COMPLEX ANALYSIS AND THEORY OF REPRODUCING KERNELS

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Abstract. The theory of reproducing kernels is very fundamental, beautiful and will have many applications in analysis, numerical analysis and data sciences. In this paper, some essential results in complex analysis derived from the theory of reproducing kernels will be introduced, simply.

Key Words: Bergman kernel, Szegö kernel, Rudin kernel, Hardy reproducing kernel, weighted Szegö kernels, Dirichlet integral, weighted Bergman kernel, conjugate analytic Hardy H_2 norm, Green function, isoperimetric inequality, general theory of reproducing kernels, positive definiteness, norm inequalities, Suita's conjecture, Saitoh's conjecture, Yamada's conjecture, deep results of Q. Guan, analytic extension formula, sampling theory, discretization, Aveiro discretization method.

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1. Introduction

In this paper, we would like to summarize simply the relationship of complex analysis and the theory of reproducing kernels. For some general source for reproducing kernels, see the books ([32, 34, 38]) and the papers ([24, 37]).

2. Origin of Reproducing Kernels in Conformal Mappings

Professor Begehr wrote in [4]:

100 years ago in summer 1922 two young mathematicians defended their Ph.D. theses at the university in Berlin, Gábor Szegö and Stefan Bergmann (later: Bergman). Szegö at that time had published a couple of papers already and defended his second thesis (habilitation thesis), [3], while Bergman, an assistant with Richard von Mises, [4], p. 472, had just started his publications in this particular year

and their details. Both famous mathematicians were interested in the practical construction of the Riemann mapping function, by using reproducing kernels, and their spirit may be represented by the kernel function theory and their impact was great. See [6, 7] and [25].

3. Three Typical Reproducing Kernels in One Complex Analysis

We introduce three reproducing kernels in one complex analysis.

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The Bergman kernel. Let D be a bounded regular domain on the complex $z = x+iy$ plane whoes boundary is composed of a finite number N of disjoint analytic Jordan curves. Let $AL_2(D)$ be a Hilbert space (Bergman space) comprising analytic functions $f(z)$ on D and with finite norms $\|f\|_{AL_2(D)} = \left\{\int\int_D |f(z)|^2 dxdy\right\}^{\frac{1}{2}}$. As we see simply, $f \to f(z)(z \in D)$ is a bounded linear functional on $AL_2(D)$. Therefore, there exists a reproducing kernel $K(z,u)$ such that for any $u \in D$ and for any function $f \in AL_2(D)$,

$$f(u) = \int\int_D f(z)\overline{K(z,u)}dxdy. \tag{3.1}$$

From $K(z,u) = \overline{K(u,z)}$, $K(z,u)$ is analytic in $\overline{u}$ (complex conjugate). So, we shall denote it as $K(z,\overline{u})$. This is the Bergman kernel of D or on D.

Let $G(z,u)$ be the Green's function for the Laplace equation on D with pole at $u \in D$. Then, we have the identity

$$K(z,\overline{u}) = -\frac{2}{\pi}\frac{\partial^2 G(z,u)}{\partial z \partial \overline{u}}. \tag{3.2}$$

Here we shall introduce the adjoint $L-$ kernel for $K(z,\overline{u})$ by

$$L(z,u) = -\frac{2}{\pi}\frac{\partial^2 G(z,u)}{\partial z \partial u}.$$

The adjoint L kernel $L(z,u)$ has one double pole at u on D such that

$$L(z,u) = \frac{1}{\pi(z-u)^2} + \text{regular terms}$$

of the second order. Furthermore, along the boundary ∂D we have the identity

$$\overline{K(z,\overline{u})dz} = -L(z,u)dz. \tag{3.3}$$

This important identity shows that the Bergman kernel is indeed an analytic differential $K(z,\overline{u})dz$ and it is continued analytically to the double - closed Riemann surface - of D, and at the symmetric point u of $\overline{u}$ it has the same double pole as in the above. So, the Bergman kernel is a fundamental differential on the closed Riemann surface.

The Szegö kernel. Let $H_2(D)$ be the Hardy 2 analytic function space on D. A member of the class has nontangential boundary values belonging to $L_2(\partial D)$. Let $AL_2(\partial D)$ denote its closed subspace in $L_2(\partial D)$, and we introduce the norm in $H_2(D)$ by

$$\left\{\int_{\partial D} |f(z)|^2 |dz|\right\}^{\frac{1}{2}} < \infty.$$

For a function $f \in AL_2(\partial D)$, as we see from the Cauchy integral formula, since $f \to f(u), u \in D$ is a bounded linear functional, there exists a reproducing kernel $\hat{K}(z,u)$ satisfying

$$f(u) = \int_{\partial D} f(z)\overline{\hat{K}(z,u)}|dz|.$$

As in the Bergman kernel, $\hat{K}(z,u)$ is analytic in $\overline{u}$, and so we shall denote it by $\hat{K}(z,\overline{u})$. This is the Szegö kernel of D or on D.

The important concept is the adjoint L kernel $\hat{L}(z,u)$ for $\hat{K}(z,\overline{u})$. That is, it is a meromorphic function on $D \cup \partial D$ and only at the point u, it has a simple pole

$$\hat{L}(z,u) = \frac{1}{2\pi(z-u)} + \text{regular terms}, \tag{3.4}$$

and along the boundary ∂D it satisfies the relation

$$\overline{\hat{K}(z,\overline{u})}|dz| = \frac{1}{i}\hat{L}(z,u)dz. \tag{3.5}$$

This adjoint L kernel is a uniquley determined by these properties on $D \cup \partial D$. From (3.5) we have the identity

$$\hat{L}(z,u) = -\hat{L}(u,z) \quad \text{on} \quad D \times D. \tag{3.6}$$

The adjoint L kernel $\hat{L}(z,u)$ is also characterized by the minimum property:

$$\min \int_{\partial D} |h(z,u)|^2 |dz| = \int_{\partial D} |\hat{L}(z,u)|^2 |dz|.$$

Here the minimum is considered on the meromorphic functions $\{h(z,u)\}$ on D with the singularity at u as in (3.4) and satisfying $\left\{\int_{\partial D} |h(z,u)|^2|dz|\right\}^{\frac{1}{2}} < \infty$.

