The fundamental reformulation of the concept of a weak solution to the Navier-Stokes problem (the preliminary version)

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Abstract.

At first we identify the main error in the formulation of the concept of the weak solution to Navier-Stokes (NS) equations which is the completely insufficient treatment of the incompressibility condition on the fluid (expressed in the standard way by div $\mathbf{u} = 0$). The repair requires the complete reformulation of the NS problem. The basic concept must be the generalized motion (i.e. the generalized flow) which replaces the standard velocity field. Here we define the generalized flow on the bases of Geometric measure theory extended to the theory of Cartesian currents and weak diffeomorphisms (see [1], [2]). Then the key concept of the **complete weak solution** to the NS problem is defined and the two conjectures (the existence and the regularity ones) concerning the complete weak solutions are formulated. In two appendices many technical details are described (concerning e.g. Cartesian currents, homology conditions, weak diffeomorphisms, etc.). Our approach is based on the unification of the standard analysis of NS equations with the methods of Geometric measure theory and of the theory of Cartesian currents.

1. Introduction

In this paper the regularity of initial and boundary conditions for the Navier-Stokes (NS) problem will be assumed.

The aim of this paper consists in the realization of following steps

(i) The division of the NS equations into two parts: the evolution part and the determinant part (expressing the volume conservation of the flow)
(ii) The analysis of the insufficient (and incomplete) formulation of the determinant part (i.e. of the equation div **u** = 0)
(iii) The introduction of the concepts of the generalized flow and of the homological conditions
(iv) The definition of the weak differentiability of the generalized flow and the definition of the associated velocity field to a given weakly differentiable generalized flow
(v) The definition of the concept of the **complete weak solution** to the NS problem, i.e. the **fundamental reformulation** of the standard weak formulation of the NS equations
(vi) The formulation of the existence and regularity conjectures for the complete weak solution to the NS problem

In conclusion we can characterized our approach in general in the following terms:

We have combined two different parts of analysis

(i) The standard analysis of NS equations, used in the evolution part
(ii) Geometric measure theory extended to the theory of Cartesian currents, homology conditions, weak diffeomorphisms etc. used in the determinant part of the NS problem

The resulting theory has a substantially richer structure than the standard analysis of NS equations and this gives a hope to arrive at a regularity. Especially the homology conditions (and their consequence – the weak convergence of determinants) can give the new input into the analysis of the NS problem. (On the other hand, the property of the weak convergence of determinants is necessary for the "right" solution to the incompressibility problem.)

The organization of the paper is following. In sect. 2 we analyze the two objections to the standard concept of the weak solution to NS equations. In sect. 3 we describe the basic concepts and facts from the theory of Cartesian currents, weak diffeomorphisms, homology conditions etc. In sect. 4 we introduce the key new concepts: the generalized flow, its weak derivative, its associated velocity field. In sect. 5 we introduce the reformulation of the NS problem, the concept of the complete weak solution and the two basic conjectures. Then in sect. 6 there is a discussion and in sect. 7 conclusions. In App. A there is a detailed definition of graph-current and of homology conditions. In App. B we describe one variant of the definition of the standard weak solution to the NS equations which is used in this paper.

2. Two critical objections to the standard concept of the weak solution to the Navier-Stokes equations

The concept of the weak solution to the Navier-Stokes (= NS) equations has two problematic points which will be analyzed in this section.

The first objection consists in the fact that the NS equations are the mixture of two different parts:

(i) the standard evolution equation for the velocity field
(ii) the (non-evolution) volume-conserving constraint containing the problem with determinants of the motion of the fluid.

These two parts are completely different in character and we assert that part with determinants is (when considered in a standard way) treated in a wrong and insufficient way.

The standard treatment of the volume-conserving condition is expressed as the divergence equation div $\mathbf{u}(x, t) = 0$ (for a.e. t, $0 \leq t \leq T$ and a.e. $x \in \Omega$) where $\mathbf{u}$ is a velocity field. But it is known from [1] and [2] that the correct treatment of determinants requires to take into account the homology properties of maps describing the motion of a fluid.

Thus the correct formulation of the NS problem requires the following

(i) To consider the standard weak solution to the NS equations
(ii) To complement the treatment of the volume-conserving constraint in a way reflecting the necessary homology properties of maps describing the weak flow of the fluid.

To realize this program it is necessary

(i) To define the concept of the generalized (i.e. weak) flow, i.e. the semigroup of weak diffeomorphisms (see [2])
(ii) To define what is the weak (time) derivative of this weak flow (i.e. the velocity field)
(iii) To define what is the volume-conserving weak flow

This will be done in the section 4.

Then it is necessary to reformulate the NS problem in terms of the weak flow (i.e. in the terms of the generalized motion of the fluid) instead of the standard formulation in terms of the velocity field (i.e. in the form of NS equations for the velocity field). This will be done in the section 5 where the basic concept of the **complete weak solution** to the NS problem is defined.

