Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited)

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Abstract

In all classical algebraic structures, the *Laws of Compositions* on a given set are well-defined. But this is a restrictive case, because there are many more situations in science and in any domain of knowledge when a law of composition defined on a set may be only partially-defined (or partially true) and partially-undefined (or partially false), that we call *NeutroDefined*, or totally undefined (totally false) that we call *AntiDefined*.

Again, in all classical algebraic structures, the *Axioms* (Associativity, Commutativity, etc.) defined on a set are totally true, but it is again a restrictive case, because similarly there are numerous situations in science and in any domain of knowledge when an Axiom defined on a set may be only partially-true (and partially-false), that we call *NeutroAxiom*, or totally false that we call *AntiAxiom*.

Therefore we open for the first time in 2019 new fields of research called *NeutroStructures* and *AntiStructures* respectively.

Keywords

Neutrosophic Triplets, (Axiom, NeutroAxiom, AntiAxiom), (Law, NeutroLaw, AntiLaw), (Associativity, NeutroAssociaticity, AntiAssociativity), (Commutativity, NeutroCommutativity, AntiCommutativity), (WellDefined, NeutroDefined, AntiDefined), (Semigroup, NeutroSemigroup, AntiSemigroup), (Group, NeutroGroup, AntiGroup), (Ring, NeutroRing, AntiRing), (Algebraic Structures, NeutroAlgebraic Structures, AntiAlgebraic Structures), (Structure, NeutroStructure, AntiStructure), (Theory, NeutroTheory, AntiTheory), S-denying an Axiom, S-geometries, Multispace with Multistructure

6.1. Introduction

For the necessity to more accurately reflect our reality, Smarandache [1] introduced for the first time in 2019 the NeutroDefined and AntiDefined Laws, as well as the NeutroAxiom and AntiAxiom, inspired from Neutrosophy ([2], 1995), giving birth to new fields of research called NeutroStructures and AntiStructures.

Let's consider a given classical algebraic Axiom. We define for the first time the neutrosophic triplet corresponding to this Axiom, which is the following: *(Axiom, NeutroAxiom, AntiAxiom)*; while the classical Axiom is 100% or totally true, the NeutroAxiom is partially true and partially false (the degrees of truth and falsehood are both > 0), while the AntiAxiom is 100% or totally false.

For the classical algebraic structures, on a non-empty set endowed with well-defined binary laws, we have properties (axioms) such as: associativity & non-associativity, commutativity & non-commutativity, distributivity & non-distributivity; the set may contain a neutral element with respect to a given law, or may not; and so on; each set element may have an inverse, or some set elements may not have an inverse; and so on.

Consequently, we construct for the first time the neutrosophic triplet corresponding to the Algebraic Structures, which is this: *(Algebraic Structure, NeutroAlgebraic Structure, AntiAlbegraic Structure).*

Therefore, we now introduce for the first time the *NeutroAlgebraic Structures* & the *AntiAlgebraic Structures*.

A (classical) Algebraic Structure is an algebraic structure dealing only with (classical) Axioms (which are totally true).

Then a NeutroAlgebraic Structure is an algebraic structure that has at least one NeutroAxiom, and no AntiAxioms.

While an AntiAlgebraic Structure is an algebraic structure that has at least one AntiAxiom.

These definitions can straightforwardly be extended from Axiom/NeutroAxiom/AntiAxiom

to any

Property/NeutroProperty/AntiProperty,
Proposition/NeutroProposition/AntiProposition,
Theorem/NeutroTheorem/AntiTheorem,
Theory/NeutroTheory/AntiTheory, etc.

and from Algebraic Structures to other Structures in any field of knowledge.

6.2. Neutrosophy

We recall that in neutrosophy we have for an item <*A*>, its opposite <*antiA*>, and in between them their neutral <*neutA*>.

We denoted by <*nonA*> = <*neutA*> ∪ <*antiA*>, where ∪ means union, and <*nonA*> means what is not <*A*>.

Or <*nonA*> is refined/split into two parts: <*neutA*> and <*antiA*>.

