Discovering and Programming the Cubic Formula

Timothy W. Jones

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Abstract

Solving a cubic polynomial using a formula is possible; a formula exists. In this article we connect various dots from a pre-calculus course and attempt to show how the formula could be discovered. Along the way we make a TI-84 CE menu driven program that allows for experiments, confirmations of speculations, and eventually a working program that solves all cubic polynomials.

Introduction

Blitzer's Algebra and Trigonometry [1] gives the formula for one root of $x^3 + mx = n$ at the beginning of Section 3.4: Zeros of Polynomial Functions. We'll confirm that the formula works for a few cases using a TI-84 CE (TI-83 family) program, see Figure 1.

Where did the formula come from? Can we derive it using no more than the contents of this algebra book? Using the chapter on polynomials, we can infer that one root of all cubics will be real; that's the intermediate value theorem (3.2). It is easily understood as all cubics will have negative values that map to negative values and positive values that map to positive values. As these functions are continuous these properties imply that the function's graph crosses the x-axis and hence that it has at least one real root. Finding that one real root should be enough: using synthetic division of polynomials (3.3), we can divide the original cubic by $(x - r)$, r the one found real (or actually any) root and obtain as the quotient a quadratic. The quadratic formula (1.5) can solve this quadratic and give us the remaining two roots.

```
001  -1/2+i√(3)/2→O
002  -1/2-i√(3)/2→S
003  Menu("CUBICS","ENTER NEW COEFTS",A,"EVALUATE",B,"CONFIRM BLITZER",C,"CARDAN 3 ROOTS",D,"QUIT",Q)
004  Lbl A
005  Prompt A,B,C,D
006  prgmCUBICS
007  Lbl B
008  Prompt X
009  AX³+BX²+CX+D→Y
010  Disp Y
011  Pause
012  prgmCUBICS
013  Lbl C
014  Prompt M,N
015  ³√(√((N/2)²+(M/3)³)-N/2)→F
016  -³√(√((N/2)²+(M/3)³)+N/2)→L
017  Disp -(F+L)
018  Pause
019  prgmCUBICS
020  Lbl D
021  C→M:-D→N
022  ³√(√((N/2)²+(M/3)³)-N/2)→F
023  -³√(√((N/2)²+(M/3)³)+N/2)→L
024  Disp -(F+L)
025  Disp -(F*O+L*S)
026  Disp -(F*S+L*O)
027  Pause
028  prgmCUBICS
029  Lbl Q
030  Stop
```

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NORMAL FLOAT AUTO a+bi RADIAN CL