The second objection to the concept of the standard weak solution to the NS equations consists in the fact that this solution is not, in general, the solution of the original NS problem, i.e. to find the (possibly generalized) motion of the fluid.

In the situation where the velocity field is not smooth, it is known that, in general, the corresponding flow is not uniquely defined and thus the motion of the fluid is not uniquely defined by the velocity field. (In the situation when the velocity field is smooth, the corresponding flow is uniquely defined.)

Thus the weak solution to the NS equations does not create the weak solution to the original NS problem.

As a conclusion we arrive at following facts

(i) The treatment of the volume-conservation condition in the standard definition of the weak solution to the NS problem is insufficient

(ii) The weak solution to the NS equations is not, in general, the solution to the NS problem (in fact, it is necessary to introduce the concept of the complete weak solution using the concept of the generalized flow).

3. The basic concepts for problems with determinants: Cartesian currents, weak diffeomorphisms, homology conditions, the weak convergence of determinants

Let us fix the bounded simply connected domain $\Omega \subset R^3$ with the smooth boundary.

Notation 3.1. We shall consider a copy $R^{3\hat{}}$ of R^3 and $\Omega\hat{} \subseteq R^{3\hat{}}$ where $\Omega\hat{}$ is isomorphic to Ω ($\Omega\hat{}$ is a copy of Ω in $R^{3\hat{}}$). We shall denote coordinates in R^3 by (x_1, x_2, x_3) and coordinates in $R^{3\wedge}$ by (y^1, y^2, y^3).

The graph G_U of the map U is defined as a 3-dimensional current in $R^3 \times R^{3\wedge}$ ([1], sect. 3.2.1 (5) and above, p. 230) and its meaning is the integration of the differential 3-form over the graph G_U of U (the explicit definition of the graph-current G_U can be found in App. A below).

The Sobolev map $U \in W^{1,1}(\Omega, R^{3\wedge})$ is called a Cartesian map (as a special case of Cartesian currents) if 2 x 2 minors of DU and detDU are integrable functions and the associated graph-current G_U is homologically closed, $\partial G_U = 0$, (i.e. $\partial G_U(\omega) = G_U(d\omega)= 0$ for each 2-form ω compactly supported in $\Omega \times R^{3\hat{}}$). We assume moreover that U satisfies the Lusin's condition (i.e. $|U(A)| = 0$ if $|A| = 0$), where |.| denotes the Lebesgue measure in R^3).

The set $dif^{1,1,vc}$ $(\Omega, \Omega\hat{})$ of weak volume-conserving diffeomorphisms in Ω is defined as following (vc = volume-conserving).

Definition 3.1. The Sobolev map $U \in W^{1,1}$ $(\Omega. R^{3\hat{}})$ is in $dif^{1,1,vc}$ $(\Omega. \Omega^{\wedge})$, i.e. U is the volume-conserving weak diffeomorphism on Ω if U satisfies the following conditions

(i) $U(\Omega) = \Omega\hat{}$ a.e., i.e. $|U(\Omega) - \Omega\hat{}| = |\Omega\hat{} - U(\Omega)| = 0$. (Here | . | denotes the Lebesgue measure in R^3.)

(ii) There exists a Sobolev map $U\hat{} \in W^{1,1}$ $(\Omega\hat{}, R^3)$ such that $U\hat{}(U(x)) = x$ for a.e. $x \in \Omega$ and $U(U\hat{}(y)) = y$ for a.e. $y \in \Omega\hat{}$ - i.e. $U\hat{}$ is the a.e. inverse of U.

(iii) Both U and $U\hat{}$ satisfy the Luzin property, i.e. if $|A| = 0$ then $|U(A)| = 0$ (and $|U\hat{}(A)| = 0$ for each $A \subseteq \Omega\hat{}$, $|A|=0$).

(iv) 2x2-minors $M^{2x2}DU$ are integrable functions and the same is true for $M^{2x2}DU\hat{}$

(v) The Jacobian $detDU(x) = 1$ for a.e. $x \in \Omega$. Correspondingly $detDU\hat{}(y) = 1$ for a.e. $y \in \Omega\hat{}$.

(vi) The current $G_U = G\hat{}_{U\hat{}}$ is closed inside of $\Omega \times \Omega\hat{}$ in the sense of [1], i.e $\partial G_U = \partial G\hat{}_{U\hat{}} = 0$ in $\Omega \times \Omega\hat{}$. (i.e. $G_U(d\omega) = 0$ for each smooth compactly supported 2-form ω on $\Omega \times \Omega\hat{}$.)

(vii) For each measurable subset A of Ω we have $|U(A)| = |A|$ and the same is true for $U\hat{}$ - i.e. the volume-conservation for U and $U\hat{}$ is satisfied.

Remark 3.2. The closeness $\partial G_U = 0$ of the graph G_U is the principal property of Cartesian currents and of weak diffeomorphisms. It is clear that this condition is weakly closed (i.e. it is passing through the weak limits) since it can be expressed as $G_U(d\omega) = 0$ for each smooth 2–form ω compactly supported in $\Omega \times \hat{\Omega}$.

