IMPROVED BOUND FOR THE NUMBER OF INTEGRAL POINTS IN A CIRCLE OF RADIUS $r > 1$

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Abstract. Using the method of compression, we prove an inequality related to the Gauss circle problem. Let $\mathcal{N}_r$ denotes the number of integral points in a circle of radius $r > 0$, then we have

$$2r^2\left(1+\frac{1}{4}\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r}) \leq \mathcal{N}_r \leq 8r^2\left(1+\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r})$$

for all $r > 1$.

1. Introduction

The Gauss circle problem is a problem that seeks to counts the number of integral points in a circle centered at the origin and of radius r. It is fairly easy to see that the area of a circle of radius $r > 0$ gives a fairly good approximation for the number of such integral points in the circle, since on average each unit square in the circle contains at least an integral point. In particular, by denoting $N(r)$ to be the number of integral points in a circle of radius r, then the following elementary estimate is well-known

$$N(r) = \pi r^2 + |E(r)|$$

where $|E(r)|$ is the error term. The real and the main problem in this area is to obtain a reasonably good estimate for the error term. In fact, it is conjectured that

$$|E(r)| \ll r^{\frac{1}{2}+\epsilon}$$

for $\epsilon > 0$. The first fundamental progress was made by Gauss [3], where it is shown that

$$|E(r)| \leq 2\pi r\sqrt{2}.$$

G.H Hardy and Edmund Landau almost independently obtained a lower bound [1] by showing that

$$|E(r)| \neq o(r^{\frac{1}{2}}(\log r)^{\frac{1}{4}}).$$

The current best upper bound (see [2]) is given by

$$|E(r)| \ll r^{\frac{131}{208}}.$$

In this paper we prove a general upper bound and lower bound for the number of integral points in a circle of radius $r > 1$. This upper bound is of the desired quality as does the Gauss circle problem, where the quest is to be obtain an error of quality as that in the following result

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Theorem 1.1 (The inequality)**.** *Let $\mathcal{N}_r$ denotes the number of integral points in a circle of radius r. Then*

$$2r^2\left(1+\frac{1}{4}\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r})\leq \mathcal{N}_r\leq 8r^2\left(1+\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r})$$

for all $r>1$.

Now we describe the steps employed in achieving these inequalities. We write them down chronologically as follows:

- (i) We pick a point in the plane with compression gap $2r$ and construct the circle of compression. This circle has radius r by virtue of the choice of compression gap.
- (ii) We first count the number of integral points on the boundary of the circle of radius r using the upper and the lower bounds of the compression gap. The error terms of the upper and the lower bound emanates from this particular analysis.
- (iii) We construct further smaller circle of compression by shrinking down the radius of each successive circle by a factor of 2. This procedure has the tendency of creating annular regions in the circle.
- (iv) For each annular region we construct an integer square grid that exactly covers the upper circle and count the number of points in the gird and in this annular region. The main terms in the inequalities follow by upper and lower bounding this count.

2. Preliminaries and background

Definition 2.1. By the compression of scale $m>0$ ($m\in\mathbb{R}$) fixed on $\mathbb{R}^n$ we mean the map $\mathbb{V}:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]=\left(\frac{m}{x_1},\frac{m}{x_2},\ldots,\frac{m}{x_n}\right)$$

for $n\geq 2$ and with $x_i\neq x_j$ for $i\neq j$ and $x_i\neq 0$ for all $i=1,\ldots,n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n\geq 2$. Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.3. *A compression of scale $1\geq m>0$ with $\mathbb{V}_m:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]=\mathbb{V}_m[(y_1,y_2,\ldots,y_n)]$, then it follows that

$$\left(\frac{m}{x_1},\frac{m}{x_2},\ldots,\frac{m}{x_n}\right)=\left(\frac{m}{y_1},\frac{m}{y_2},\ldots,\frac{m}{y_n}\right).$$

It follows that $x_i=y_i$ for each $i=1,2,\ldots,n$. Surjectivity follows by definition of the map. Thus the map is bijective. $\square$

2.1. **The mass of compression.** In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.4. By the mass of a compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) - k} \leq \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i, j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is such that $x_i \leq x_j$ for $1 \leq i, j \leq n$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m > 0$.

Proposition 2.6. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for each $1 \leq i \leq n$ and $x_i \neq x_j$ for $i \neq j$, then the estimates holds*

$$m \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \leq \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \leq m \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right) + m$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) = m\sum_{j=1}^{n}\frac{1}{x_j}$$
$$\geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.$$

□

Definition 2.7. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq 0$ for all $i=1,2\ldots,n$. Then by the gap of compression of scale $m>0$, denoted $\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]$, we mean the expression

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)] = \left|\left|\left(x_1-\frac{m}{x_1},x_2-\frac{m}{x_2},\ldots,x_n-\frac{m}{x_n}\right)\right|\right|$$

Definition 2.8. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq 0$ for all $1\leq i\leq n$. Then by the ball induced by $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ under compression of scale $m>0$, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ we mean the inequality

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},x_2+\frac{m}{x_2},\ldots,x_n+\frac{m}{x_n}\right)\right|\right| < \frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)].$$

A point $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n=2$.

