Two Notes on Regular Polygons: Geometric Motivation of the π Constant

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Abstract. In the paper we prove that the ratio between the circumference of the incircle of the regular n-gon and its perimeter is equivalent to the ratio of their areas, respectively. These ratios are constants for regular n-gons. Also, it is shown that the ratio of circumference of the excircle and perimeter of the regular n-gon is not the same as the ratio of areas of the excircle and this regular n-gon.

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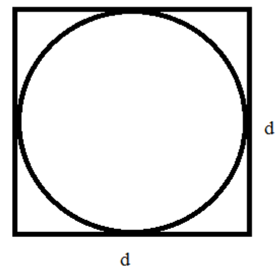
1. INTRODUCTION.

Importance of the number π is well known beyond mathematics and even recognized by the assigned designated day of celebration in the scientific calendar: March 14th is the international pi day. Despite a huge amount of investigations related to constant π (see e.g. [1],[2],[3]) there is a logical gap for fans of mathematics. The presence of this greatest mathematical constant still leaves unanswered questions about discovery of its definition: ratio of circumference and diameter. What geometric meaning was put behind this ratio?. While searching for the answer new properties of inscribed and circumscribed regular polygons are obtained.

In terms of elementary geometry we prove two basic properties of regular n-gons in relation with their incircles and excircles. The first theorem says that the ratio of the circumference of an incircle of a regular n-gon and its perimeter is equal to the ratio of areas of this incircle and regular n-gon, respectively. Also, it is noticed that for any regular n-gons (with fixed number $n \geq 3$) those ratios are π related constants. In the second theorem it is shown that the ratio of circumference of an excircle to the perimeter of the subscribed regular n-gon is not equal to the ratio of the areas of this excircle and this regular n-gon. Although, these ratios are still π related different constants.

2. RESULTS AND PROOFS.

Let's take a look at the definition of π again: $\pi = \frac{C(d)}{d}$, where $C(d)$ is the circumference of a circle and d is the diameter. Since a circle is a smooth geometric figure we can call its circumference a perimeter. It is reasonable to find the relationship between circumference (perimeter) and area of a circle and same parameters (perimeter and area) of circumscribed non-smooth figures. For this intent we are comparing the value of $C(d)$ to the perimeter of the circumscribed square around the circle. This relation most clearly represents the essence of describing constant π number. Let's rewrite the original equation $\pi = \frac{C(d)}{d}$ in the equivalent view: $\frac{\pi}{4} = \frac{C(r)}{4d}$. Here $4d$ is the perimeter of the circumscribed square. Hence $\frac{\pi}{4} = \frac{C(d)}{P_4(d)}$, where $P_4(d)$ is the perimeter of the circumscribed square. Let's denote the area of incircle by $A(d)$ and the area of a square by $A_4(d)$. Then $\frac{A(d)}{A_4(d)} = \frac{\frac{\pi d^2}{4}}{d^2} = \frac{\pi}{4}$.

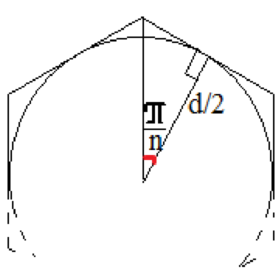


We landed on previously unknown property in relation with π number which can be written as: $\frac{C(d)}{P_4(d)} = \frac{A(d)}{A_4(d)} = const. = \frac{\pi}{4}$.

This hidden property about ratios of perimeters and areas of circle and circumscribed square raises a question: Are the ratios of perimeters and areas of a circle and circumscribed and inscribed n-gons also constant?

Theorem 1. (Constant ratios): Let $P_n(d)$ and $A_n(d)$ be perimeter and area of a regular n-gon and d be the diameter of an incircle. Then the following proportion is valid

$$\frac{C(d)}{P_n(d)} = \frac{A(d)}{A_n(d)} = \frac{\pi}{ntan(\frac{\pi}{n})}.$$



Proof: Since $\frac{C(d)}{P_n(d)} = \frac{\pi d}{ndtan(\frac{\pi}{n})} = \frac{\pi}{ntan(\frac{\pi}{n})}$ and $\frac{A(d)}{A_n(d)} = \frac{\frac{\pi d^2}{4}}{\frac{1}{2}n\left(\frac{d^2}{2}\right)tan(\frac{\pi}{n})} = \frac{\pi}{ntan(\frac{\pi}{n})}$ we get the equality $\frac{C(d)}{P_n(d)} = \frac{A(d)}{A_n(d)} = \frac{\pi}{ntan(\frac{\pi}{n})}$. Q.E.D.