The map U conserves the orientation a.e. This follows from the condition (v).

Remark 3.3. In fact, the condition (iv) from the definition 3.1. (the integrability of minors) is superfluous since it follows from the other conditions from this definition.

Proof. 2×2 minor of DU can be expressed as a derivative of $\hat{U}$ ($\hat{U}$ is an inverse map to the map U), $M^{2x2}(DU(x)) = M^{2x2}D\hat{U}(y)/\det DU(\hat{U}y)$, $y = U(x)$. Since $\det DU = 1$ a.e. we have $M^{2x2}(DU(x)) = D\hat{U}(y)$. The transformation from variables x onto variables y is governed by the property that $\det DU = 1$ a.e.

As a consequence we obtain that detDU (i.e. that the distributional determinant of DU) is represented by the function identically equal to 1 a.e.

Homological conditions are defined as $G_U(d\omega)$ for each 2-form ω compactly supported in $\Omega \times \hat{\Omega}$. There are 3 types of forms ω's depending on the number of differentials dy: 0, 1, 2. The details of the homological conditions are described in App. A, part 2.

The weak continuity of determinants means that the weak convergence of DU_k to DU implies the weak convergence of $\det DU_k$ to detDU. This is the quite non-trivial but fundamental property. Without this property the variational problems with determinants cannot be solved. The weak continuity of determinants contains two properties (we assume that DU_k converge weakly to DU)

(i) $\det DU_k$ are integrable functions and $\det DU_k$ converges weakly to some integrable function w

(ii) $w = \det DU$

The property (i) is not connected directly to the homology conditions while the property (ii) is fundamentally based on homology conditions. The details of the weak continuity of determinants can be found in [2].

In general, the weak continuity of determinants is the central property for the solution of problems with determinants. Homological conditions are the true basis for proving the weak continuity of determinants. None of these properties are considered in the standard treatment of the NS equations and this is the main drawback (insufficient property) of the standard formulation of the weak solution to the NS equations.

4. The new concepts: the generalized (weak) flow, its weak derivative and the associated velocity field

At first we shall define the concept of a one-dimensional semi-group of weak diffeomorphisms which we shall denote as a weak (i.e. generalized) flow.

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set in $\mathbb{R}^3$ with a smooth boundary.

Let U, V, W be weak diffeomorphism defined on Ω with values in Ω. We say that W is a composition of U and V (denoted as $U \circ V$) if for a.e. $x \in \Omega$ the relation $V(U(x)) = W(x)$ is satisfied.

Let $\{U^{rs}\}_{0\le r\le s\le T}$ be a set of weak diffeomorphisms defined on Ω with values in Ω, T > 0.

Definition 4.1. The set $\{U^{rs}\}_{0\le r\le s\le T}$ of weak diffeomorphisms will be called the weak flow if the following conditions are satisfied

(i) U^{rr} is an identity map on Ω for each $0 \le r \le T$

(ii) For each $0 \le r \le s \le t \le T$ the diffeomorphism U^{rt} is the composition of U^{rs} and U^{st}, i.e.

$$U^{rt} = U^{rs} \circ U^{rs}$$

Definition 4.2. The weak flow $\{U^{rs}\}_{0\le r\le s\le T}$ will be called the volume-conserving weak flow if for a.e. r and s the map U^{rs} is the volume-conserving weak diffeomorphism.

In the NS problem only such weak flows are considered which have an associated velocity field $\mathbf{u}(x, t)$ where $x \in \Omega$, $t \in [0, T]$. The definition of the associated velocity field requires certain attention.

Definition 4.3. We shall say that the weak flow $\{U^{rs}\}_{0\le r\le s\le T}$ is weakly differentiable if there exists an L^1-vector field $\mathbf{u}(x, t) \in R^3$, $x \in \Omega$, $t \in [0, T]$ such that for a.e. t, r, s ∈ [0, T], 0<t<r<s we have

$$\int_\Omega dx\, \varphi(x)\, U^{ts}(x) = \int_\Omega dx\, \varphi(x)\, U^{tr}(x) + \int_\Omega \int_{r\le w\le s} dx\, \varphi(x)\, dw\, \mathbf{u}(U^{tw}(x), w)$$

for each smooth function φ(x) on Ω.

The vector field $\mathbf{u}(x, t)$ is then called the *velocity field* of the weak flow $\{U^{rs}\}_{0\le r\le s\le T}$ (assuming that this weak flow is weakly differentiable).