The neutrosophic triplet of <*A*> is:

$(\langle A\rangle, \langle neutA\rangle, \langle antiA\rangle)$, with $\langle neutA\rangle \cup \langle antiA\rangle = \langle nonA\rangle$.

6.3. Definition of Neutrosophic Triplet Axioms

Let $\mathcal{U}$ be a universe of discourse, endowed with some well-defined laws, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and an Axiom α, defined on S, using these laws. Then:

1) If all elements of $\mathcal{S}$ verify the axiom α, we have a *Classical Axiom*, or simply we say *Axiom*.
2) If some elements of $\mathcal{S}$ verify the axiom α and others do not, we have a *NeutroAxiom* (which is also called *NeutAxiom*).
3) If no elements of $\mathcal{S}$ verify the axiom α, then we have an *AntiAxiom*.

The Neutrosophic Triplet Axioms are:

(Axiom, NeutroAxiom, AntiAxiom)

with

NeutroAxiom ⋃ *AntiAxiom = NonAxiom,*

and *NeutroAxiom* ⋂ *AntiAxiom* = φ (empty set),

where ⋂ means intersection.

Theorem 1

The Axiom is 100% true, the NeutroAxiom is partially true (its truth degree > 0) and partially false (its falsehood degree > 0), and the AntiAxiom is 100% false.

Proof is obvious.

Theorem 2

Let

$$d : \{Axiom, NeutroAxiom, AntiAxiom\} \rightarrow [0 , 1]$$

represent the **degree of negation function**.

The NeutroAxiom represents a degree of partial negation { $d \in (0, 1)$ } of the Axiom, while the AntiAxiom represents a degree of total negation { $d = 1$ } of the Axiom.

Proof is also evident.

6.4. Neutrosophic Representation

We have:

$\langle A \rangle$ = *Axiom;*

$\langle neutA \rangle$ = *NeutroAxiom (or NeutAxiom);*

$\langle antiA \rangle$ = *AntiAxiom;*

and $\langle nonA \rangle$ = *NonAxiom.*

Similarly as in *Neutrosophy, NonAxiom* is refined/split into two parts: *NeutroAxiom* and *AntiAxiom*.

6.5. Application of NeutroLaws in Soft Science

In *soft sciences* the laws are interpreted and re-interpreted; in social and political legislation the laws are flexible; the same law may be true from a point of view, and false from another point of view. Thus the law is partially true and partially false (it is a neutrosophic law).

For example, "gun control". There are people supporting it because of too many crimes and violence (and they are right), and people that oppose it because they want to be able to defend themselves and their houses (and they are right too).

We see two opposite propositions, both of them true, but from different points of view (from different criteria/parameters; plithogenic logic may better be used herein). How to solve this? Going to the middle, in between opposites (as in neutrosophy): <u>allow</u> military, police, security, registered hunters to bear arms; <u>prohibit</u> mentally ill, sociopaths, criminals, violent people from bearing arms; and <u>background check</u> on everybody that buys arms, etc. Definition of Classical Associativity

Let $\mathcal{U}$ be a universe of discourse, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, endowed with a well-defined binary law $*$. The law $*$ is associative on the set $\mathcal{S}$, iff $\forall\ a, b, c\ \in \mathcal{S}$, $a * (b * c) = (a * b) * c$.

6.6. Definition of Classical NonAssociativity

Let $\mathcal{U}$ be a universe of discourse, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, endowed with a well-defined binary law $*$. The law $*$ is non-associative on the set $\mathcal{S}$, iff $\exists a, b, c\ \in \mathcal{S}$, such that $a * (b * c) \neq (a * b) * c$.

So, it is sufficient to get a single triplet a, b, c (where a, b, c may even be all three equal, or only two of them equal) that doesn't satisfy the associativity axiom.

Yet, there may also exist some triplet $d, e, f \in \mathcal{S}$ that satisfies the associativity axiom: $d * (e * f) = (d * e) * f$.