A deep and important property of the adjoint L kernel $\hat{L}(z,u)$ is that it does not have any zero point on $D \cup \partial D$. From this fact and (3.5),

$$f_0(z;u) = \frac{\hat{K}(z,\overline{u})}{\hat{L}(z,u)}$$

is the **Ahlfors function** of D with respect to u. That is, $f_0(z;u)$ is an analytic function on D, among the analytic functions satisfying the properties

$$f(u;u) = 0, f'(u;u) \geq 0 \quad \text{and} \quad |f(z;u)| \leq 1 \quad \text{on} \quad D,$$

it satisfies the extremal property $f_0'(u;u) = \max f'(u;u)$. - This may be related to the **analytic capacity**. Then, as we see from (3.5) and the principle of argument, the function $f_0(z;u)$ maps D onto the unit disc and the disc is covered N times. Here, we assume D is N-ply connected.

From (3.5) along ∂D we see

$$\overline{\hat{K}(z,\overline{u})^2 dz} = -\hat{L}(z,u)^2 dz. \tag{3.7}$$

From this relation, we see that the square of the Szegö kernel is an analyic differential, it is continued to the double of D analytically that is a closed Riemann surface as in the Bergman kernel. In this sense, the Szegö kernel is a half order differential on the closed Riemann surface and so its properties are very involved for the sake of multi-valuedness on the closed Riemann surface. The profound theory for these properties was done by D. A. Hejhal [22] and J. D. Fay [18] by using the **Riemann theta functions** and the **Klein prime forms**. As we see from (3.5), the Szegö kernel is a Cauchy kernel on the Riemann surface and so it is very important reproducing kernel.

The Hardy reproducing kernel. Let $H_2(D)(\hat{H}_2(D), \text{resp.})$ be the Hilbert space equipped the norm in the space $H_2(D)$:

$$\left\{\frac{1}{2\pi}\int_{\partial D}|f(z)|^2\frac{\partial G(z,t)}{\partial \nu_z}|dz|\right\}^{\frac{1}{2}} \tag{3.8}$$

$$\left(\left\{\frac{1}{2\pi}\int_{\partial D}|f(z)|^2\left(\frac{\partial G(z,t)}{\partial \nu_z}\right)^{-1}|dz|\right\}^{\frac{1}{2}}, \text{resp.}\right). \tag{3.9}$$

Here, $\partial/\partial\nu_z$ is the inner normal derivative with respect to D. $\partial G(z,t)/\partial\nu_z$ is a positive continuous function on ∂D and so as in the Szegö space, we can consider the reproducing kernel $K_t(z,\overline{u})(\hat{K}_t(z,\overline{u}), \text{resp.})$ satisfying

$$f(u) = \frac{1}{2\pi}\int_{\partial D} f(z)\overline{K_t(z,\overline{u})}\frac{\partial G(z,t)}{\partial \nu_z}|dz|,$$

$$\left(f(u) = \frac{1}{2\pi}\int_{\partial D} f(z)\overline{\hat{K}_t(z,\overline{u})}\left(\frac{\partial G(z,t)}{\partial \nu_z}\right)^{-1}|dz|, \text{resp.}\right).$$

We call $K_t(z,\overline{u})$ and $\hat{K}_t(z,\overline{u})$ the Hardy reproducing kernel and its conjugate reproducing kernel of D.

For an arbitrary open Riemann surface S, we can introduce the space $H_2(S)$ similarly and the Hardy reproducing kernel. The reproducing kernel $K_t(z,\overline{u})$ is introduced by W. Rudin in 1955.

Let $G^*(z,t)$ be the conjugate harmonic function of $G(z,t)$ and when we form the multi-valued meromorphic function $W(z,t) = G(z,t) + iG^*(z,t)$, $idW(z,t)$ is a single-valued meromorphic differential, and it satisfies along ∂D,

$$\frac{\partial G(z,t)}{\partial \nu_z}|dz| = idW(z,t).$$

Therefore, the integral (3.9) is represented by

$$\frac{1}{2\pi}\int_{\partial D}\frac{|f(z)dz|^2}{idW(z,t)}.$$

This will mean that $\hat{H}_2(D)$ is indeed comprised of analytic differentials $f(z)dz$. Therefore, the conjugate space may be considered on bordered Riemann surfaces with some good boundaries.

For $K_t(z,\overline{u})(\text{resp.}\hat{K}_t(z,\overline{u}))$, the adjont L kernel $L_t(z,u)(\text{resp.}\hat{L}_t(z,u))$, that is a meromorphic function on $D\cup\partial D$ with only one simle pole at u with residue 1, is characterized by the identity, along ∂D

$$\overline{K_t(z,\overline{u})}idW(z,t) = \frac{1}{i}L_t(z,u)dz \tag{3.10}$$

$$\left(\overline{\hat{K}_t(z,\overline{u})dz} = \frac{1}{i}\hat{L}_t(z,u)idW(z,t) \quad \text{along} \quad \partial D, \text{resp.}\right). \tag{3.11}$$

From (3.10) and (3.11), we have a very important property:

$$L_t(z,u) = -\hat{L}_t(u,z) \quad \text{on} \quad D\times D. \tag{3.12}$$

The relations (3.10), (3.11) and (3.12) show that $L_t(z,u)dz$ is a meromorphic differential with respect to z, a meromorphic function in u, and along $z=u$ with one simple pole with residue 1; that is the Cauchy kernel of D. Therefore, these reproducing kernels are very important as in the Bergman and Szegö kernels as the third kind of reproducing kernels in one complex variable analysis ([32, 34]). These reproducing kernels were examined deeply on Riemann surfaces by J. D. Fay [18] and A. Yamada [40],[41] in terms of the Riemann theta functions and the Klein prime forms.

The norms of Bergman spaces are given by surface integrals on the domains and so, we can consider them on some very general domains as in on regular domains, meanwhile, for the Szegö spaces, their norms are given in terms of boundary integrals and so, the Szegö spaces may be considered for some very respective domains for their boundary integrals. In this viewpoint, it will be very interested in the H_2 spaces, because the norms are given by boundary integrals and however, for any open domains we can consider them.