Remark 4.1. It is clear that the velocity field is uniquely determined by the corresponding generalized flow (assuming that this flow is weakly differentiable). In fact, if we have for a.e. $x \in \Omega$

$$U^{ts}(x) = U^{tr}(x) + \int_{r \le w \le s} dw\, \mathbf{u}_k(U^{tw}(x), w)\ , k = 1,2$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are two possible velocity fields, then we obtain for a.e. $r < s$ and for a.e. $x \in \Omega$ that

$$\int_{r \le w \le s} dw\, [\mathbf{u}_1(U^{tw}(x), t) - \mathbf{u}_2(U^{tw}(x), w)] = 0.$$

Thus $\mathbf{u}_1(y, w) = \mathbf{u}_2(y, w)$ for a.e. $y \in \Omega$ and a.e. $w \in [t, T]$ since $U^{tw}(\Omega)$ covers almost all of Ω for a.e. w.

Let us remark that the inverse transformation from the velocity field to the weak flow is not, in general, uniquely defined. In fact, to a given (non-smooth) velocity field there may exist many corresponding weak flows and also may exist no corresponding weak flow.

Remark 4.2. There is a question considering the integrability of the last integral in the Definition 4.3. Let us consider the following integral (Φ is a smooth function)

$$\int_\Omega \int_{r \le w \le s} dx'\, dw\, \Phi((U^{tw})^{-1}x')\, \mathbf{u}(x', w)\ .$$

This integral is convergent since the function $\Phi((U^{tw})^{-1}x')$ is bounded and $\mathbf{u}$ is integrable. Then we can make a transformation $x' = U^{tw}(x)$ in the integral which is volume-conserving. We obtain

$$\int_\Omega \int_{r \le w \le s} dx\, dw\, \Phi(x)\, \mathbf{u}(U^{tw}(x), t)\ .$$

5. The reformulation of the NS problem: the concept of a complete weak solution to the NS problem and the two conjectures

Our reformulation of NS problem is based on the concept of the complete weak solution to the NS problem. This **new type** of the weak solution to the NS problem is formulated using the weak flow as a primitive object **instead** of the velocity field.

Definition 5.1. Assume that the smooth initial and boundary data are given.

We shall say that the weak flow $\{U^{rs}\}_{0\le r\le s\le T}$ is a complete weak solution to the NS problem if the following conditions are satisfied

(i) The weak flow $\{U^{rs}\}_{0\le r\le s\le T}$ is the volume-conserving weak flow

(ii) The weak flow $\{U^{rs}\}_{0\le r<s\le T}$ is weakly differentiable. Let $\mathbf{u}(x, t)$ be the velocity field associated to this flow .

(iii) The velocity field $\mathbf{u}(x, t)$ is the standard Leray-Hopf (see [3] and App. B) weak solution to NS equations.

(iv) The (smooth) initial and boundary conditions for the velocity field are satisfied.

The main novity in our approach is two-fold:

(i) The standard central object – the velocity field is substituted by the weak flow describing the (generalized) motion of the fluid

(ii) The divergence equation div $\mathbf{u}$ = 0 is expressed by the requirement of the volume-conservation of this weak flow. This allows (and requires) the use of the homology conditions (and Cartesian currents, in general) which will be new elements in the study of the NS problem. This new input may enable us (possibly) to arrive at the statement that the corresponding complete weak solution exists and is smooth.

Thus we can state the following two conjectures.

The Conjecture 1. (the existence conjecture).

For each smooth initial and boundary conditions there exists a complete weak solution to the NS problem.

The Conjecture 2. (the regularity conjecture).

Each complete weak solution to the NS problem (assuming the smooth initial and boundary conditions) is such that its associated velocity field is smooth.

We assume that the new input containing the homology conditions and the correct formulation of the volume-conservation condition make possible that these two conjectures will be true.

Even in the situation where the regularity proof is not available but the Conjecture 1. would be proved we obtain the strong advantage by having the complete weak solution to NS problem since it gives the weak solution to the complete NS problem (i.e. it defines the weak motion of the fluid).

6. The discussion

It is quite probable that the Conjecture 1. is true since it is completely natural. But this does not mean that the proof must be simple.

On the other side, the Conjecture 2. seems to be rather open.

We have only an idea why the standard regularity conjecture should not be true: since the standard volume-conserving constrain (div **u** = 0) is rather insufficient (i.e. it is not considering correctly distributional minors and determinants). But this is not the argument for the validity of the regularity of the complete weak solution. This argument gives only the possibility of the regularity of the complete weak solution to the NS problem.

7. Conclusions

As a conclusion we can consider the following definitions and findings

- The NS problem must be divided into two parts: the evolution part and the determinant (incompressibility and non-evolution) part
- The determinant part is in the standard way treated in a wrong and insufficient way (i.e. by the condition div $\mathbf{u} = 0$)
- The right treatment of the determinant part must be based on methods from the Geometric measure theory and from the theory of Cartesian currents (see [1], [2], consider weak diffeomorphisms, homology conditions and other concepts from this theory)
- Our approach is based on the union of two rather different parts of real analysis: the theory of NS equations and the Geometric measure theory extended to the theory of Cartesian currents
- This approach requires the change of the basic object of the study: the velocity field must be replaced by the weak (i.e. generalized) flow which is a semigroup of weak diffeomorphisms (introduced in [2])
- The weak differentiability of the weak flow must be carefully defined and also the velocity field associated to the weak flow must be defined in a unique way
- The (hidden) main problem stays in the fact that in the situation when the velocity field is not smooth then the corresponding weak flow can be non-unique and can be also non-existent
- The NS equations are reformulated in a completely new way based on the concept of a weak flow
- The central concept of the **complete weak solution** to the NS problem is defined

