free: $p_A(v) \neq q(v)^2$ for some polynomial. We proceed by reduction. For $n = 2$ in $C\ell_{2,0}$ and $A = a_1 + e_1a_2 + e_2a_3 + e_3a_4$ we compute $p_A(v) = \left(a_1^2 - a_2^2 - a_3^2 + a_4^2 - 2a_1v + v^2\right)^2$ and a similar result holds also for the other signatures of $C\ell_2$ . Therefore, we have a contradiction and the dimension can be reduced to $k = n/2$. In the same way, suppose that $n$ is odd the characteristic polynomial is square free. However, for $n = 3$ in $C\ell_{3,0}$ and $A = a_1 + e_1a_2 + e_2a_3 + e_3a_4 + a_5e_{12} + a_6e_{13} + a_7e_{23} + a_8e_{123}$ it is established that $p_A(v) = q(v)^2$ for $q(v) = (a_1^2 - a_2^2 - a_3^2 - a_4^2 + a_5^2 + a_6^2 + a_7^2 - a_8^2 + 2i(-a_4a_5 + a_3a_6 - a_2a_7 + a_1a_8) - 2a_1v - 2ia_8v + v^2)(a_1^2 - a_2^2 - a_3^2 - a_4^2 + a_5^2 + a_6^2 + a_7^2 - a_8^2 + 2i(a_4a_5 - a_3a_6 + a_2a_7 - a_1a_8) - 2a_1v + 2ia_8v + v^2)$. Similar results hold also for the other signatures of $C\ell_3$ . Therefore, we have a contradiction and the dimension can be reduced to $k = (n+1)/2$. Therefore, overall, one can reduce the number of steps to $k = 2^{\lceil n/2\rceil}$.

As a second case, suppose that $\mathrm{gr}[A] = r$. Let $E_r$ be the set of all blades of grade $\leq r$. We compute the restricted multiplication tables $\mathbf{M}(E_r)$ and respectively $\mathbf{G}(E_r)$ and form the restricted map $\pi_r$. Then

$$\pi_r(AA^{-1}) = \pi_r(A)\pi_r(A^{-1}) = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n, \quad n = 2^r$$

Therefore, the FVS algorithm terminates in $k = 2^r$ steps. Observe that $\pi^{-1} : \mathbf{A}\mathbf{M}_k \mapsto Am_k$. Therefore, $\mathrm{tr}[\mathbf{A}\mathbf{M}_k]$ will map to $nA * m_k$ by eq. 1. Now, suppose that $t_k \neq 0$; then for the last step of the algorithm we obtain:

$$Am_{k-1} - t_k = 0 \Rightarrow A\frac{m_{k-1}}{t_k} = 1 \Rightarrow A^{-1} = \frac{m_{k-1}}{t_k}$$

Furthermore, we can always embed the Clifford number A into higher dimensional algebra according to the above congruence, therefore the number of steps can be ultimately reduced to $k = 2^{\lceil r/2\rceil}$.

## 5 Implementation

Computations are performed using the *Clifford* package in Maxima [5]. The function below returns the inverse (if it exists) and the characteristic polynomial $p_A(v)$ of the Clifford expression $A$.

**Listing 1.1.** FVS Maxima code for the Clifford package

```
fadlevicg2cp(A, v):=block([M:1,K,i:1,n,k:maxgrade(A),cq,c,ss],
    n:2^(ceiling(k/2)),
    array(c,n+1),  for r:0 thru n+1 do c[r]:1,
    A:rat(A),  ss:c[1]*v^^n,
    while i<n and K#0 do (
        K:dotsimpc(expand (A.M)),
        cq:-n/i*scalarpart(K),
        if _debug1=all then print("t_{",i,"}=",cq," m_{",i,"}=
            ",K,"\\\\"),
        if K#0 then
            M: rat(K + cq),
        c[i+1]:cq,  ss:ss+c[i+1]*v^^(n-i),
        i:i+1
    ),
    K:dotsimpc(expand(A.M)),
    cq:-n/i*scalarpart(K),
    if _debug1=all then print("t_{",i,"}=",cq," m_{",i,"}=",K,
        "\\\\"),
    ss:ss+cq,
    if cq=0 then cq:1,  M:factor(-(M)/cq),
    [M, ss]
);
```

## 6 Experiments

*Example 1.* Let us compute a rational example in $C\ell_{2,5}$ . Let $A = 1 + 5e_1e_3e_4 - 2e_1e_5$. Then for the maximal representation we have $k = 2^3 = 8$ steps.

$$
\begin{array}{ll}
t_1 = -8 & m_1 = 1 + 5(e_1e_3e_4) - 2(e_1e_5) \\
t_2 = 112 & m_2 = -28 - 30(e_1e_3e_4) + 12(e_1e_5) \\
t_3 = -560 & m_3 = 210 + 390(e_1e_3e_4) - 156(e_1e_5) \\
t_4 = 3976 & m_4 = -1988 - 1360(e_1e_3e_4) + 544(e_1e_5) \\
t_5 = -12320 & m_5 = 7700 + 8580(e_1e_3e_4) - 3432(e_1e_5) \\
t_6 = 54208 & m_6 = -40656 - 14520(e_1e_3e_4) + 5808(e_1e_5) \\
t_7 = -85184 & m_7 = 74536 + 53240(e_1e_3e_4) - 21296(e_1e_5) \\
t_8 = 234256 & m_8 = 234256
\end{array}
$$