For the conjugate Hardy norm

$$\frac{1}{2\pi}\int_{\partial D}|f(z)|^2\left(\frac{\partial G(z,t)}{\partial \nu_z}\right)^{-1}|dz| = \frac{1}{2}\int_{\partial D}\frac{|f(z)dz|^2}{idW(z,t)},$$

we obtain the best possible norm inequality:

$$\int\int_D |f'(z)|^2 dxdy \leq \frac{1}{2}\int_{\partial D}\frac{|f'(z)dz|^2}{idW(z,t)}, \tag{3.13}$$

and from this we see the naturality of the norm looked curiously. This inequality is not so simple to derive and for its proof we must examine deeply the relations among the Hardy reproducing kernel, its conjugate kernel and the Bergman kernel ([29]).

4. Suita Conjecture and its Great Extension by Q. Guan

For the Bergman kernel and the Szegö kernel on D, we have the basic and deep relation

$$K(z,\overline{u}) >> 4\pi\hat{K}(z,\overline{u})^2$$

– the left minus the right is a positive definite quadratic form function – which was given by D. A. Hejhal [23]. This profound result was given on the long historical lines as in

G.F.B. Riemann (1826-1866); F. Klein (1849-1925); S. Bergman; G. Szegö; Z. Nehari; M.M. Schiffer; P.R. Garabedian (1949 published); D.A. Hejhal (1972 published).

It seems that any elementary proof is impossible, however, the result will, in particular, mean the fairly simple inequality:

For two functions φ and ψ of $H_2(D)$, we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi}\int\int_D|\varphi(z)\psi(z)|^2dxdy \leq \frac{1}{2\pi}\int_{\partial D}|\varphi(z)|^2|dz|\frac{1}{2\pi}\int_{\partial D}|\psi(z)|^2|dz|, \tag{4.1}$$

and we can determine completely the cases holding the equality here. In the thesis [28] of the author published in 1979 the result was given. The author realized the importance of the abstract and general theory of reproducing kernels by N. Aronszajn. In the paper, the core part was to determine the equality statement in the above inequality, surprisingly enough, some deep and general independing poof was appeared 26 years later in A. Yamada (see the appendix of [38]). A. Yamada was developed deeply equality problems for some general norm inequalities derived by the theory of reproducing kernels and it was published in the book appendix of [38].

Of course, in the thesis we can find some fundamental idea for nonlinear transforms. In particular, for the special case $\varphi \equiv \psi \equiv 1$, for the plane measure $m(D)$ of D and the length ℓ of the boundary we have the isoperimetric inequality

$$4\pi m(D) \leq \ell^2.$$

Meanwhile, for the Hardy reproducing kernel we can give the conjecture in the converse direction, from the case of doubly connected domains

$$\pi K(z,\overline{u}) << K_t(z,\overline{u})\hat{K}_t(z,\overline{u}) \tag{4.2}$$

([28]). From the relation we can derive various results, the conjecture will be, however, very difficult to solve.

The author attacked to this problem several years, it was, however, impossible to solve it and in those days, the author derived the above result applying the result of Hejhal conversely to the general theory of reproducing kernels by Aronszajn. The author thinks for the conjecture we must use the deep theory of J. D. Fay, however, his profound theory seems to be too deep and great for the author. So, from the viewpoints of fundamental results and applicable analysis, the author turned his research interest for the general theory of reproducing kernels.

Basic Open Problems

For the space $H_2(D)$, by exchanging

$$\frac{1}{2\pi}\frac{\partial G(z,t)}{\partial \nu_z}$$

with a general positive continuous function ρ, we consider the weighed Szegö space $H_2^\rho(D)$ and the corresponding the weighted Szegö kernel $K_\rho(z,\overline{u})$ and the conjugate kernel $L_\rho(z,u)$; that is, for the space

$$\left\{ \int_{\partial D} |f(z)|^2 \rho(z)|dz| \right\}^{\frac{1}{2}} < \infty$$

with the weighted norm, we consider the reproducing kernel $K_\rho(z,\overline{u})$ satisfying

$$f(u) = \int_{\partial D} f(z)\overline{K_\rho(z,\overline{u})}\rho(z)|dz|.$$

and

$$\overline{K_\rho(z,\overline{u})}\rho(z)|dz| = \frac{1}{i}L_\rho(z,u)dz, \tag{4.3}$$

along ∂D. In particular, note that $L_\rho(z,u)$ has a similar singularity at $z=u$ as $\hat{L}(z,u)$.

For the weight ρ^{-1}, we see that the very important relations as in the Hardy case

$$L_\rho(z,u) = -L_{\rho^{-1}}(u,z)$$

and

$$\overline{K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u})dz} = -L_\rho(z,u)L_{\rho^{-1}}(u,z)dz. \tag{4.4}$$

From the boundary relations (3.3) and (4.4) on kernels, for any simply connected regular domain we have the identity

$$4\pi K(z,\overline{u}) \equiv K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u}). \tag{4.5}$$

We do not know whether the case (4.5) happens for $N>1$ for some ρ. Note, in particular, that for this case we have the similar result in generalizations of the inequality (4.5) for the weighted Szegö norms.

Now, we can state the basic open problems:

Basic big open problems:

(A) Look for the condition of ρ such that

$$4\pi K(z,\overline{u}) >> K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u}) \tag{4.6}$$

is valid.

(B) Look for the condition of ρ such that

$$4\pi K(z,\overline{u}) << K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u}) \tag{4.7}$$

is valid.

By the deep result of D. A. Hejhal and our example, this open problems are valid, in general, as very difficult problems. We can state the partial open problems as in

Partial big open problems:

(A′) Look for the condition of ρ such that

$$4\pi K(z,\overline{z}) > K_\rho(z,\overline{z})K_{\rho^{-1}}(z,\overline{z}) \tag{4.8}$$

is valid.