Then we formulate two basic conjectures

- The existence conjecture: for smooth data the complete weak solution exists
- The regularity conjecture: (assuming smooth initial and boundary data) the velocity field associated to the complete weak solution is regular

In two appendices technical parts concerning Cartesian currents, homological conditions and existence of the standard weak solutions to NS equations the situation is explained In more details.

Appendix A.

In this section we describe in more details the definition of the graph-current G_U and the intuitive content of homology conditions. Also the weak continuity of determinants is discussed.

1. The detailed description of the graph-current G_U .

To each map U with integrable minors of its Jacobi matrix we can define the corresponding 3-dimensional current G_U in the space $\mathbb{R}^3 \times \mathbb{R}^{3\wedge}$ (here $\mathbb{R}^{3\wedge}$ is the copy of $\mathbb{R}^3$).

We shall use the following useful notation (useful only in $\mathbb{R}^3$)

$$1'=2,\ 1''=3,\ 2'=3.\ 2''=1,\ 3'=1,\ 3''=2, \qquad \partial x = \partial/\partial x$$

in our formulas. The indexes i. j. .. α, β, .. will take values in {1, 2, 3}.

At first we have to classify 3-forms in $\mathbb{R}^3$ where x_1, x_2, x_3 will be coordinate variables in $\mathbb{R}^3$ and y^1, y^2, y^3 will be coordinates in $\mathbb{R}^{3\wedge}$. The 3-form φ can be decomposed as $\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3$, where

$\varphi_0 = a_0 (x, y)\, dx$, $dx = dx_1 \wedge dx_2 \wedge dx_3$

$\varphi_1 = \Sigma\, b^{\alpha}_i (x, y)\, dx_{i'} \wedge dx_{i''} \wedge dy^{\alpha}$

$\varphi_2 = \Sigma\, c^{\beta}{}_j (x, y)\, dx_j \wedge dy^{\beta'} \wedge dy^{\beta''}$

$\varphi_3 = a_3 (x, y)\, dy^1 \wedge dy^2 \wedge dy^3$.

The graph-current G_U is defined as an integration of the 3-form φ over the graph of U. Thus we obtain (assuming that the map U is smooth)

$G_U (\varphi_0) = \int_{\Omega} a_0 (x, U(x))\, dx$

$G_U (\varphi_1) = \Sigma \int b^{\alpha}_i (x, U(x))\, dx_{i'} \wedge dx_{i''} \wedge \partial x_k U^{\alpha}(x)\, dx_k = \Sigma \int b^{\alpha}_i (x, U(x))\, \partial x_i U^{\alpha}(x)\, dx$

$G_U (\varphi_2) = \Sigma \int c^{\beta}_j(x, U(x))\, dx_j \wedge \partial x_m U^{\beta'}(x)\, dx_m \wedge \partial x_k U^{\beta''}(x)\, dx_k$

$= \Sigma \int c^{\beta}_j(x, U)\, dx_j \wedge [\partial x_{j'} U^{\beta'}\, dx_{j'} \wedge \partial x_{j''} U^{\beta''}\, dx_{j''} + \partial x_{j''} U^{\beta'}\, dx_{j''} \wedge \partial x_{j'} U^{\beta''}\, dx_{j'}] = \Sigma \int c^{\beta}_j(x, U)\, [\partial x_{j'} U^{\beta'}\, \partial x_{j''} U^{\beta''} - \partial x_{j''} U^{\beta'}\, \partial x_{j'} U^{\beta''}]\, dx_j \wedge dx_{j'} \wedge dx_{j''} =$

$= \Sigma \int c^{\beta}_j(x, U(x))\, M^{\beta',\beta''}{}_{j',j''}(DU(x))\, dx$

$G_U (\varphi_3) = \Sigma \int a_3 (x, U(x))\, \partial x_i U^1(x)\, dx_i \wedge \partial x_j U^2(x)\, dx_j \wedge \partial x_k U^3(x)\, dx_k$

$= \Sigma \int a_3 (x, U)\, \partial x_i U^1\, \partial x_j U^2\, dx_j\, \partial x_k U^3\, \varepsilon_{ijk}\, dx = \int a_3(x, U(x))\, \det DU(x)\, dx$.

In the situation where U is a Sobolev map from $W^{1,1}(\Omega)$ with the integrable minors of the Jacobi matrix, we define the graph-current G_U by the formula described above (i.e. by the same formula as in the regular case).