The classical definition of NonAssociativity does not make a distinction between a set $(\mathcal{S}_1, *)$ whose all triplets $a, b, c \in \mathcal{S}_1$ verify the non-associativity inequality, and a set $(\mathcal{S}_2, *)$ whose some triplets verify the non-associativity inequality, while others don't.

6.7. NeutroAssociativity & AntiAssociativity

If $\langle \mathrm{A} \rangle$ = (classical) Associativity, then $\langle \mathrm{nonA} \rangle$ = (classical) NonAssociativity.

But we refine/split $\langle \mathrm{nonA} \rangle$ into two parts, as above:

$\langle neutA \rangle$ = *NeutroAssociativity;*

$\langle antiA \rangle$ = *AntiAssociativity.*

Therefore,

NonAssociativity = *NeutroAssociativity* ∪ *AntiAssociativity*.

The Associativity's neutrosophic triplet is:

<*Associativity*, *NeutroAssociativity*, *AntiAssociativity*>.

6.8. Definition of NeutroAssociativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.

The set $(\mathcal{S}, *)$ is NeutroAssociative if and only if:

there exists at least one triplet $a_1, b_1, c_1 \in \mathcal{S}$ such that:

$a_1 * (b_1 * c_1) = (a_1 * b_1) * c_1$;

and there exists at least one triplet $a_2, b_2, c_2 \in \mathcal{S}$ such that:

$a_2 * (b_2 * c_2) \neq (a_2 * b_2) * c_2$.

Therefore, some triplets verify the associativity axiom, and others do not.

6.9. Definition of AntiAssociativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.

The set $(\mathcal{S},*)$ is AntiAssociative if and only if:

for any triplet $a, b, c \in \mathcal{S}$ one has $a * (b * c) \neq (a * b) * c$.

Therefore, none of the triplets verify the associativity axiom.

6.10. Example of Associativity

Let $N = \{0, 1, 2, \ldots, \infty\}$, the set of natural numbers, be the universe of discourse, and the set $\mathcal{S} = \{0, 1, 2, \ldots, 9\} \subset$ N, also the binary law $*$ be the classical addition modulo 10 defined on N.

Clearly the law * is well-defined on S, and associative since:

$a + (b + c) = (a + b) + c$ (mod 10), for all $a, b, c \in \mathcal{S}$.

The degree of negation is 0%.

6.11. Example of NeutroAssociativity

$\mathcal{S} = \{0, 1, 2, \ldots, 9\}$, and the well-defined binary law $*$ constructed as below:

$a * b = 2a + b$ (mod 10).

Let's check the associativity:

$a * (b * c) = 2a + (b * c) = 2a + 2b + c$

$(a * b) * c = 2(a * b) + c = 2(2a + b) + c = 4a + 2b + c$

The triplets that verify the associativity result from the below equality:

$2a + 2b + c = 4a + 2b + c$

or $2a = 4a$ (mod 10)

or $0 = 2a$ (mod 10), whence $a \in \{0, 5\}$.

Hence, two general triplets of the form:

$\{(0, b, c), (5, b, c), \text{where } b, c \in \mathcal{S}\}$

verify the associativity.

The degree of associativity is $\frac{2}{10} = 20\%$, corresponding to the two numbers $\{0, 5\}$ out of ten.

While the other general triplet:

$\{(a, b, c), \text{where } a \in \mathcal{S} \setminus \{0, 5\}, \text{while } b, c \in \mathcal{S}\ \}$

do not verify the associativity.

The degree of negation of associativity is $\frac{8}{10} = 80\%$.

6.12. Example of AntiAssociativity

$\mathcal{S} = \{a, b\}$, and the binary law $*$ well-defined as in the below Cayley Table:

$*$	a	b
a	b	b
b	a	a

Theorem 1.

For any $x, y, z \in \mathcal{S}$, $x * (y * z) \neq (x * y) * z$.

Proof.

We have $2^3 = 8$ possible triplets on $\mathcal{S}$:

1) (a, a, a)

$a * (a * a) = a * b = b$

while $(a * a) * a = b * a = a \neq b$.

2) (a, a, b)

$a * (a * b) = a * b = b$

$(a * a) * b = b * b = a \neq b$.