(B′) Look for the condition of ρ such that

$$4\pi K(z,\overline{z}) < K_\rho(z,\overline{z})K_{\rho^{-1}}(z,\overline{z}) \tag{4.9}$$

is valid.

For doubly connected domains $N=2$, since the differentials

$$4\pi K(z,\overline{u})dz\overline{du} - K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u})dz\overline{du}$$

is one dimensional first order differential that is complex Hermitian form of $dz\overline{du}$, the above basic big open problems and partial big open problems are the same problems.

We do not know whether they are always the same problems or not.

It is very interested in the result of S. R. Bell and B. Gustafsson ([5]) that for the Hejhal case, also for $N = 3$ both problems are same.

Q. Guan's results

Surprisingly enough, since 40 years later after publication of [28], for some open question proposed there, an entirely unexpected partial solution was published in [14] that is an entirely new result.

For the conjecture (4.2), he proved that

$$\pi K(z, \overline{z}) \leq \hat{K}_z(z, \overline{z}) \tag{4.10}$$

and surprisingly enough, he completely determined the equality problem. Here note the important fact

$$K_t(z, \overline{t}) \equiv 1.$$

His interest is on the long and large topics on the Oikawa-Sario's problems and the Suita conjecture that determine the magnitudes of many conformal invariant quantities.

We recall the following solution of the famous conjecture posed by Suita [39].

Theorem ([11]). *Let $c_\beta(z_0) = \lim_{z\to z_0} \exp(G(z, z_0) - \log|z - z_0|)$. Then $(c_\beta(z_0))^2 \leq \pi K(z_0, \overline{z_0})$ and $(c_\beta(z_0))^2 = \pi K(z_0, \overline{z_0})$ holds for some $z_0 \in D$ if and only if D conformally equivalent to the unit disc, i.e. $N = 1$.*

Note that

$$(c_\beta(z_0))^2 \leq \pi K(z_0, \overline{z_0})$$

that was proved by Blocki in [8] for planar domains D. We can see the great impact of the Suita conjecture from many internet sites like Wikipedia; Suita conjecture, however, we see its frontier information in [26] and [40]. In particular, it seems that the impact of T. Ohsawa was great that was introduced and connected to several complex analysis group and to several complex analysis. For a complete version of Yamada, see [41].

For some detail and global information on the Suita conjecture, see Ohsawa ([27]), the last 5 stories.

Then, Guan derived the inequalities

$$(c_\beta(z_0))^2 \leq \pi K(z_0, \overline{z_0}) \leq \hat{K}_{z_0}(z_0, \overline{z_0})$$

that mean that the values $(c_\beta(z_0))^2$ and $\hat{K}_{z_0}(z_0, \overline{z_0})$ are the extremals for the value $K(z_0, \overline{z_0})$ and he solved completely the equality problems for these inequalities by an elementary means.

Surprisingly enough, he derived there the following identity:

For any fixed $t \in D$ and for any fixed analytic function f on D which is continuous on $D \cup \partial D$, the identity

$$\lim_{r \to 1-0} \frac{1}{1-r} \int\int_{\{e^{-2G(z,t)} \geq r\}} |f(z)|^2 dxdy \tag{4.11}$$

$$= \frac{1}{2} \int_{\partial D} |f(z)|^2 (\partial G(z,t)/\partial \nu)^{-1} |dz|$$

holds.

Main Result

The following inequalities seem to be interesting on its own sense:

Theorem: *For any given $\epsilon > 0$ and for any fixed analytic function $f(z)$ on $D \cup \partial D$, there exists a (large) r in $(0 < r < 1)$ satisfying the inequality*

$$\int\int_D |f'(z)|^2 dxdy - \epsilon \leq \frac{1}{1-r} \int\int_{\{e^{-2G(z,t)} \geq r\}} |f'(z)|^2 dxdy.$$

This inequality may be looked as an isoperimetric inequality, because the Dirichlet integral on a domain is estimated (restricted) by the Dirichlet integral on some small boundary neighborhood of the domain. Here, the neighborhood size and estimation are stated by the level curve of the Green function, precisely.

Even the case of the identity function $f(z) = z$, we can enjoy the senses of the estimation and the result.

Out Line of Proof of the Main Result

The integral of the conjugate analytic Hardy $H^2(D)$ norm seems to be not popular, however, the norm will have a beautiful structure and the norm is conformally invariant. Furthermore, we obtain the inequality:

For any analytic function $f(z)$ on $D \cup \partial D$, we have the inequality

$$\int\int_D |f'(z)|^2 dxdy \leq \frac{1}{2} \int_{\partial D} |f'(z)|^2 (\partial G(z,t)/\partial \nu)^{-1} |dz|. \tag{4.12}$$

This result was derived from some complicated theory of reproducing kernels in ([29]) on some comparison of the magnitudes of the Bergman and the Rudin (Hardy H^2) reproducing kernels. The equality problem in the inequality was also established; that is, equality holds if and only if the domain is simply-connected and the function $f'(z)$ is expressible in the form $CK(z, \overline{t})$ for the Bergman reproducing kernel $K(z, \overline{u})$ on the domain D and for a constant C. Its source was given by [28] and then the topics was cited in the book [32] in detail.

We can obtain the main result by combining of the Guan identity and this inequality.

Remarks

We note the following interesting problems:

Problem 1 : *How will be some generalization of the Guan identity for a general weight for* $(\partial G(z,t)/\partial \nu)^{-1}$*?*

Problem 2 : *We can consider similar inequalities for various function spaces. For example, how will be the case for the harmonic Dirichlet integrals?*

Problem 3 : *The structure and proof of the inequality (4.12) are very complicated (involved) and the Guan identity is very unique. So, we are interested in some direct proof of the theorem.*

Problem 4 : *The theorem seems to be valid for a general domain* D *and for general analytic functions with finite Dirichlet integrals on* D *apart from the proof in this paper. However, in the theorem* r *is depending on the function* f *and therefore the generalization of the theorem is not simple.*

For the inequality (4.12), note the inequality

$$\left(\frac{1}{\pi}\int\int_D |f'(z)|^2 dxdy\right)^2 = \left(\frac{1}{2\pi i}\int_{\partial D} \overline{f(z)} f'(z) dz\right)^2$$
$$\leq \frac{1}{2\pi}\int_{\partial D} |f(z)|^2 \frac{\partial G(z,t)}{\partial \nu}|dz| \frac{1}{2\pi}\int_{\partial D} |f'(z)|^2 (\partial G(z,t)/\partial \nu)^{-1}|dz|.$$

Note that the relation of the Bergman norm and the weighted H^2 norm is very delicate. See [28, 29, 32].