CUBICS
1:ENTER NEW COEFTS
2:EVALUATE
3:CONFIRM BLITZER
4:CARDAN 3 ROOTS
5:QUIT
```
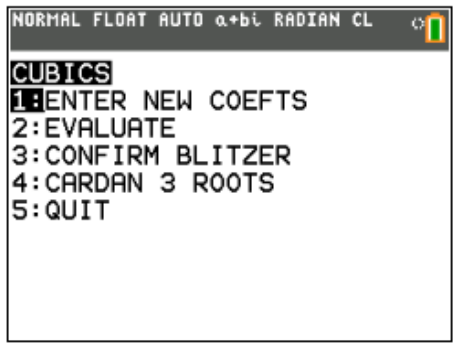
Figure 1: 1) Enter coefficients for general cubic; 2) evaluate at a specified X; 3) confirm Blitzer's formula; 4) show 3 roots per Cardan.

We do need a result not in the book: an identity. But the identity we need is close to one the book does have (P5): $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$. The book's identity can be glossed as a way to factor some cubic polynomials, our theme: for example, $x^3 + 2^3 = (x+2)(x^2 - 2x + 4)$. We extend this identity to the sum of three cubics and combine it with the cube roots of unity (7.5) to give an identity that factors, hence solves a *depressed* cubic (one lacking a x^2 term). This identity is in Chrystal [2] and other books and it is referenced (obliquely) in Wikipedia's article on cubics. Our derivation might be new; this latter reference puts it in the context of a discrete Fourier transform and Chrystal makes it a problem; see Wikipedia's Lagrange's method in their article on cubics. We just use a few inferences from the factored form of a cubic to its coefficient form, nothing too fancy for a high school student: we think it is about at the level of deriving the quadratic formula, a little hard.

Other references of interest are Tignol's *Galois' Theory of Algebraic Equations* and Herstein's *Topics in Algebra* [3, 4]. The first gives an historical evolution of solving the cubic and puts it in a broader context of Galois the-

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ory. Herstein presents a highly condensed, fast formulae for all three roots to a general cubic, page 251. He builds into his formulae the transformations that *depress* a general cubic. We'll touch on transformations in our last section.

Formula confirmation

Figure 2: Option 3 of the CUBICS program with M=1 and N=30 gives a root of 3 which is correct.

Before going forward, Blitzer specifies that

$$x = \sqrt[3]{\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} + \frac{n}{2}} - \sqrt[3]{\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} - \frac{n}{2}} \qquad (1)$$

is one root of $x^3 + mx = n$. Taking $m = 1$ and $n = 30$, we should find a root of 3.

Using our laboratory program we confirm the formula works for this case. The pertinent lines of code are 013 to 017, Figure 1. Figure 2 gives the confirmation. We've got one of three roots.

An important cubic

The easiest cubic equation that has three calculable roots is $x^3 - 1 = 0$. Its roots are $\omega, \omega^2, \omega^3 = 1$ where $\omega = (\cos 120 + i \sin 120)$. As powers rotate points by multiples of 120, we get $3*120 = 360$, $6*120 = 2\cdot360$ per DeMoivre's theorem (7.5), implying each is a root of unity. We can also translate these

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complex numbers into radical forms by solving $x^3 + (-1)^3 = (x-1)(x^2+x+1)$ using the quadratic formula and an earlier identity:

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \ \text{and} \ \omega^3 = 1.$$

We gain some experience by confirming these results with a little algebra:

$$\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - 2\left(\frac{1}{2}\right)\left(i\frac{\sqrt{3}}{2}\right) - \frac{3}{4}$$

$$-\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

and

$$\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 = -\frac{1}{8} + 3\frac{1}{4}i\frac{\sqrt{3}}{2} - 3\frac{1}{2}\left(\frac{i\sqrt{3}}{2}\right)^2 + \left(\frac{i\sqrt{3}}{2}\right)^3$$

$$= -\frac{1}{8} + \frac{3}{8}i\sqrt{3} + \frac{9}{8} - \frac{3}{8}i\sqrt{3}$$