3) (a, b, a)

$a * (b * a) = a * a = b$

$(a*b)*a = b*a = a \neq b$.

4) (b, a, a)

$b*(a*a) = b*b = a$

$(b*a)*a = a*a = b \neq a$.

5) (a, b, b)

$a*(b*b) = a*a = b$

$(a*b)*b = b*b = a \neq b$.

6) (b, a, b)

$b*(a*b) = b*b = a$

$(b*a)*b = a*b = b \neq a$.

7) (b, b, a)

$b*(b*a) = b*a = a$

$(b*b)*a = a*a = b \neq a$.

8) (b, b, b)

$b*(b*b) = b*a = a$

$(b*b)*b = a*b = b \neq a$.

Therefore, there is no possible triplet on $\mathcal{S}$ to satisfy the associativity. Whence the law is AntiAssociative. The degree of negation of associativity is $\frac{8}{8} = 100\%$.

6.13. Definition of Classical Commutativity

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$. The law $*$ is Commutative on the set $\mathcal{S}$, iff $\forall\, a, b \in \mathcal{S}$, $a*b = b*a$.

6.14. Definition of Classical NonCommutativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$. The law $*$ is NonCommutative on the set $\mathcal{S}$, iff $\exists a, b \in \mathcal{S}$, such that $a * b \neq b * a$.

So, it is sufficient to get a single duplet $a, b \in \mathcal{S}$ that doesn't satisfy the commutativity axiom.

However, there may exist some duplet $c, d \in \mathcal{S}$ that satisfies the commutativity axiom: $c * d = d * c$.

The classical definition of NonCommutativity does not make a distinction between a set $(\mathcal{S}_1,*)$ whose all duplets $a, b \in \mathcal{S}_1$ verify the NonCommutativity inequality, and a set $(\mathcal{S}_2,*)$ whose some duplets verify the NonCommutativity inequality, while others don't.

That's why we refine/split the NonCommutativity into NeutroCommutativity and AntiCommutativity**.**

6.15. NeutroCommutativity & AntiCommutativity

Similarly to Associativity we do for the Commutativity:

If ⟨A⟩ = (classical) *Commutativity*, then ⟨nonA⟩ = (classical) *NonCommutativity*.

But we refine/split ⟨nonA⟩ into two parts, as above:

$\langle neutA \rangle$ = *NeutroCommutativity;*

$\langle antiA \rangle$ = *AntiCommutativity.*

Therefore,

NonCommutativity =

= *NeutroCommutativity* ∪ *AntiCommutativity*.

The Commutativity's neutrosophic triplet is:

<*Commutativity*, *NeutroCommutativity*, *AntiCommutativity*>.

In the same way, Commutativity means all elements of the set commute with respect to a given binary law, NeutroCommutativity means that some elements commute while others do not, while AntiCommutativity means that no elements commute.

6.16. Example of NeutroCommutativity

$\mathcal{S} = \{a, b, c\}$, and the well-defined binary law $*$.

*	a	b	c
a	b	c	c
b	c	b	a
c	b	b	c

$a * b = b * a = c$ (commutative);

$\begin{cases} a * c = c \\ c * a = b \neq c \end{cases}$ (not commutative);

$\begin{cases} b * c = a \\ c * b = b \neq a \end{cases}$ (not commutative).

We conclude that $(\mathcal{S},*)$ is $\frac{1 \text{ pair}}{3 \text{ pairs}} \approx 33\%$ commutative, and $\frac{2 \text{ pair}}{3 \text{ pairs}} \approx 67\%$ not commutative.

Therefore, the degree of negation of the commutativity of $(\mathcal{S},*)$ is 67%.

6.17. Example of AntiCommutativity

$\mathcal{S} = \{a, b\}$, and the below binary well-defined law $*$.

*	a	b
a	b	b
b	a	a

where $a * b = b$, $b * a = a \neq b$ (not commutative)

Other pair of different element does not exist, since we cannot take $a * a$ nor $b * b$. The degree of negation of commutativity of this $(\mathcal{S},*)$ is 100%.