Remark 2.9. In the geometry of balls under compression of scale $m>0$, we will assume implicitly that $1\geq m>0$. The circle induced by points under compression is the ball induced on points when we take $n=2$.

Proposition 2.10. *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ for $n\geq 2$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2 = \mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]+m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]-2mn.$$

In particular, we have the estimate

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2 = \mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]-2mn+O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]\right)$$

for $\vec{x}\in\mathbb{N}^n$, where $m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]$ is the error term in this case.

Lemma 2.11 (Compression estimate)**.** *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ for $n\geq 2$ and $x_i\neq x_j$ for $i\neq j$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2 \leq n\sup(x_j^2)+m^2\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)^2}\right)-2mn$$

and

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2 \geq n\mathrm{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\sup(x_j^2)}\right)^{-1}-2mn.$$

Theorem 2.12. *Let $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$. Then $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if*

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$, then it follows that $||\vec{y}||>||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}||\leq||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.10 that $||\vec{z}||<||\vec{y}||$. It follows that

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|&<\left|\left|\vec{y}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|\\&=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

Theorem 2.13. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$. If $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 2.12 that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]&>\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\\&\geq\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\\&>\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 2.14. Theorem 2.13 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

2.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 2.15. Let $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with $y_i \neq y_j$ for all $1 \leq i < j \leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| = \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

Remark 2.16. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball.

Theorem 2.17. *The point $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \neq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 2.12, we obtain the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{y}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{z}] < \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.10 that $||\vec{x}|| < ||\vec{y}||$ or $||\vec{y}|| < ||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y}$ is an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. The latter equality follows from assertion that two balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lives on the outer of the indistinguishable balls and must satisfy the inequality

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &= \left|\left|\vec{z} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &= \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}] = \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right|$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. □

Remark 2.18. We note that we can replace the set $\mathbb{N}^n$ used in our construction with $\mathbb{R}^n$ at the compromise of imposing the restrictions $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_i > 1$ for all $1 \leq i \leq n$ and $x_i \neq x_j$ for $i \neq j$. The following construction in our next result in the sequel employs this flexibility.

Theorem 2.19 (The inequality)**.** *Let $\mathcal{N}_r$ denotes the number of integral points in a circle of radius r. Then*

$$2r^2\left(1+\frac{1}{4}\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r})\leq \mathcal{N}_r\leq 8r^2\left(1+\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r})$$

for all $r>1$.

Proof. Pick arbitrarily a point $(x_1,x_2)=\vec{x}\in \mathbb{R}^2$ with $x_i>1$ for $1\leq i\leq 2$ and $x_1\neq x_2$ such that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=2r$. This ensures the circle induced under compression is of radius r. Next we apply the compression of fixed scale $m:=m(r)\leq 1$, given by $\mathbb{V}_m[\vec{x}]$ and construct the circle induced by the compression given by

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])}{2}=r$. By appealing to Theorem 2.17 admissible points $\vec{x}_k\in \mathbb{R}^2$ $(\vec{x}_k\neq\vec{x})$ of the circle of compression induced must satisfy the condition $\mathcal{G}\circ\mathbb{V}_m[\vec{x}_k]=2r$. Also by appealing to Theorem 2.12 points $\vec{x}_l\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ must satisfy the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]\leq \mathcal{G}\circ\mathbb{V}_m[\vec{x}]=2r.$$

In particular points in $\vec{x}_l\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contained in the $2r\times 2r$ grid that covers this circle must satisfy for their coordinates

$$\max_{\vec{x}_l\in 2r\times 2r}\sup(x_{l_i})_{i=1}^{2}=2r+\frac{1}{\log r}$$

for all $r>1$ so that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]\leq 2r$. We note that all points in the ball

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])}{2}=r$ constructed can be classified according to the values of the compression gap $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=s$ for all $1\leq s\leq 2r$. Let us choose $0<m:=m(r)=\frac{1}{2\log^2 r}\leq 1$, then the number of integral points contained in the circle is the sum

$$\begin{aligned}\mathcal{N}_r &= \sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]\leq 2r}}1\\ &= \sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]<2r}}1+\sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]=2r}}1\\ &= \sum_{1\leq k\leq\lfloor\frac{\log r}{\log 2}\rfloor}\sum_{\substack{\vec{x}_j\in\lfloor\frac{2r}{2^{k-1}}\rfloor\times\lfloor\frac{2r}{2^{k-1}}\rfloor\\ \frac{r}{2^{k-1}}\leq\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]<\frac{2r}{2^{k-1}}}}1+\sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]=2r}}1.\end{aligned}$$