Therefore, $A^{-1} = \frac{1}{22}(1 - 5e_1e_3e_4 + 2e_1e_5)$ and $p_A(v) = 234256 - 85184v + 54208v^2 - 12320v^3 + 3976v^4 - 560v^5 + 112v^6 - 8v^7 + v^8$. On the other hand, for the reduced algorithm will run in $k = 2^{\lceil 3/2 \rceil} = 4$ steps.

$$
\begin{array}{ll}
t_1 = -4 & m_1 = 1 + 5(e_1e_3e_4) - 2(e_1e_5) \\
t_2 = 48 & m_2 = -24 - 10(e_1e_3e_4) + 4(e_1e_5) \\
t_3 = -88 & m_3 = 66 + 110(e_1e_3e_4) - 44((e_1e_5) \\
t_4 = 484 & m_4 = -484
\end{array}
$$

and $p_A(v) = 484 - 88v + 48v^2 - 4v^3 + v^4$, which squared gives the above polynomial.

*Example 2.* Consider $C\ell_{5,2}$ and let $A = 1 - e_2 + e_1\, e_2\, e_3\, e_4\, e_5\, e_6\, e_7$. Then

$$A^{-1} = \frac{1}{5} - \frac{1}{5}\, e_2 - \frac{3}{5}\, (e_1\, e_2\, e_3\, e_4\, e_5\, e_6\, e_7) + \frac{2}{5}\, (e_1\, e_3\, e_4\, e_5\, e_6\, e_7)$$

## A Supporting results

**Definition 5 (Sparsity property).** *A matrix has the sparsity property if it has exactly one non-zero element per column and exactly one non-zero element per row. Such a matrix we call sparse.*

**Lemma 1 (Sparsity lemma).** *If the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are sparse then so is* $\mathbf{C} = \mathbf{AB}$. *Moreover,*

$$c_{ij} = \begin{cases} 0 \\ a_{iq} b_{qj} \end{cases}$$

*(no summation!) for some index* $q$.

*Proof.* Consider two sparse square matrices $\mathbf{A}$ and $\mathbf{B}$ of dimension $n$. Let $c_{ij} = \sum_\mu a_{i\mu} b_{\mu j}$. Then as we vary the row index $i$ then there is only one index $q \leq n$, such that $a_{iq} \neq 0$. As we vary the column index $j$ then there is only one index $q \leq n$, such that $b_{qj} \neq 0$. Therefore, $c_{ij} = (0; a_{iq} b_{qj})$ for some $q$ by the sparsity of $\mathbf{A}$ and $\mathbf{B}$. As we vary the row index $i$ then $c_{qj} = 0$ for $i \neq q$ for the column $j$ by the sparsity of $\mathbf{A}$. As we vary the column index $j$ then $c_{iq} = 0$ for $j \neq q$ for the row $i$ by the sparsity of $\mathbf{B}$. Therefore, $\mathbf{AB}$ is sparse.

**Lemma 2 (Multiplication Matrix Structure).** *For the multi-index disjoint sets* $S \prec T$ *the following implications hold for the elements of* $\mathbf{M}$ *:*

$$\begin{tikzcd}
m_{\mu\lambda}\, e_S \arrow[r, "\exists \lambda' > \lambda"] \arrow[d, "\exists"'] & m_{\mu\lambda'} e_T \arrow[d] & \\
m_{\lambda\mu} e_S \arrow[r, "\exists"] & m_{\lambda\mu} m_{\mu\lambda'} e_{S \triangle T} \arrow[r, "\exists \lambda'' = \lambda'"] & m_{\lambda\lambda''} e_{S \triangle T}
\end{tikzcd}$$

*so that* $m_{\lambda\lambda'} = m_{\lambda\mu} \sigma_\mu m_{\mu\lambda'}$ *for some index* $\mu$.

*Proof.* Suppose that the ordering of elements is given in the construction of $C\ell_{p,q,r}$ . To simplify presentation, without loss of generality, suppose that $e_s$ and $e_t$ are some generators. By the properties of $\mathbf{M}$ there exists an index $\lambda' > \lambda$, such that $e_M e_{L'} = m_{\mu\lambda'}\, e_t$, $L' \backslash M = T$ for $L \prec L'$. Choose $M$, s.d. $L \prec M \prec L'$. Then for $L \prec M \prec L'$ and $S \prec T$

$$\begin{aligned} e_M e_L &= m_{\mu\lambda}\, e_s, \quad L \triangle M = S \Leftrightarrow e_L e_M = m_{\lambda\mu}\, e_s \\ e_M e_{L'} &= m_{\mu\lambda'}\, e_t, \quad L' \triangle M = T \end{aligned}$$

Suppose that $e_s e_t = e_{st}, st = S \cup T = S \triangle T$. Multiply together the diagonal nodes in the matrix

$$e_L \underbrace{e_M e_M}_{\sigma_\mu} e_{L'} = m_{\lambda\mu} m_{\mu\lambda'}\, e_{st}$$

Therefore, $s \in L$ and $t \in L'$. We observe that there is at least one element (the algebra unity) with the desired property $\sigma_\mu \neq 0$.