Addition with the New Paper [15] by Q. Guan and Z. Yuan

The result of Q. Guan was very surprised from the line of D. A. Hejhal and J. D. Fay of the viewpoint of positive definiteness of reproducing kernels. On the line of Oikawa-Sario problems Q. Guan and Z. Yuan got very deep and surprising results [15]. This direction was suggested by the conjecture of A. Yamada [40] that is a generalization of the conjecture of Suita:

Let u be a harmonic function on D, and let $\rho = e^{2u}$. For the weighted Bergman kernel $K_\rho^B(z, \overline{u})$

Yamada Conjecture:

$$(c_\beta(z))^2 \leq \pi \rho(z) K_\rho^B(z, \overline{z}).$$

In [11], Guan-Zhou proved the Yamada conjecture and more general weighted versions were developed surprisingly and deeply in [12] and [13].

Following this line, Q. Guan and Z. Yuan is developing the corresponding result with the weighted Hardy H_2 reproducing kernels and their results are too deep and complicated to state them simply. A typical case, however, may be stated as follows as a definite result:

Let λ be any positive continuous function on ∂D. By solving the Dirichlet problem, there exists the continuous harmonic function u on the closed domain of D satisfying that $u = -\frac{1}{2}\log\lambda$.

Then, they obtain surprisingly:

$$K_{\lambda\left(\frac{\partial G(\cdot,z)}{\partial\nu.}\right)^{-1}}(z,\overline{z}) \geq K^B_{e^{-2u}}(z,\overline{z}).$$

In particular, they are developing a detail and deep analysis for the identity (4.11). However, not identities, but estimates.

Q. Guan and Z. Yuan gave a very delicate and profound property of

$$\frac{\partial G(z,t)}{\partial\nu}$$

for a very general situation for the Green function from the view point of the identity (4.11). For its complicated structure, it seems that the result is at this moment abstract in nature. However, as a typical case they were able to get the result for the general weighted Hardy H_2 norms.

This line was not an expected direction, however, we see that the results show more wide field for our mathematics. For example, how will be the values

$$K_{\left(\lambda\left(\frac{\partial G(\cdot,z)}{\partial\nu.}\right)^{-1}\right)^{-1}}(z,\overline{z})$$

and

$$K_{\lambda\left(\frac{\partial G(\cdot,z)}{\partial\nu.}\right)^{-1}}(z,\overline{z})K_{\left(\lambda\left(\frac{\partial G(\cdot,z)}{\partial\nu.}\right)^{-1}\right)^{-1}}(z,\overline{z})$$

?

Moore and Surprising Addition with the New Paper [16] by Q. Guan and Z. Yuan

They furthermore consider the complete version for the product space $D\times D\times D\times\cdots\times D\subset C^n$; indeed, they surprisingly formulate the complete version and complete results containing the equalities problems for the related estimates with the great paper.

See also [17] for a weighted version of Suita conjecture for higher derivatives.

In addition, S. R. Bell and B. Gustafsson [5] is discussing on the Hejhal's result and the Suita conjecture from a very interesting viewpoint.

5. Applications to Analytic Extension Formulas

We will introduce typical applications to analytic extension formulas of reproducing kernels. We obtained many and many analytic extension formulas and applications. Its principle is the restriction and extension of reproducing kernels; because reproducing kernels determine the uniquely determined Hilbert spaces. See the book [34] and many references therein.

Now, let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space, and consider E to be an abstract set and $\mathbf{h}$ a Hilbert $\mathcal{H}$-valued function on E. Then, a very general linear transform from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued functions on E may be considered by

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{5.1}$$

in the framework of Hilbert spaces. Many general linear mappings may be considered in this framework by many modifications and arrangements. For this recall the Schwartz kernel theorem in connection with the distribution theory. In particular, recall that distribution may be considered as functions by considering integrals. Further, restrictions of functions and weighted norms are usual techniques. However, the theory of reproducing kernels may be considered essentially and favorably in the framework of Hilbert spaces. However, even for the theory of Banach spaces, we can discuss the relation between the spaces of Banach and Hilbert, and we can find many related references.

In order to investigate the linear mapping (5.1), we form a positive definite quadratic form function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \quad \text{on} \quad E \times E. \tag{5.2}$$

Then, the following fundamental results are valid:

Proposition 5.1. (I) *The range of the linear mapping* (5.1) *by* $\mathcal{H}$ *is characterized as the reproducing kernel Hilbert space* $H_K(E)$ *admitting the reproducing kernel* $K(p,q)$.

(II) *In general, the inequality*

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}$$

holds. Here, for any member f *of* $H_K(E)$ *there exists a uniquely determined* $\mathbf{f}^* \in \mathcal{H}$ *satisfying*

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \quad \text{on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}. \tag{5.3}$$

(III) *In general, the inversion formula in* (5.1) *in the form*

$$f \mapsto \mathbf{f}^* \tag{5.4}$$

in (II) *holds, by using the reproducing kernel Hilbert space* $H_K(E)$.

When we consider the linear mapping (5.1), Proposition 5.1 gives the image identification that is a very fundamental and important. Furthermore, we can obtain many and many reproducing kernels from the general linear mappings. When we consider the inversion of the linear mapping (5.1), the typical ill-posed problem (5.1) looking for its inversion becomes a well-posed problem, because the image space of (5.1) is characterized as the reproducing kernel Hilbert space $H_K(E)$ with the isometric identity (4.3), which may be considered as a generalization of the *Pythagorean* theorem.