6.18. Definition of Classical Unit Element

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$ and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.

The set $\mathcal{S}$ has a classical unit element $e \in \mathcal{S}$, iff e is unique, and for any $x \in \mathcal{S}$ one has $x * e = e * x = x$.

6.19. Partially Negating the Definition of Classical Unit Element

It occurs when at least one of the below statements occurs:

1) There exists at least one element $a \in \mathcal{S}$ that has no unit element.

2) There exists at least one element $b \in \mathcal{S}$ that has at least two distinct unit elements $e_1, e_2 \in \mathcal{S}$, $e_1 \neq e_2$, such that:

$b * e_1 = e_1 * b = b$,

$b * e_2 = e_2 * b = b$.

3) There exists at least two different elements $c, d \in \mathcal{S}$, $c \neq d$, such that they have different unit elements $e_c, e_d \in \mathcal{S}$, $e_c \neq e_d$, with $c * e_c = e_c * c = c$, and $d * e_d = e_d * d = d$.

6.20. Totally Negating the Definition of Classical Unit Element

The set $(\mathcal{S},*)$ has *AntiUnitElements*, if:

1) Each element $x \in \mathcal{S}$ has either no unit element, or two or more unit elements (unicity of unit element is negated);

2) If some elements $x \in \mathcal{S}$ have only one unit element each, then these unit elements are different two by two.

6.21. Definition of NeutroUnitElements

The set $(\mathcal{S},*)$ has *NeutroUnit Elements*, if:

1) [Degree of Truth] There exist at least an element that has a single unit-element.

2) [Degree of Falsehood] There exist at least one element that either has no unit-element, or has two or more unit-elements.

6.22. Definition of AntiUnit Elements

The set $(\mathcal{S},*)$ has *AntiUnit Elements*, if:

Each element $x \in \mathcal{S}$ has either no unit-element, or two or more distinct unit-elements.

6.23. Example of NeutroUnit Elements

$\mathcal{S} = \{a, b, c\}$, and the well-defined binary law $*$:

*	a	b	c
a	b	b	a
b	b	b	a
c	a	b	c

Since,

$a * c = c * a = a$

$c * c = c$

the common unit-element of a and c is c (two distinct elements a $\neq c$ have the same unit element c).

From

$b * a = a * b = b$

$b * b = b$

we see that the element b has two distinct unit-elements a and b.

Since only one element b does not verify the classical unit axiom (i.e. to have a unique unit), out of 3 elements, the degree of negation of unit element axiom is $\frac{1}{3} \approx 33\%$, while $\frac{2}{3} \approx 67\%$ is the degree of truth (validation) of the unit element axiom.

6.24. Example of AntiUnit Elements

$\mathcal{S} = \{a, b, c\}$, endowed with the well-defined binary law $*$ as follows:

$*$	a	b	c
a	a	a	a
b	a	c	b
c	a	c	b

Element a has 3 unit elements: a, b, c, because:

$a * a = a$

$a * b = b * a = a$

and $a * c = c * a = a$.

Element b has no unit element, since:

$b * a = a \neq b$

$b * b = c \neq b$

and $b * c = b$, but $c * b \neq b$.

Element c has no unit element, since:

$c * a = a \neq c$

$c * b = c$, but $b * c = b \neq c$,

and $c * c = b \neq c$.

The degree of negation of the unit element axiom is $\frac{3}{3} = 100\%$.

6.25. Definition of Classical Inverse Element

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$ and a non $-$ empty set $\mathcal{S} \subseteq \mathcal{U}$.

Let $e \in \mathcal{S}$ be the classical unit element, which is unique.

For any element $x \in \mathcal{S}$, there exists a unique element, named the inverse of x, denoted by x^{-1}, such that:

$x * x^{-1} = x^{-1} * x = e$.

6.26. Partially Negating the Definition of Classical Inverse Element

It occurs when at least one statement from below occurs:

1) There exist at least one element $a \in \mathcal{S}$ that has no inverse with respect to no ad-hoc unit-elements;

or

2) There exist at least one element $b \in \mathcal{S}$ that has two or more distinct inverses with respect to some ad-hoc unit-elements.