2. Homology conditions.

The general form of homology conditions is expressed by $\partial G_U(\omega) = G_U(d\omega) = 0$, where ω is any compactly supported 2-form in $\Omega \times \mathbb{R}^{3\wedge}$. In this subsection we analyze the content of homology conditions in details.

The 2-form ω can be expressed as a sum of the following 2-forms

$\omega^0{}_i = f(x, y)\, dx_{i'} \wedge dx_{i''}$, $\qquad i \in \{1, 2, 3\}$

$\omega^{1,\alpha}{}_j = g(x, y)\, dx_j \wedge dy^\alpha$, $\qquad \alpha, i \in \{1, 2, 3\}$

$\omega^{2,\beta} = h(x, y)\, dy^{\beta'} \wedge dy^{\beta''}$. $\qquad \beta \in \{1, 2, 3\}$

Then we obtain

$d\omega^0{}_i = \partial x_i f(x, y)\, dx_i \wedge dx_{i'} \wedge dx_{i''} + \Sigma_\alpha\, \partial y^\alpha f(x, y)\, dy^\alpha \wedge dx_{i'} \wedge dx_{i''}$,

$d\omega^{1,\alpha}{}_j = \partial x_{j'} g(x, y)\, dx_{j'} \wedge dx_j \wedge dy^\alpha + \partial x_{j''} g(x, y)\, dx_{j''} \wedge dx_j \wedge dy^\alpha + \partial y^{\alpha'} g(x, y)\, dy^{\alpha'} \wedge dx_j \wedge dy^\alpha + \partial y^{\alpha''} g(x, y)\, dy^{\alpha''} \wedge dx_j \wedge dy^\alpha$

$d\omega^{2,\beta} = \Sigma_k\, \partial x_k h^\beta(x, y)\, dx_k \wedge dy^{\beta'} \wedge dy^{\beta''} + \partial y^\beta h^\beta(x, y)\, dy^\beta \wedge dy^{\beta'} \wedge dy^{\beta''}$.

The application of the Cartesian current G_U onto the 3-form dω gives the rather complication expression, so that we choose the special form of 2-forms ω's

$\omega^{0,\gamma}{}_i = \varphi(x)\, y^\gamma\, dx_{i'} \wedge dx_{i''}$, $\qquad i, \gamma \in \{1, 2, 3\}$

$\omega^{1,\alpha,\gamma}{}_j = \varphi(x)\, y^\gamma\, dx_j \wedge dy^\alpha$, $\qquad \alpha, \gamma, i \in \{1, 2, 3\}$

$\omega^{2,\beta,\gamma} = \varphi(x)\, y^\gamma\, dy^{\beta'} \wedge dy^{\beta''}$. $\qquad \beta, \gamma \in \{1, 2, 3\}$

Then we obtain

$d\omega^{0,\gamma}{}_i = \partial x_i \varphi(x)\, y^\gamma\, dx_i \wedge dx_{i'} \wedge dx_{i''} + \varphi(x)\, dy^\gamma \wedge dx_{i'} \wedge dx_{i''}$

$d\omega^{1,\alpha,\gamma}{}_j = \partial x_{j'} \varphi(x)\, y^\gamma\, dx_{j'} \wedge dx_j \wedge dy^\alpha + \partial x_{j''} \varphi(x)\, y^\gamma\, dx_{j''} \wedge dx_j \wedge dy^\alpha + \varphi(x)\, dy^{\alpha'} \wedge dx_j \wedge dy^\alpha\, \delta_{\gamma\alpha'} + \varphi(x)\, dy^{\alpha''} \wedge dx_j \wedge dy^\alpha\, \delta_{\gamma\alpha''}$.

$d\omega^{2,\beta,\gamma} = \Sigma\, \partial x_k \varphi(x)\, y^\gamma\, dx_k \wedge dy^{\beta'} \wedge dy^{\beta''} + \varphi(x)\, dy^\beta \wedge dy^{\beta'} \wedge dy^{\beta''}\, \delta_{\gamma\beta}$.

Then applying the graph-current G_U to these forms we obtain

$G_U(d\omega^{0,\gamma}{}_i) = \int \partial x_i \varphi(x)\, U^\gamma(x)\, dx_i \wedge dx_{i'} \wedge dx_{i''} + \varphi(x)\, \partial x_i U^\gamma\, dx_i \wedge dx_{i'} \wedge dx_{i''} = \int [\partial x_i \varphi(x)\, U^\gamma + \varphi(x)\, \partial x_i U^\gamma]\, dx = \int \partial x_i [\varphi(x)\, U^\gamma]\, dx = 0$

