Navier-Stokes Equations Analytic 3D Solution for Incompressible Viscous Fluids in the absence of external Forces for a given periodic initial velocity vector

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Abstract

This study proves the existence of smooth periodic solutions for Navier-Stokes three-dimensional equations under the assumption of a given periodic initial velocity vector field with positive viscosity. The solution proposed solves the equation by utilizing a Fourier series representation of periodic initial velocity vector fields and predicting the velocity vector field at all times. The significance of this finding is that it contributes positively towards understanding the behavior of solutions of Navier-Stokes equations and suggests that smooth periodic solutions for the given problem can indeed exist under certain conditions. Additionally, the authors suggest that their solution can be used to settle the Clay Mathematics Millennium Prize Problem, which seeks to find a solution for Navier-Stokes equations meeting specific criteria. It is important to note, however, that this study does not provide a complete solution to the problem, but it provides a significant contribution to the understanding of the behavior of solutions of Navier-Stokes equations. Overall, this research demonstrates that the smooth periodic solutions for Navier-Stokes equations can exist for a given initial velocity vector field with positive viscosity, and it presents a new approach for the Navier-Stokes equation.

1 Introduction

The Navier-Stokes equations are a set of partial differential equations that govern the behavior of fluids, including air and water. These equations are used extensively in many different fields, including fluid mechanics, meteorology, and engineering. Despite their importance, there are only a few cases in which the

Navier-Stokes equations have a known analytic solution. In this paper, I will explore the current state of research on the three-dimensional analytic solution to the Navier-Stokes equations.

The Navier-Stokes equations are nonlinear equations that describe the motion of a fluid in terms of its velocity, pressure, and density. These equations are based on the principle of conservation of mass, momentum, and energy. They are difficult to solve analytically because of their complexity and the nonlinear nature of the equations. However, there are some special cases in which analytic solutions can be found.

One of the most famous examples of an analytic solution to the Navier-Stokes equations is the solution for the flow of an incompressible fluid between two infinitely long parallel plates, known as the Couette flow. This solution was originally found by Claude-Louis Navier in 1823 and was later refined by Stokes in 1845. This solution is relatively simple because it assumes that the fluid flow is linear and the viscosity is constant. Another example of an analytic solution to the Navier-Stokes equations is the solution for the flow of a fluid around a sphere, known as the Stokes flow. This solution was also discovered by Stokes and assumes that the fluid flow is very slow and the viscosity is again constant. This solution is used to model many different problems, including fluid flow around aircraft wings and the behavior of microorganisms in liquid.

Despite these examples of analytic solutions, there is no known analytical solution for the three-dimensional Navier-Stokes equations in general. This means that numerical methods must be used to approximate the solution. Although numerical methods can provide accurate solutions in many cases, they are computationally expensive and can require large amounts of computational resources.

In recent years, there have been many researchers working on finding analytic solutions to the three-dimensional Navier-Stokes equations. One of the most promising approaches is through the use of symmetries. By identifying the symmetries of the equations, researchers can simplify the problem and reduce the number of variables needed to describe the flow. This approach has led to some important breakthroughs in recent years, including the discovery of new analytic solutions for the Navier-Stokes equations.

In conclusion, the analytic solution to the three-dimensional Navier-Stokes equations remains an active area of research in fluid mechanics. While some special cases have been solved analytically, the general case remains unsolved. Recent advances in the use of symmetries have shown promising results, and it is possible that new analytical solutions will be discovered in the coming years. The study of the Navier-Stokes equations is a fascinating area of research that has important applications in many different fields, and it will undoubtedly continue to be an active area of research for years to come.

1.1 Statement of the problem

Let $a, b, c \in \mathbb{R}$ and $\vec{\nabla}g = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$. Also, let $\vec{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Therefore, $\vec{R} \cdot \vec{\nabla}g = g(x, y, z) = ax + by + cz$.

Navier-Stokes equation, which describes the motion of a fluid. The left-hand side of the equation represents the acceleration of the fluid, which is the rate of change of velocity with respect to time and the convective accel- eration due to the advection of the fluid by itself. The right-hand side of the equation represents the forces acting on the fluid, which include the viscous forces due to the diffusion of momentum and the pressure gradient force due to the variation of pressure in the fluid. Overall, the Navier-Stokes equation is a fundamental equation in fluid mechanics and is used to model a wide range of phenomena, from the flow of air over an airplane wing to the circulation of blood in the human body.

$$\frac{\partial}{\partial t}\vec{U} + (\vec{U} \cdot \vec{\nabla})\vec{U} = \nu\nabla^2\vec{U} - \vec{\nabla}P + \vec{F} \tag{1}$$

Equation (1) is the Navier Stokes three dimensional equation by considering density of the fluid unitary,

$$\vec{\nabla} \cdot \vec{U} = 0 \tag{2}$$

Equation (2) shows divergence free condition.

$$\vec{U}\Big|_{t=0} = \vec{U}^0 \tag{3}$$

Equation (3) Initial velocity vector field derived by setting time zero.

$$\vec{U}^0(\vec{R} \cdot \vec{\nabla}g) = \vec{U}^0((\vec{R} + \vec{\nabla}g) \cdot \vec{\nabla}g)$$

$$\vec{U}^0(g(x, y, z)) = \vec{U}^0(g(x, y, z) + 1) \tag{4}$$

Equation (4) Shows periodic function of initial velocity vector field.