6.27. Totally Negating the Definition of Classical Inverse Element

Each element has either no inverse, or two or more distinct inverses with respect to some ad-hoc unit-elements respectively.

6.28. Definition of NeutroInverse Elements

The set $(\mathcal{S},*)$ has NeutroInverse Elements if:

1) [Degree of Truth] There exist at least an element that has an inverse with respect to some ad-hoc unit-element.

2) [Degree of Falsehood] There exists at least one element that does not have any inverse with respect to no ad-hoc unit-element, or has at least two or more distinct inverses with respect to some ad-hoc unit-elements.

6.29. Definition of AntiInverse Elements

The set $(\mathcal{S},*)$ has AntiInverse Elements, if: each element has either no inverse with respect to no ad-hoc unit-element, or two or more distinct inverses with respect to some ad-hoc unit-elements.

6.30. Example of NeutroInverse Elements

$S = \{a, b, c\}$, endowed with the binary well-defined law * as below:

$*$	a	b	c
a	a	b	c
b	b	a	a
c	b	b	b

Because $a * a = a$, hence its ad-hoc unit/neutral element $neut(a) = a$ and correspondingly its inverse element is $inv(a) = a$.

Because $b * a = a * b = b$, hence its ad-hoc inverse/neutral element $neut(b) = a$;

from $b * b = a$, we get $inv(b) = b$.

No $neut(c)$, hence no $inv(c)$.

Hence *a* and *b* have ad-hoc inverses, but *c* doesn't.

6.31. Example of AntiInverse Elements

Similarly, $S = \{a, b, c\}$, endowed with the binary well-defined law * as below:

$*$	a	b	c
a	b	b	c
b	a	a	a
c	c	a	a

There is no *neut(a)* and no *neut(b),* hence: no *inv(a)* and *no inv(b).*

$c * a = a * c = c$, hence: $neut(c) = a$.

$c * b = b * c = a$, hence: $inv(c) = b$;

$c * c = c * c = a,$ hence: $inv(c) = c$; whence we get two inverses of c.

6.32. Cases When Partial Negation (NeutroAxiom) Does Not Exist

Let's consider the classical geometric *Axiom*:

On a plane, through a point exterior to a given line it's possible to draw a single parallel to that line.

The *total negation* is the following *AntiAxiom*:

On a plane, through a point exterior to a given line it's possible to draw either no parallel, or two or more parallels to that line.

The *NeutroAxiom* does not exist since it is not possible to partially deny this classical axiom.

6.33. Connections between the neutrosophic triplet *(Axiom, NeutroAxiom, AntiAxiom)* and the *S-denying an Axiom*

The *S-denying of an Axiom* was first defined by Smarandache [3, 4] in 1969 when he constructed *hybrid geometries* (or *S-geometries*) [5 – 18].

6.34. Definition of S-denying an Axiom

An *Axiom* is said *S-denied* [3, 4] if in the same space the axiom behaves differently (i.e., validated and invalided; or only invalidated but in at least two distinct ways).

Therefore, we say that an axiom is partially negated (or there is a degree of negation of an axiom): http://fs.unm.edu/Geometries.htm.

6.35. Definition of S-geometries

A geometry is called *S-geometry* [5] if it has at least one S-denied axiom.

Therefore, the Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries were united altogether for the first time, into the same space, by some S-geometries. These S-geometries could be partially Euclidean and partially Non-Euclidean, or only Non-Euclidean but in multiple ways.

The most important contribution of the S-geometries was the introduction of the *degree of negation of an axiom* (and more general the degree of negation of any theorem, lemma, scientific or humanistic proposition, theory, etc.).

Many geometries, such as pseudo-manifold geometries, Finsler geometry, combinatorial Finsler geometries, Riemann geometry, combinatorial Riemannian geometries, Weyl geometry, Kahler geometry are particular cases of S-geometries. (Linfan Mao)

6.36. Connection between S-denying an Axiom and NeutroAxiom / AntiAxiom

“Validated and invalidated” Axiom is equivalent to NeutroAxiom. While “only invalidated but in at least two distinct ways” Axiom is part of the AntiAxiom (depending on the application).