In Proposition 5.1, when we know the isometrical mapping between the domain $\mathcal{H}$ and the reproducing kernel Hilbert space $H_K(E)$, we can determine the system or linear mapping (5.1) by representing the system function $\mathbf{h}$ in terms of the linear mapping and the reproducing kernel.

Here, we shall introduce the typical image identifications and isometric identities that are obtained by considering the integral representations in the heat conduction.

We consider the simple heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{on} \quad \mathbb{R} \times T_+ \quad (T_+ \equiv \{t > 0\}) \tag{5.5}$$

subject to the initial condition

$$u_F(\cdot, 0) = F \in L^2(\mathbb{R}) \quad \text{on} \quad \mathbb{R}. \tag{5.6}$$

Using the Fourier transform, we obtain a representation of the solution $u_F(x,t)$

$$u_F(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi \tag{5.7}$$

at least in the formal sense.

For any fixed $t > 0$, we first examine the integral transform $F \mapsto u_F$ and we shall characterize the image function $u_F(x,t)$.

We write

$$k(x;t) \equiv \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \quad (x \in \mathbb{R},\, t > 0) \tag{5.8}$$

and

$$K(x,x';t) \equiv \int_{\mathbb{R}} k(x-\xi;t)k(x'-\xi;t)d\xi = \frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{x^2}{8t} - \frac{x'^2}{8t} + \frac{xx'}{4t}\right). \tag{5.9}$$

Then, we obtained surprisingly the initial results.

Proposition 5.2. *Let $t > 0$ fix. A function f takes the form $u_F(\cdot,t)$ for some $F \in L^2(\mathbb{R})$ if and only if f admits analytic extension $\tilde{f}$ to $\mathbb{C}$ and satisfies*

$$\sqrt{\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |\tilde{f}(x+iy)|^2 \exp\left(\frac{-y^2}{2t}\right) dxdy} < \infty. \tag{5.10}$$

In this case, $f \in H_K(\mathbb{R})$ and the norm is given by:

$$\|f\|_{H_K(\mathbb{R})} = \sqrt{\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |\tilde{f}(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy}.$$

Proposition 5.3. *Let $t > 0$ fix. In the integral transform $F \mapsto u_F(\cdot,t)$ of $L^2(\mathbb{R})$ functions F, the images $u_F(\cdot,t)$ extend analytically onto $\mathbb{C}$ to a function, which we still write $u_F(\cdot,t)$. Furthermore, we have the isometrical identity*

$$\int_{\mathbb{R}} |F(\xi)|^2 d\xi = \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |u_F(z,t)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy, \tag{5.11}$$

for any fixed $t > 0$.

Proposition 5.4. *If a C^∞-function $f : \mathbb{R}^n \to \mathbb{C}$ has a finite integral on the right-hand side in* (5.11)*, then f is extended analytically onto $\mathbb{C}$ and*

$$\sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 dx = \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |f(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy. \tag{5.12}$$

Proof. For fixed $t > 0$, the solution operator $F \mapsto u_F$ can be regarded as

$$u_F(x,t) = \langle F, k(x - \cdot; t)\rangle_{L^2(\mathbb{R})}.$$

So with

$$E = \mathbb{R}, \quad \mathcal{H} = L^2(\mathbb{R}), \quad \mathbf{h}(x) \equiv k(x - \cdot; t) \text{ and } L = [F \in \mathcal{H} \mapsto u_F(\cdot, t) \in \mathcal{F}(E)],$$

the reproducing Hilbert kernel space $\mathcal{R}(L)$ is given by:

$$\mathcal{R}(L) = \{u_F(\cdot, t) \,:\, F \in L^2(\mathbb{R})\}.$$

Note that for any fixed $t > 0$, the system

$$\{k(\cdot - \xi, t); \xi \in \mathbb{R}\} \tag{5.13}$$

spans a dense subspace in $L^2(\mathbb{R})$. Therefore, L is isometric.

Now let us view this mapping from the point of complex analysis. Note that the kernel $K(x, x'; t)$ extends analytically to $\mathbb{C} \times \overline{\mathbb{C}}$;

$$K(z, \overline{u}; t) = \frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{z^2}{8t} - \frac{\overline{u}^2}{8t} + \frac{z\overline{u}}{4t}\right). \tag{5.14}$$

Observe that (5.14) stands for

$$K(z, \overline{u}; t) = \int_{\mathbb{R}} k(z - \xi; t)\overline{k(u - \xi; t)}\, d\xi. \tag{5.15}$$

Consequently, the extended kernel K is again a positive definite function. Denote by $H_K(\mathbb{C})$ the RKHS associated with K. The following is a description of $H_K(\mathbb{C})$; we can see directly

$$H_K(\mathbb{C}) = \left\{ f \in \mathcal{O}(\mathbb{C}) \,:\, \|f\|_{H_K(\mathbb{C})} = \sqrt{\iint_{\mathbb{C}} \frac{|\tilde{f}(x+iy)|^2}{\sqrt{2\pi t}} \exp\left(\frac{-y^2}{2t}\right) dxdy} < \infty \right\}. \tag{5.16}$$

However, (5.16) was derived naturally and simply from the reproducing structure of (5.14) by using Proposition 3.6.

Meanwhile, the norm (5.10) is also expressible in terms of the trace $f(x)$ of $\tilde{f}(z)$ to the real line.

By using the identity

$$k(x - \xi, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\{-p^2 t + ip(x - \xi)\} dp, \tag{5.17}$$

we have

$$K(x, x'; t) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\{-2p^2 t + ip(x - x')\} dp. \tag{5.18}$$

This implies that any member $f(x)$ of $H_K(\mathbb{R})$ is expressible in the form

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} g(p)\exp(ipx - 2p^2 t)dp = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[g\cdot\exp(-2p^2 t)](x) \tag{5.19}$$

for a function g satisfying

$$\int_{\mathbb{R}} |g(p)|^2 \exp(-2p^2 t)dp < \infty \tag{5.20}$$

and we have the isometrical identity

$$\|f\|_{H_K(\mathbb{R})} = \sqrt{\frac{1}{2\pi}\int_{\mathbb{R}} |g(p)|^2 \exp(-2p^2 t)dp}. \tag{5.21}$$

Meanwhile, by the Fourier transform and (5.19), we have

$$g(p) = \sqrt{2\pi}\mathcal{F}f(p)\exp(2p^2 t) \tag{5.22}$$

in $L^2(\mathbb{R})$. Hence, we obtain

$$\|f\|_{H_K(\mathbb{R})}{}^2 = \int_{\mathbb{R}} |\mathcal{F}f(p)|^2 \exp(-2p^2 t)dp = \sum_{j=0}^{\infty}\frac{(2t)^j}{j!}\int_{\mathbb{R}} |f^{(j)}(x)|^2 dx, \tag{5.23}$$

by virtue of the monotone convergence theorem and the Parseval–Plancherel identity.