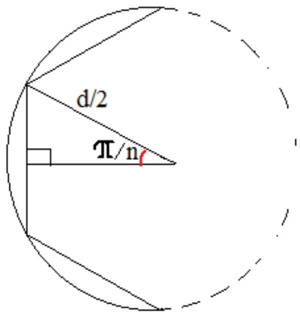
Theorem 1 directly implies that $\lim_{n\to\infty}\left(\frac{\pi}{ntan(\frac{\pi}{n})}\right) = 1$. Therefore, if n approaches to infinity then $\frac{C(d)}{P_n(d)} = \frac{A(d)}{A_n(d)}$=1, i.e. a regular n-gon with an infinitely many number of sides is a good approximation of a circle.

Now we examine whether a similar result is valid for n-gons inscribed into the circle, i.e. if the ratio of the perimeters is equal to the ratio of the areas?. The answer to this question is given below.

Theorem 2. (Different ratios): Let $P_n(d)$ and $A_n(d)$ be perimeter and area of a regular n-gon and d be the diameter of the excircle on this n-gon. Then, we have:

$$\frac{C(d)}{P_n(d)} = \frac{\pi}{nsin(\frac{\pi}{n})},$$

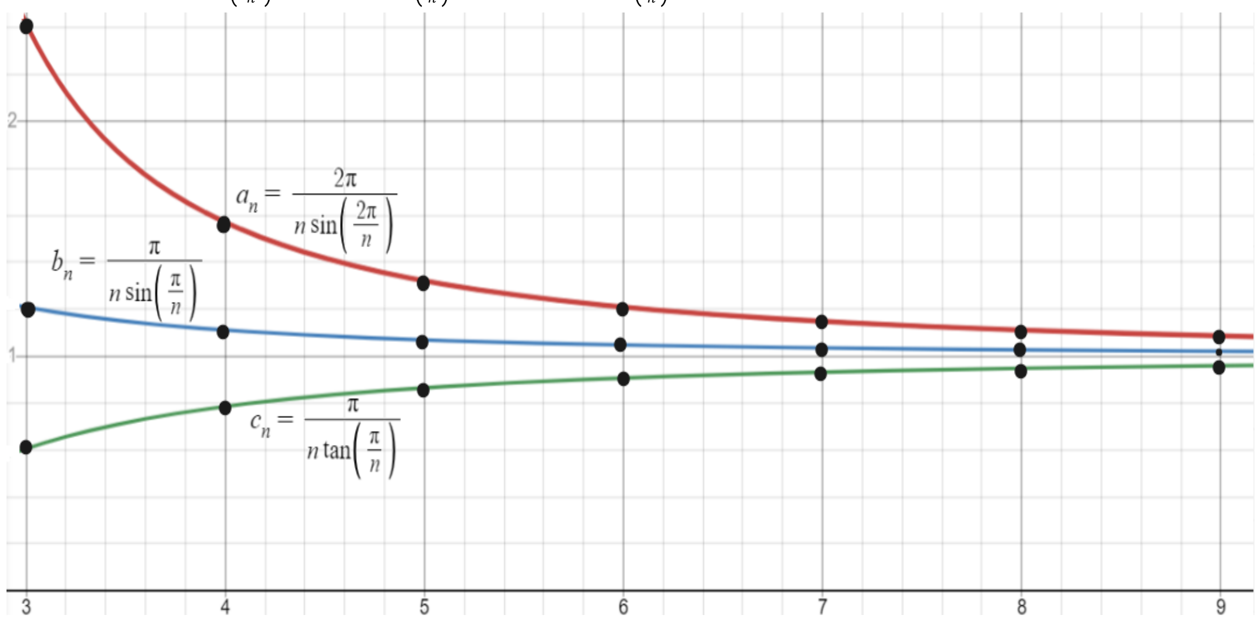
$$\frac{A(d)}{A_n(d)} = \frac{2\pi}{n\sin(\frac{2\pi}{n})}.$$



Proof: It is obvious that $\frac{C(d)}{P_n(d)} = \frac{\pi d}{nd\sin(\frac{\pi}{n})} = \frac{\pi}{n\sin(\frac{\pi}{n})}$. For the second equality have $\frac{A(d)}{A_n(d)} = \frac{\pi\frac{d^2}{4}}{\frac{1}{2}n(\frac{d^2}{4})\sin(\frac{2\pi}{n})} = \frac{2\pi}{n\sin(\frac{2\pi}{n})}$. Q.E.D.

Let's notice that $\lim_{n\to\infty}\left(\frac{\pi}{n\sin(\frac{\pi}{n})}\right) = 1$ and $\lim_{n\to\infty}\left(\frac{2\pi}{n\sin(\frac{2\pi}{n})}\right) = 1$, i.e. if n approaches to infinity then $\frac{C(d)}{P_n(d)} = \frac{A(d)}{A_n(d)}$=1.

Finally, for the convenience of readers we marked above established three constant ratios as follows: $a_n = \frac{2\pi}{n\sin\left(\frac{2\pi}{n}\right)}$, $b_n = \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}$ and $c_n = \frac{\pi}{n\tan\left(\frac{\pi}{n}\right)}$, where $n \geq 3$.



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