$$\vec{U}(\vec{R} \cdot \vec{\nabla}g, k(t)) = \vec{U}((\vec{R} + \vec{\nabla}g) \cdot \vec{\nabla}g, k(t))$$

$$\vec{U}(g(x, y, z), k(t)) = \vec{U}(g(x, y, z) + 1, k(t)) \tag{5}$$

Equation (5) Shows periodic velocity vector field in time t

$$\vec{F} = 0 \tag{6}$$

Equation (6) Shows the absence of external force applied on the fluid

1.2 Initial velocity vector field

$$\vec{U}^0 = \vec{a}_0 + \sum_{n=1}^{\infty} \Big(\vec{a}_n \cos(2n\pi g) + \vec{b}_n \sin(2n\pi g) \Big) \tag{7}$$

Where,

$$\vec{a}_0 = \frac{1}{2} \int_{-1}^{1} \vec{U}^0(g) dg \tag{8}$$

$$\vec{a}_n = \int_{-1}^{1} \vec{U}^0(g) \cos(2n\pi g) dg \tag{9}$$

$$\vec{b}_n = \int_{-1}^{1} \vec{U}^0(g) \sin(2n\pi g) dg \tag{10}$$

2 Velocity vector field and Pressure solution.

2.1 Proposed three dimensional velocity vector field solution

$$\vec{U}(x, y, z, t) = \vec{a}_0 h(t) + \sum_{n=1}^{\infty} H_n(t) \vec{S}_n(x, y, z, t) \tag{11}$$

Where,

$$\vec{S}_n(x, y, z, t) = \vec{a}_n \cos(2n\pi g + k(t)) + \vec{b}_n \sin(2n\pi g + k(t)) \tag{12}$$

$$H_n(t) = e^{-\nu (2n\pi)^2 t} \tag{13}$$

$$k(t) = \int_0^t (\vec{U} \cdot \vec{\nabla} g) d\tau \tag{14}$$

2.2 Proposed three dimensional pressure solution

$$-p(x, y, z, t) = \vec{a}_0 \cdot \vec{R} \frac{d}{dt} h(t) \tag{15}$$

2.3 Prove that the proposed solutions satisfy Navier Stokes equations

Derivatives of equations (11) with respect to time t

$$\frac{\partial \vec{U}}{\partial t} = \vec{a}_0 \frac{dh(t)}{dt} + \sum_{n=1}^{\infty} \frac{dH_n}{dt} \vec{S}_n + \frac{dk(t)}{dt} \sum_{n=1}^{\infty} \frac{d\vec{S}_n}{dg} H_n \tag{16}$$

Derivate equation (13) and equation (14) with respect to time t and substitute to equation (16) results

$$\frac{\partial \vec{U}}{\partial t} = \vec{a}_0 \frac{dh(t)}{dt} - \nu \sum_{n=1}^{\infty} (2n\pi)^2 H_n \vec{S}_n - (\vec{U} \cdot \vec{\nabla} g) \sum_{n=1}^{\infty} \frac{d\vec{S}_n}{dg} H_n \tag{17}$$

The term of equation (1), the convective acceleration due to the advection of the fluid by itself is simplified to

$$(\vec{U} \cdot \vec{\nabla})\vec{U} = (\vec{U} \cdot \vec{\nabla} g) \sum_{n=1}^{\infty} \frac{d\vec{S}_n}{dg} H_n \tag{18}$$

Simplifying by substituting equation (11) to the viscous forces due to the diffusion of momentum term from equation (1) results

$$\nu \nabla^2 \vec{U} = \nu \sum_{n=1}^{\infty} \frac{d^2 \vec{S}_n}{dg^2} H_n \tag{19}$$

Second derivate equation (12) and substitute to to equation (19) results

$$\nu \nabla^2 \vec{U} = -\nu \sum_{n=1}^{\infty} (2n\pi)^2 \vec{S}_n H_n \tag{20}$$

Gradient of equation (15) results

$$-\vec{\nabla} p = \vec{a}_0 \frac{d}{dt} h(t) \tag{21}$$

Substitute equations (17),(18),(20),(21) to equation (1) results

$$\cancel{\vec{a}_0 \frac{dh(t)}{dt}} - \nu \cancel{\sum_{n=1}^{\infty} (2n\pi)^2 H_n \vec{S}_n} - \cancel{(\vec{U} \cdot \vec{\nabla} g) \sum_{n=1}^{\infty} \frac{d\vec{S}_n}{dg} H_n} + \cancel{(\vec{U} \cdot \vec{\nabla} g) \sum_{n=1}^{\infty} \frac{d\vec{S}_n}{dg} H_n}$$

$$= -\nu \cancel{\sum_{n=1}^{\infty} (2n\pi)^2 \vec{S}_n H_n} + \cancel{\vec{a}_0 \frac{d}{dt} h(t)}$$

$$0 = 0$$

Therefore, we conclude that the proposed velocity vector field and scale pressure solutions in three dimensions satisfy equation (1).

2.4 Prove that the proposed solutions satisfy divergence free velocity vector field condition

Divergence of equation (11) is simplified to

$$\vec{\nabla} \cdot \vec{U} = \frac{d}{dg}(\vec{U} \cdot \vec{\nabla} g) \tag{22}$$

Using divergence free condition of equation (2) and equation (22) we can get

$$\vec{U} \cdot \vec{\nabla} g = \phi(t) \tag{23}$$

We can conclude that the equation (23) has to be constant in space and variable in time.

2.5 Prove that the proposed solutions satisfy initial velocity vector field condition

Substituting $t = 0$, to equation (12), (13), (14) reduces equation (11) to equation (7) which implies equation (3) is satisfied for a $h(0) = 1$.

3 Conclusion

For a given periodic initial velocity vector field periodic on a linear three dimensional function, there exist a periodic velocity vector field solution periodic over a linear function of three dimensions and there exist too aperiodic scalar pressure function of space and time solution.

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