□

As in those typical cases, we obtained many isometric identities and analytic extension formulas. Define the right-half plane by: $R^+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

Proposition 5.5. *Let $q > \frac{1}{2}$. Then, for $f \in H_{K_q}(R^+)$, admitting the Bergman Selberg reproducing kernel $\Gamma(2q)/(z+\overline{u})^{2q}$, we have the identity*

$$\begin{aligned}\|f\|_{H_{K_q}(R^+)} &\equiv \sqrt{\frac{1}{\Gamma(2q-1)\pi}\iint_{R^+} |f(z)|^2 (2x)^{2q-2} dxdy} \\ &= \sqrt{\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+2q+1)}\int_{\mathbb{R}} |(xf'(x))^{(n)}|^2 x^{2n+2q-1} dx}.\end{aligned} \tag{5.24}$$

Conversely, any $C^\infty(0,\infty)$-function f with convergent summation in (5.24) *extends analytically onto the right half plane R^+. The analytic extension $f(z)$ satisfying $\lim_{x\to\infty} f(x) = 0$ belongs to $H_{K_q}(R^+)$ and the identity* (5.24) *is valid.*

We shall write $S(r) \equiv \{z \in \mathbb{C} : 0 < \arg(z) < r\}$ for the open sector and its boundary $\partial S(r) \equiv \{z \in \mathbb{C} : z = 0 \text{ or } \arg(z) = \pm r\}$.

Proposition 5.6. *Let $r \in (0, \pi/2)$. For an analytic function f on the open sector $S(r)$, we have the identity*

$$\iint_{S(r)} |f(x+iy)|^2 dxdy = \sin(2r)\sum_{j=0}^{\infty}\frac{(2\sin r)^{2j}}{(2j+1)!}\int_{\mathbb{R}} x^{2j+1}|f^{(j)}(x)|^2 dx. \tag{5.25}$$

Conversely, if any $f \in C^\infty(S(r))$ has a convergent sum in the right-hand side in (5.25)*, then the function $f(x)$ can be extended analytically onto the sector $S(r)$ in the form $f(z)$ and the identity* (5.25) *is valid.*

In the Szegö space, we have the following formula:

Proposition 5.7. *Let $r \in (0, \pi/2)$. For any member f in the Szegö space on the open sector $S(r)$, we have the identity*

$$\oint_{\partial S(r)} |f(z)|^2 |dz| = 2\cos r \sum_{j=0}^{\infty} \frac{(2\sin r)^{2j}}{(2j)!} \int_{\mathbb{R}} x^{2j} |f^{(j)}(x)|^2 dx, \tag{5.26}$$

where $f(x)$ means the nontangential Fatou limit on $\partial S(r)$ for $x \in \mathbb{R}$. Conversely, if any $f \in C^\infty(0,\infty)$ has a convergent sum in the right-hand side in (5.26)*, then the function $f(x)$ extends analytically onto the open sector $\Delta(r)$ and the identity* (5.26) *is valid.*

These results were applied to investigate analyticity properties of the solutions of nonlinear partial differential equations. In particular, an analogue is applied to the proof of the unique existence of the Schödinger equation by N. Hayashi. H. Aikawa considered the class $W(c_j;\mathbb{R})$ by changing $\frac{(2\sin r)^{2j}}{(2j)!}$ with other general positive sequences, where he proved that some function can not be extended beyond a sector.

These Propositions may be looked at the relations of the correspondent RKHSs for the restrictions of RKs, because the both sides are reproducing kernel Hilbert spaces. That is the relation of analytic extension and the restriction (extension) of reproducing kernels.

6. Conclusion

In connection with the relation of complex analysis to the theory of reproducing kernels, we will recall the famous sampling theory. However, the sampling theory is not restricted to analytic functions, but the theory holds for some very general functions. Indeed, it depends on some very general discretization principle that is very favorable in the theory of reproducing kernels. Typically, its concrete realization was established as the **Aveiro discretization method**. See [38]. Another typical problem may be considered in the real inversion formula of the Laplace transform, because **we have to get analyticity by discrete data.** However, we were able to overcome this very difficult problem as a general theory and we published its principle in the book [38] globally.

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References

[1] H. Aikawa, *Infinite order Sobolev spaces, analytic continuation and polynomial expansions*, Complex Variables. Theory and Application. **18** (1992), 253–266.

[2] H. Aikawa, N. Hayashi and S. Saitoh, *Isometrical identities for the Bergman and the Szegö spaces on a sector*, J. Math. Soc., Japan **43** (1991), 195–201.

[3] N. Aronszajn, *Theory of reproducing kernels,* Trans. Amer. Math. Soc., **68** (1950), 337–404.

[4] H. Begehr and S. Saitoh, International Journal of Reproducing Kernels, Vol. 1 No. 1 (2022), Preface.

[5] S. R. Bell and B. Gustafsson, *Ruminations on Hejhal's theorem about the Bergman and Szego kernels*, arXiv:2112.06877 (2021).

[6] S. Bergman, *The kernel function and conformal mapping,* Amer. Math. Soc., Providence, R. I. (1950, 1970).

[7] B. Bergman and M. M. Schiffer, *Kernel functions and differential equations*, Academic Press. New York, 1953.

[8] Z. Blocki, Suita conjecture and the Ohsawa–Takegoshi extension theorem, Invent. Math., **193** (2013), 149–158.