This defines the distributional derivative of U^γ. Then we have

$G_U(d\omega^{1,\alpha,\gamma}{}_j) = \int \partial x_{j'} \varphi\, U^\gamma\, dx_{j'} \wedge dx_j \wedge \partial x_{j''} U^\alpha\, dx_{j''} + \partial x_{j''} \varphi\, U^\gamma\, dx_{j''} \wedge dx_j \wedge \partial x_{j'} U^\alpha\, dx_{j'} + \varphi dx_j \wedge [(\partial x_{j'} U^{\alpha'}\, dx_{j'} + \partial x_{j''} U^{\alpha'}\, dx_{j''})]\, \delta_{\gamma\alpha'} \wedge [(\partial x_{j'} U^\alpha\, dx_{j'} + \partial x_{j''} U^\alpha\, dx_{j''})] + \varphi dx_j \wedge [(\partial x_{j'} U^{\alpha''} dx_{j'} + \partial x_{j''} U^{\alpha''} dx_{j''})]\, \delta_{\gamma\alpha''} \wedge [(\partial x_{j'} U^\alpha\, dx_{j'} + \partial x_{j''} U^\alpha\, dx_{j''})]$

$= \int [\partial x_{j'} \varphi\, U^\gamma\, \partial x_{j''} U^\alpha\, (-1) + \partial x_{j''} \varphi\, U^\gamma\, \partial x_{j'} U^\alpha] dx + \varphi\, \delta_{\gamma\alpha'}\, [\partial x_{j'} U^{\alpha'}\, \partial x_{j''} U^\alpha - \partial x_{j''} U^{\alpha'}\, \partial x_{j'} U^\alpha] + \varphi\, \delta_{\gamma\alpha''}\, [\partial x_{j'} U^{\alpha''}\, \partial x_{j''} U^\alpha - \partial x_{j''} U^{\alpha''}\, \partial x_{j'} U^\alpha]\, dx$

$= \int U^\gamma\, [\partial x_{j''} \varphi\, \partial x_{j'} U^\alpha - \partial x_{j'} \varphi\, \partial x_{j''} U^\alpha] dx + \varphi \delta_{\gamma\alpha'} M^{\alpha',\alpha}{}_{j',j''}(DU) dx + \varphi \delta_{\gamma\alpha''} M^{\alpha'',\alpha}{}_{j',j''}(DU) dx$

Here the only interesting homology conditions are those which contain minors of different order – i.e. where $\delta_{\gamma\alpha'}$ or $\delta_{\gamma\alpha''}$ are nonzero – i.e. where $\gamma=\alpha'$ or $\gamma=\alpha''$. In this way we obtain two homology conditions

(i) $G_U(d\omega^{1,\alpha,\alpha'}{}_j) = \int U^{\alpha'} [\partial x_{j''} \varphi\, \partial x_{j'} U^\alpha - \partial x_{j'} \varphi\, \partial x_{j''} U^\alpha]\, dx + \varphi\, M^{\alpha',\alpha}{}_{j',j''}(DU)\, dx$

(ii) $G_U(d\omega^{1,\alpha,\alpha''}{}_j) = \int U^{\alpha''} [\partial x_{j''} \varphi\, \partial x_{j'} U^\alpha - \partial x_{j'} \varphi\, \partial x_{j''} U^\alpha]\, dx + \varphi\, M^{\alpha'',\alpha}{}_{j',j''}(DU)\, dx$

Both expressions must be equal to zero and this gives the distributional definition of 2 x 2 minors of DU in terms of DU.

$G_U(d\omega^{2,\beta,\gamma}) = \int \Sigma\, \partial x_k \varphi\, U^\gamma\, dx_k \wedge [\partial x_{k'} U^{\beta'}\, dx_{k'} + \partial x_{k''} U^{\beta'}\, dx_{k''}] \wedge [\partial x_{k'} U^{\beta''}\, dx_{k'} + \partial x_{k''} U^{\beta''}\, dx_{k''}] + \varphi\, \Sigma\, \partial x_i U^\beta\, dx_i \wedge \partial x_j U^{\beta'} dx_j \wedge \partial x_k U^{\beta''} dx_k\, \delta_{\gamma\beta} = \int \Sigma\, \partial x_k \varphi\, U^\gamma\, [\partial x_{k'} U^{\beta'}\, \partial x_{k''} U^{\beta''} - \partial x_{k''} U^{\beta'}\, \partial x_{k'} U^{\beta''}]\, dx + \varphi\, \Sigma\, \partial x_i U^\beta\, \partial x_j U^{\beta'}\, \partial x_k U^{\beta''}\, \varepsilon_{ijk}\, dx\, \delta_{\gamma\beta}$.

This condition is non-trivial only when $\delta_{\gamma\beta} = 1$ (i.e. $\gamma=\beta$) and in this case it gives the distributional definition of the determinant of DU using 2 x 2 minors of DU.

$G_U(d\omega^{2,\beta,\beta}) = \int \Sigma\, \partial x_k \varphi\, U^\beta\, M^{\beta',\beta''}{}_{k',k''}(DU)\, dx + \varphi\, \det DU\, dx = 0$.

In the NS problem we have detDU = 1 a.e. (i.e. the volume-conservation) and thus the last equation is the condition required for 2 x 2 minors of DU.

Appendix B.

The explicit definition of the standard weak solution to the NS equations will be described in this section (all statements in this appendix are standard and we take them from [3]).