“Partially negated” (or $0 < d < 1$, where d is the degree of negation) is referred to NeutroAxiom. While “there is a degree of negation of an axiom” is referred to both NeutroAxiom (when $0 < d < 1$) and AntiAxiom (when $d = 1$).

6.37. Connection between NeutroAxiom and MultiSpace

In any domain of knowledge, a *S-multispace with its multistructure* is a finite or infinite (countable or uncountable) union of many spaces that have various structures (Smarandache, 1969, [19]). The multi-spaces with their multi-structures [20, 21] may be non-disjoint. The multispace with multistructure form together a *Theory of Everything*. It can be used, for example, in the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak, and strong interactions in physics.

Therefore, a NeutroAxiom splits the set S, which it is defined upon, into two subspaces: one where the Axiom is true and another where the Axiom is false. Whence S becomes a BiSpace with BiStructure (which is a particular case of MultiSpace with MultiStructure).

6.38. (Classical) WellDefined Binary Law

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and a binary law $*$ defined on U. For any $x, y \in \mathcal{S}$, one has $x * y \in \mathcal{S}$.

6.39. NeutroDefined Binary Law

There exist at least two elements (that could be equal) $a, b \in \mathcal{S}$ such that $a * b \in \mathcal{S}$. And there exist at least other two elements (that could be equal too) $c, d \in \mathcal{S}$ such that $c, d \notin \mathcal{S}$.

6.40. Example of NeutroDefined Binary Law

Let $U = \{a, b, c\}$ be a universe of discourse, and a subset $\mathcal{S} = \{a, b\}$, endowed with the below NeutroDefined Binary Law $*$:

$*$	a	b
a	b	b
b	a	c

We see that: $a * b = b \in S$, $b * a = a \in S$, but $b * b$ = c $\notin S$.

6.41. AntiDefined Binary Law

For any $x, y \in \mathcal{S}$ one has $x * y \notin \mathcal{S}$.

6.42. Example of AntiDefined Binary Law

Let $U = \{a, b, c, d\}$ a universe of discourse, and a subset $\mathcal{S} = \{a, b\}$, and the below binary well-defined law $*$.

$*$	a	b
a	c	d
b	d	c

where all combinations between *a* and *b* using the law * give as output *c* or *d* who do not belong to S.

6.43. Theorem of the Degenerate Case

If a set is endowed with AntiDefined Laws, all its algebraic structures based on them will be AntiStructures.

6.44. WellDefined n-ary Law

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and a n-ary law, for n integer, $n \geq 1$, defined on $\mathcal{U}$.

$L: \mathcal{U}^n \to \mathcal{U}$.

For any $x_1, x_2, \dots, x_n \in \mathcal{S}$, one has $L(x_1, x_2, \dots, x_n) \in \mathcal{S}$.

6.45. NeutroDefined n-ary Law

There exists at least a n-plet $a_1, a_2, \dots, a_n \in \mathcal{S}$ such that $L(a_1, a_2, \dots, a_n) \in \mathcal{S}$. The elements $a_1, a_2, \dots, a_n$ may be equal or not among themselves.

And there exists at least a n-plet $b_1, b_2, \dots, b_n \in \mathcal{S}$ such that $L(b_1, b_2, \dots, b_n) \notin \mathcal{S}$. The elements $b_1, b_2, \dots, b_n$ may be equal or not among themselves.

6.46. AntiDefined n-ary Law

For any $x_1, x_2, \dots, x_n \in \mathcal{S}$, one has $L(x_1, x_2, \dots, x_n) \notin \mathcal{S}$.

6.47. WellDefined n-ary HyperLaw

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subset_{\neq} \mathcal{U}$, and a n-ary hyperlaw, for *n* integer, $n \geq 1$:

$H: \mathcal{U}^n \to \mathcal{P}(\mathcal{U})$, where $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$.