We now analyze the contribution of each of the sums. We note that the right-hand sum contributes the error term. We notice that we can write

$$\begin{aligned}
\sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]=2r}} 1 &= \sum_{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor} \frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j])^2}{4r^2}\\
&\leq \sum_{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor} \frac{2(\sup(x_{j_i}^2)_{1\leq i\leq 2}+m^2\log\left(1+\frac{1}{\mathrm{Inf}(x_{j_i})^2}\right)-4m}{4r^2}\\
&\leq \sum_{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor} \frac{2\max_{\vec{x}_l\in 2r\times 2r}\sup(x_{j_i}^2)_{1\leq i\leq 2}+m^2\log\left(1+\frac{1}{\mathrm{Inf}(x_{j_i})^2}\right)-4m}{4r^2}\\
&= \sum_{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor} \frac{2(2r+\frac{1}{\log r})^2+\frac{1}{\log^4 r}\log\left(1+\frac{1}{\mathrm{Inf}(x_{j_i})^2}\right)-\frac{2}{\log^2 r}}{4r^2}\\
&= 2\sum_{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor} 1+O(\frac{r}{\log r})\\
&= 8r^2+O(\frac{r}{\log r}).
\end{aligned}$$

Now, we evaluate the first sum which contributes the main term of the upper bound

$$\begin{aligned}
\sum_{1\leq k\leq\lfloor\frac{\log r}{\log 2}\rfloor}\ \sum_{\substack{\vec{x}_j\in\lfloor\frac{2r}{2^{k-1}}\rfloor\times\lfloor\frac{2r}{2^{k-1}}\rfloor\\ \frac{r}{2^{k-1}}\leq\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]<\frac{2r}{2^{k-1}}}} 1 &\leq 2\sum_{1\leq k\leq\lfloor\frac{\log r}{\log 2}\rfloor}\ \sum_{\vec{x}_j\in\lfloor\frac{2r}{2^{k-1}}\rfloor\times\lfloor\frac{2r}{2^{k-1}}\rfloor} 1\\
&\leq 8r^2\sum_{1\leq k\leq\lfloor\frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}.
\end{aligned}$$

For the lower bound, we only count the number of integral points with their smallest coordinates satisfying

$$\min_{\vec{x}_l\in 2r\times 2r}\inf(x_{l_i})_{i=1}^2 > r+\frac{1}{\log r}$$

for all $r>1$ so that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]\gtrsim r$ so that we obtain the lower bound

$$\begin{aligned}\sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]=2r}} 1 &> \sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]=2r\\ \min_{\vec{x}_l\in 2r\times 2r}\inf(x_{j_i})_{i=1}^2>r+\frac{1}{\log r}}} 1\\
&= \sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \min_{\vec{x}_l\in 2r\times 2r}\inf(x_{l_i})_{i=1}^2>r+\frac{1}{\log r}}} \frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j])^2}{4r^2}\\
&\geq \sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \min_{\vec{x}_l\in 2r\times 2r}\inf(x_{l_i})_{i=1}^2>r+\frac{1}{\log r}}} \frac{2(\inf(x_{j_i}^2)_{1\leq i\leq 2}+m^2\log\left(1+\frac{1}{\mathrm{Inf}(x_{j_i})^2}\right)-4m}{4r^2}\\
&\geq \sum_{\substack{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor\\ \inf(x_{l_i})_{i=1}^2>r+\frac{1}{\log r}}} \frac{2\min_{\vec{x}_l\in 2r\times 2r}\inf(x_{j_i}^2)_{1\leq i\leq 2}+m^2\log\left(1-\frac{1}{\sup(x_{j_i})^2}\right)^{-1}-4m}{4r^2}\\
&= \sum_{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor} \frac{2(r+\frac{1}{\log r})^2+\frac{1}{\log^4 r}\log\left(1+\frac{1}{\mathrm{Inf}(x_{j_i})^2}\right)-\frac{2}{\log^2 r}}{4r^2}\\
&= \frac{1}{2}\sum_{\vec{x}_j\in\lfloor 2r\rfloor\times\lfloor 2r\rfloor} 1+O(\frac{r}{\log r})\\
&= 2r^2+O(\frac{r}{\log r}).\end{aligned}$$

For the main term of the lower bound, we have

$$\begin{aligned}\sum_{1\leq k\leq\lfloor\frac{\log r}{\log 2}\rfloor}\sum_{\substack{\vec{x}_j\in\lfloor\frac{2r}{2^{k-1}}\rfloor\times\lfloor\frac{2r}{2^{k-1}}\rfloor\\ \frac{r}{2^{k-1}}\leq\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]<\frac{2r}{2^{k-1}}}} 1 &\geq \frac{1}{2}\sum_{1\leq k\leq\lfloor\frac{\log r}{\log 2}\rfloor}\sum_{\vec{x}_j\in\lfloor\frac{2r}{2^{k-1}}\rfloor\times\lfloor\frac{2r}{2^{k-1}}\rfloor} 1\\
&\geq \frac{r^2}{2}\sum_{1\leq k\leq\lfloor\frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}.\end{aligned}$$

By piecing these estimate together the lower bound also follows. $\square$

References

[1] Hardy, Godfrey Harold, *On the expression of a number as the sum of two squares*, Quart. J. Math., vol. 46, 1915, pp 263–283.

[2] Huxley, MN, *Integer points, exponential sums and the Riemann zeta function*, AK Peters/CRC Press, 2002, pp 109.

[3] Hardy, Godfrey Harold, *Ramanujan: Twelve lectures on subjects suggested by his life and work*, American Mathematical Soc., vol. 136 ,1999.

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