Further, we observe that there exists unique index $\lambda''$ such that $m_{\lambda\lambda''} e_{st}$. Since $\lambda$ is fixed. This implies that $L'' = L' \Rightarrow \lambda'' = \lambda'$. Therefore,

$$e_L e_{L'} = m_{\lambda\lambda'} e_{st}, \quad L' \triangle L = \{s, t\}$$

which implies the identity $m_{\lambda\lambda'}\, e_{st} = m_{\lambda\mu} \sigma_\mu m_{\mu\lambda'}\, e_{st}$. For higher graded elements $e_S$ and $e_T$ we should write $e_{S\triangle T}$ instead of $e_{st}$.

**Proposition 3.** *Consider the multiplication table* $\mathbf{M}$. *All elements* $m_{kj}$ *are different for a fixed row* $k$. *All elements* $m_{iq}$ *are different for a fixed column* $q$.

*Proof.* Fix $k$. Then for $e_K, e_J \in \mathbf{B}$ we have $e_K e_J = m_{kj} e_S, \quad S = K \triangle J$. Suppose that we have equality for 2 indices $j, j'$. Then $K \triangle J' = K \triangle J = S$. Let $\delta = J \cap J'$; then

$$K \triangle (J \cup \delta) = K \triangle J = S \Rightarrow K \triangle \delta = S \Rightarrow \delta = \emptyset$$

Therefore, $j = j'$. By symmetry, the same reasoning applies to a fixed column $q$.

**Proposition 4.** *For* $e_s \in \mathbf{E}$ *the matrix* $\mathbf{A}_s = C_s(\mathbf{M})$ *is sparse.*

*Proof.* Fix an element $e_s \in \mathbf{E}$. Consider a row $k$. By Prop. 3 there is a $j$, such $e_{kj} = e_s$. Then $a_{kj} = m_{kj}$, while for $i \neq j$ $a_{ki} = 0$.

Consider a column $l$ By Prop. 3 there is a $j$, such $e_{jl} = e_s$. Then $a_{jl} = m_{jl}$, while for $i \neq j$ $a_{il} = 0$. Therefore, $\mathbf{A}_s$ has the sparsity property.

**Proposition 5.** *For generator elements* $e_s$ *and* $e_t$ $\mathbf{E}_s \mathbf{E}_t + \mathbf{E}_t \mathbf{E}_s = \mathbf{0}$.

*Proof.* Consider the basis elements $e_s$ and $e_t$. By linearity and homomorphism of the $\pi$ map (Th. 1): $\pi : e_s e_t + e_t e_s = 0 \mapsto \pi(e_s e_t) + \pi(e_t e_s) = \mathbf{0}$. Therefore, for two vector elements $\mathbf{E}_s \mathbf{E}_t + \mathbf{E}_t \mathbf{E}_s = \mathbf{0}$.

**Proposition 6.** $\mathbf{E_s E_s} = \sigma_s \mathbf{I}$

*Proof.* Consider the matrix $\mathbf{W} = \mathbf{GA_s GA_s}$. Then $w_{\mu\nu} = \sum_\lambda \sigma_\mu \sigma_\lambda a_{\mu\lambda} a_{\lambda\nu}$ element-wise. By Lemma 1 $\mathbf{W}$ is sparse so that $w_{\mu\nu} = (0; \sigma_\mu \sigma_q a_{\mu q} a_{q\nu})$.

From the structure of $\mathbf{M}$ for the entries containing the element $e_S$ we have the equivalence

$$\begin{cases} e_M e_Q = a^s_{\mu q} e_S, & S = M \triangle Q \\ e_Q e_M = a^s_{q\mu} e_S, \end{cases}$$

After multiplication of the equations we obtain $e_M e_Q e_Q e_M = a^s_{\mu q} e_S a^s_{q\mu} e_S$, which simplifies to the *First fundamental identity*:

$$\sigma_q \sigma_\mu = a^s_{\mu q} a^s_{q\mu} \sigma_s \tag{4}$$

We observe that if $\sigma_\mu = 0$ or $\sigma_q = 0$ the result follows trivially. In this case also $\sigma_s = 0$. Therefore, let's suppose that $\sigma_s \sigma_q \sigma_\mu \neq 0$. We multiply both sides by $\sigma_s \sigma_q \sigma_\mu$ to obtain $\sigma_s = \sigma_q \sigma_\mu a^s_{\mu q} a^s_{q\mu}$. However, the RHS is a diagonal element of $\mathbf{W}$, therefore by the sparsity it is the only non-zero element for a given row/column so that $\mathbf{W} = \mathbf{E_s^2} = \sigma_s \mathbf{I}$.

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