For any $x_1, x_2, \dots, x_n \in \mathcal{S}$, one has $H(x_1, x_2, \dots, x_n) \in \mathcal{P}(\mathcal{S})$.

6.48. NeutroDefined n-ary HyperLaw

There exists at least a n-plet $a_1, a_2, \dots, a_n \in \mathcal{S}$ such that $H(a_1, a_2, \dots, a_n) \in \mathcal{P}(\mathcal{S})$. The elements $a_1, a_2, \dots, a_n$ may be equal or not among themselves.

And there exists at least a n-plet $b_1, b_2, \dots, b_n \in \mathcal{S}$ such that $H(b_1, b_2, \dots, b_n) \notin \mathcal{P}(\mathcal{S})$. The elements $b_1, b_2, \dots, b_n$ may be equal or not among themselves.

6.49. AntiDefined n-ary HyperLaw

For any $x_1, x_2, \dots, x_n \in \mathcal{S}$, one has $H(x_1, x_2, \dots, x_n) \notin \mathcal{P}(\mathcal{S})$.

*

The most interesting are the cases when the composition law(s) are well-defined (classical way) and neutro-defined (neutrosophic way).

6.50. WellDefined NeutroStructures

Are structures whose laws of compositions are well-defined, and at least one axiom is NeutroAxiom, and one has no AntiAxiom.

6.51. NeutroDefined NeutroStructures

Are structures whose at least one law of composition is NeutroDefined, and all other axioms are NeutroAxioms or Axioms.

6.52. Example of NeutroDefined NeutroGroup

Let U = {a, b, c, d} be a universe of discourse, and the subset

$\mathcal{S} = \{a, b, c\}$, endowed with the binary law $*$:

$*$	a	b	c
a	a	c	b
b	c	a	c
c	a	c	d

NeutroDefined Law of Composition:

Because, for example: a*b = c ∈ S, but c*c = d ∉ S.

NeutroAssociativity:

Because, for example:

b*(c*b) = b*c = c and (b*c)*b = c*b = c;

while, for example:

a*(a*b) = a*c = b and (a*a)*b = a*b = c ≠ b.

NeutroCommutativity:

Because, for example:

a*b = b*a = c, but a*c = b while c*a = a ≠ c.

NeutroUnit Element:

There exists a unit element b for c, since $c*b = b*c = c$; and there is a unit element a for a, since $a*a = a$.

But there is no unit element for b, because $b*x = a$ or c, not b, for any $x \in S$ (according to the above Cayley Table)

NeutroInverse Element:

There exists an inverse element for a, which is a, because $a*a = a$.

But there is no inverse element for b, since b has no unit element.

Therefore (S, *) is a NeutroDefined NeutroCommutative NeutroGroup.

6.53. WellDefined AntiStructures

Are structures whose laws of compositions are well-defined, and have at least one AntiAxiom.

6.54. NeutroDefined AntiStructures

Are structures whose at least one law of composition is NeutroDefined and no law of composition is AntiDefined, and has at least one AntiAxiom.

6.55. AntiDefined AntiStructures

Are structures whose at least one law of composition is AntiDefined, and has at least one AntiAxiom.

6.56. Conclusion

The neutrosophic triplet (<A>, <neutA>, <antiA>), where <A> may be an “Axiom”, a “Structure”, a “Theory” and so on, <antiA> the opposite of <A>, while <neutA> (or <neutroA>) their neutral in between, are studied in this paper.

The NeutroAlgebraic Structures and AntiAlgebraic Structures are introduced now for the first time, because they have been ignored by the classical algebraic structures. Since, in science and technology and mostly in applications of our everyday life, the laws that characterize them are not necessarily well-defined or well-known, and the axioms / properties / theories etc. that govern their spaces may be only partially true and partially false (as <*neutA*> in neutrosophy, which may be a blending of truth and falsehood).

Mostly in idealistic or imaginary or abstract or perfect spaces we have rigid laws and rigid axioms that totally apply (that are 100% true). But the laws and the axioms should be more flexible in order to comply with our imperfect world.

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