We shall use the formulation and the existence theorem of the standard weak solution to the NS equations from the recent paper [3]. For the formulation of the NS problem some function spaces must be used [3]. More details can be found in [3].

At a start we shall define spaces with zero divergence used here. Let Ω be a bounded domain in R^3 with the Lipschitz boundary.

The basic space $L^2_{0,\mathrm{div}}(\Omega)$ is defined as a closure of the space $\{\mathbf{u} \in (C^\infty_0(\Omega))^3 ; \mathrm{div}\, \mathbf{u} = 0\}$ in the L^2 norm ([3], 2.3.3).

The analogous Sobolev space is defined by ([3], 2.3.2)

$$W^{1,2}_{0,\mathrm{div}}(\Omega) = \{\mathbf{u} \in (W^{1,2}_0 (\Omega))^3 ; \mathrm{div}\, \mathbf{u} = 0\}$$

and its dual space $(W^{1,2}_{0,\mathrm{div}}(\Omega)^*$ is defined in the standard way. If $f \in (W^{1,2}_{0,\mathrm{div}}(\Omega)^*$ and $u \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, then the duality between f and u will be denoted by f [u].

The Bochner spaces are used. Let I = (0, T) be a finite interval on the time axis and let X be a Banach space. We shall say that the function $f : I \rightarrow X$ is a simple function if there exists a measurable decomposition $\{O_1, .. , O_k\}$ of an interval I such that f is constant on each O_i. A function $f : I \rightarrow X$ is called strongly measurable if there exists a sequence of simple functions f_n such that $\lim_{n\rightarrow\infty} ||f_n(t) - f(t)||_X = 0$ for a.e. $t \in I$. Then we define the Bochner space $L^1(I; X)$ as a set of all strongly measurable functions $f : I \rightarrow X$ such that the function $||f(.)||_X$ is integrable over I ([3], 2.2, 2.2.1).

Analogously, the Bochner space $L^2(I, X)$ is defined as a set of all functions $f \in L^1(I; X)$ such that $(||f(.)||_X)^2$ is integrable over I and the Bochner space $L^\infty(I, X)$ is defined as a set of all functions $f \in L^1(I; X)$ such that ess sup $||f(.)||_X$ is bounded over I ([3], 2.2.1).

Bochner spaces will be used only when X is $L^2_{0,\mathrm{div}}(\Omega)$ or $W^{1,2}_{0,\mathrm{div}}(\Omega)$ or $(W^{1,2}_{0,\mathrm{div}}(\Omega))^*$. (More details on useful function spaces can be found in [3], Chap. 2.)

Now we are ready to formulate the concept of the standard (Leray-Hopf) weak solution to the NS equations. We shall call this weak solution a standard weak solution since we have introduced above a new concept of a complete weak solution to the NS problem. (See [3], 3.1.)

Let $\mathbf{f} \in L^2(I; (W^{1,2}_{0,\mathrm{div}}(\Omega))^*)$, $\mathbf{u}_0 \in L^2_{0,\mathrm{div}}(\Omega)$.

The function $\mathbf{u} \in L^2(I; W^{1,2}_{0,div}(\Omega)) \cap L^\infty(I; L^2_{0,div}(\Omega))$ with $\partial\mathbf{u}/\partial t \in L^1(I; (W^{1,2}_{0,div}(\Omega))^*)$ is called a standard (Leray-Hopf) weak solution to the Navier–Stokes equations if

(i) $\partial\mathbf{u}/\partial t[\boldsymbol{\phi}] + \int_\Omega(\mathbf{u} \cdot \nabla\mathbf{u})\cdot\boldsymbol{\phi}\, dx + \nu \int_\Omega \nabla\mathbf{u}:\nabla\boldsymbol{\phi}\, dx = \mathbf{f}[\boldsymbol{\phi}]$, $\forall\boldsymbol{\phi} \in W^{1,2}_{0,div}(\Omega)$ and a.e. $t \in I$,
(The evolution equation for the velocity field; here ν denotes the viscosity.)

(ii) $\lim_{t\to0+} \int_\Omega \mathbf{u}(t, \cdot) \cdot \boldsymbol{\phi}\, dx = \int_\Omega\mathbf{u}_0 \cdot \boldsymbol{\phi}\, dx$, $\forall\boldsymbol{\phi} \in L^2_{0,div}(\Omega)$.
(Initial conditions.)

(iii) $\int_\Omega |\mathbf{u}(t)|^2\, dx + 2\nu \int_{(0,t)} \int_\Omega |\nabla\mathbf{u}|^2\, dx\, d\tau \leq \int_\Omega |\mathbf{u}_0|^2\, dx + 2 \int_{(0,t)} \mathbf{f}[\mathbf{u}]\, d\tau$ for a.e. $t \in I$.
(So-called energy inequality.)

Then in [3, 3.1.2] the existence of the standard (Leray-Hopf) weak solution to the NS equations is proved.

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