[9] L. P. Castro and S. Saitoh, *Fractional functions and their representations*, Complex Anal. Oper. Theory **7** (2013) no. 4: 1049-1063.

[10] L. P. Castro, S. Saitoh and A. Yamada, *Solutions of Tikhonov Functional Equations and Applications to Multiplication Operators on Szegö Spaces*, Complex Analysis and Operator Theory **10** (2016), 1705–1723.

[11] Q. A. Guan and X. Y. Zhou, A solution of an L_2 extension problem with an optimal estimate and applications, Ann. of Math. (2), **181** (2015), 1139–1208.

[12] Q. A. Guan and Z. T. Mi, Concavity of minimal L_2 integrals related to multiplier ideal sheaves, arXiv:2106.05089v2.

[13] Q. A. Guan and Z. Yuan, Concavity property of minimal L_2 integrals with Lebesgue measurable gain, preprint. https://www.researchgate.net/publication/353794984.

[14] Q. Guan, *A proof of Saitoh's conjecture for conjugate Hardy H_2 kernels,* J. Math. Soc. Japan, **71**, No. 4 (2019), 1173–1179.

[15] Q. Guan and Z. Yuan, *The weighted version of Saitoh's conjecture,* arXiv:2207.10976 [math.CV] (or arXiv:2207.10976v1 [math.CV] for this version), https://doi.org/10.48550/arXiv.2207.10976.

[16] Q. Guan and Z. Yuan, *Hardy space, kernel function and Saitoh's conjecture on products of planar domains,* arXiv:2210.14579. (2022). 85 pages.

[17] Q. Guan, X. Sun and Z. Yuan, *A remark on a weighted version of Suita conjecture for higher derivatives*, arXiv:2212.02713 2022 , 22 pages.

[18] J. D. Fay, *Theta functions on Riemann surfaces,* Lecture Notes in Math., Vol. **352**, Springer-Verlag (1973).

[19] H. Fujiwara, T. Matsuura, S. Saitoh and Y. Sawano, Numerical real inversion of the Laplace transform by using a high-accuracy numerical method. Further Progress in Analysis: 574–583 World Sci. Publ., Hackensack, NJ (2009).

[20] B. A. Fuks, *Special chapters in the theory of analytic functions of several complex variables*, Transl. Math. Mono., **14** Amer. Math. Soc., Providence, R.I. (1965).

[21] N. Hayashi and S. Saitoh, *Analyticity and smoothing effect for the Schrödinger equation*, Ann. Inst. Henri Poincaré **52** (1990), 163–173.

[22] N. Hayashi and S. Saitoh, *Analyticity and global existence of small solutions to some nonlinear Schrödinger equation*, Commun. Math. Phys., **139** (1990), 27–41.

[23] D. A. Hejhal, *Theta functions, kernel functions and Abel integrals*, Memoirs of Amer. Math. Soc., Providence, R. I. **129**(1972).

[24] T. Matsuura and S. Saitoh, *What is a Reproducing Kernel? - Some Essences for the New Journal,* International Journal of Reproducing Kernels, **1** (2022), (63 pages).
[25] Z. Nehari, *Conformal mapping,* McGraw-Hill Book Company Inc., New York (1952).
[26] T. Ohsawa, The Bergman kernel and the Suita conjecture on Riemann surfaces (Japanese), The theory of reproducing kernels and their applications (Kyoto, 1998), Surikaisekikenky-usho Kokyuroku, 1067 (1998), 89-95.
[27] T. Ohsawa, Function Theory Gaiden - 100 years of Bergman Kernel - Book – , Gendai Suugakusya (2022), in Japanese.
[28] S. Saitoh, The Bergman norm and the Szegö norm, Trans. Amer. Math. Soc., **249** (1979)(2), 261–279.
[29] S. Saitoh, The Dirichlet norm and the norm of Szegö type, Trans. Amer. Math. Soc., **254** (1979), 355– 364.
[30] S. Saitoh, *Some inequalities for analytic functions with a finite Dirichlet integral on the unit disc,* Math. Ann., **246** (1979), no. 1, 69–77.
[31] S. Saitoh, *Some inequalities for entire functions,* Proc. Amer. Math. Soc., **80** (1980), no. 2, 254–258.
[32] S. Saitoh, Theory of reproducing kernels and its applications. Pitman Research Notes in Mathematics Series, 189. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1988. x+157 pp. ISBN: 0-582-03564-3.
[33] S. Saitoh, *Natural norm inequalities in nonlinear transforms*, in General inequalities, 7 (Oberwolfach, 1995), 39–52, Internat. Ser. Numer. Math., 123, Birkhaüser, Basel, 1997.
[34] S. Saitoh, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Res. Notes in Math. Series **369**, Addison Wesley Longman Ltd, UK, (1998). CRC Press, Taylor & Francis Group, Boca Raton London, New York (in hard cover).
[35] S. Saitoh, *Analytic extension formulas, integral transforms and reproducing kernels,* Analytic Extension Formulas and their Applications, (2001), Kluwer Academic Publishers, 207–232.
[36] S. Saitoh, N. Hayashi and M. Yamamoto (eds.), *Analytic Extension Formulas and their Applications,* (2001), Kluwer Academic Publishers.
[37] S. Saitoh, *Theory of reproducing kernels: Applications to approximate solutions of bounded linear operator equations on Hilbert spaces*, Amer. Math. Soc. Transl. Ser. 2: **230** (2010), 107–137.
[38] S. Saitoh and Y. Sawano, Theory of Reproducing Kernels and Applications, Developments in Mathematics **44**, (2016), (Springer).
[39] N. Suita, Capacities and kernels on Riemann surfaces, Arch. Ration. Mech. Anal., **46** (1972), 212–217.
[40] A. Yamada, Topics related to reproducing kernels, theta functions and the Suita conjecture (Japanese), The theory of reproducing kernels and their applications (Kyoto, 1998), Surikaisekikenky-usho Kokyuroku, 1067 (1998), 39-47.
[41] A. Yamada, *Fay's trisecant formula and Hardy H^2 reproducing kernels,* Reproducing Kernels and their Applications (1999), Kluwer Academic Publishers, 165–188.

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