$$= 1.$$

We note that

$$\omega + \omega^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2} = -1. \tag{2}$$

We'll use this later. The two values ω and ω^2 are stored as O and S; see code lines 001 and 002 of Figure 1.

A real identity

There is a relationship between the root form (left hand side of (3)) and the coefficient form (right hand side of (3)) of a cubic: $p_3(x) =$

$$(x-r_1)(x-r_2)(x-r_3) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3. \tag{3}$$

As $p_3(r_1) = p_3(r_2) = p_3(r_3) = 0$, we can derive an identity by summing the following three equations

$$p_3(r_1) = r_1^3 - (r_1 + r_2 + r_3)r_1^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)r_1 - r_1 r_2 r_3 = 0$$

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$$p_3(r_2) = r_2^3 - (r_1 + r_2 + r_3)r_2^2 + (r_1r_2 + r_1r_3 + r_2r_3)r_2 - r_1r_2r_3 = 0$$

$$p_3(r_3) = r_3^3 - (r_1 + r_2 + r_3)r_3^2 + (r_1r_2 + r_1r_3 + r_2r_3)r_3 - r_1r_2r_3 = 0.$$

That sum implies

$$r_1^3 + r_2^3 + r_3^3 - 3r_1r_2r_3 = (r_1 + r_2 + r_3)(r_1^2 + r_2^2 + r_3^2 - r_1r_2 - r_1r_3 - r_2r_3). \quad (4)$$

A complex identity

We can use (4) with (2) to arrive at

$$-r_1r_2 - r_1r_3 - r_2r_3 = (\omega + \omega^2)r_1r_2 + (\omega + \omega^2)r_1r_3 + (\omega + \omega^2)r_2r_3. \quad (5)$$

Adding $r_1^2 + r_2^2 + r_3^2$ to the right hand side of (5) and multiplying by $r_1 + r_2 + r_3$ gives the complex identity:

$$r_1^3 + r_2^3 + r_3^3 - 3r_1r_2r_3 = (r_1 + r_2 + r_3)(r_1 + \omega r_2 + \omega^2 r_3)(r_1 + \omega^2 r_2 + \omega r_3). \quad (6)$$

It might be the easiet derivation to just note

$$(r_1 + \omega r_2 + \omega^2 r_3)(r_1 + \omega^2 r_2 + \omega r_3)$$

$$= r_1^2 + r_2^2 + r_3^2 + (\omega + \omega^2)r_1r_2 + (\omega + \omega^2)r_1r_3 + (\omega + \omega^2)r_2r_3,$$

by just doing the multiplication and remembering that $\omega^3 = 1$.

Exploration

We can gloss the identity (6) as the factoring of a depressed cubic; just make $r_1 = x$ for

$$x^3 - 3r_2r_3x + r_2^3 + r_3^3 = (x + r_2 + r_3)(x + \omega r_2 + \omega^2 r_3)(x + \omega^2 r_2 + \omega r_3). \quad (7)$$

Let's experiment. Let $r_2 = 2$, and $r_3 = 3$ in (6) then (7) becomes

$$x^3 - 3 \cdot 2 \cdot 3x + 2^3 + 3^3 = x^3 - 18x + 35.$$

That's a depressed cubic and we can find a root using Blitzer's (actually Cardan's) formula. We get a root of -5. We can then do a division, get a quadratic and solve the quadratic: Figure 3. After putting in coefficients

$$\gt \frac{\left(x^3 - 18 \cdot x + 35\right)}{(x+5)};$$

$$\frac{x^3 - 18x + 35}{x+5}$$

$$\gt simplify(\%);$$

$$x^2 - 5x + 7$$

$$\gt solve(x^2 - 5 \cdot x + 7, x);$$

$$\frac{5}{2} + \frac{I\sqrt{3}}{2}, \frac{5}{2} - \frac{I\sqrt{3}}{2}$$

Figure 3: Given any one root r of a cubic, we can divide by $x - r$ and then solve the quotient quadratic.

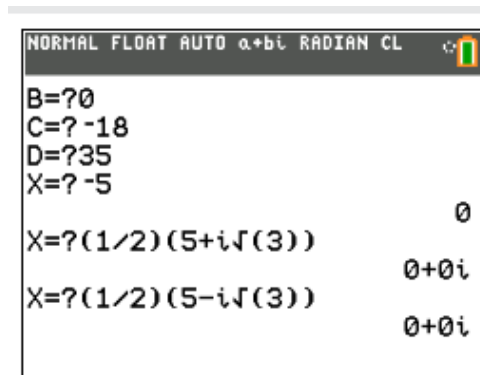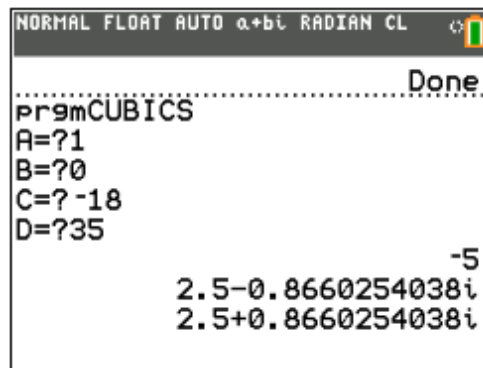Figure 4: Using option 1) and 2) the CUBICS program confirms roots.

Figure 5: The CUBICS program correctly finds all roots of $x^3 - 18x + 35 = 0$.

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$A = 1$, $B = 0$, $C = -18$, and $D = 35$ and, using Option 2, Evaluate, we confirm $1/2(5 + i\sqrt{3})$ and $1/2(5 - i\sqrt{3})$ are roots: Figure 4. Our calculator program encodes (7) in lines 021 through 026. Using Option 4) we should see these three roots. Bingo: Figure 5 confirms that for this cubic our formula found the roots. We see that the TI-84 dropped the symbolic, exact solutions and presented decimal forms: $\sqrt{3}/2 = 0.8660254038$.

Depressed cubic formula

We can infer from this example that the central puzzle to solving a cubic given in the form $x^3 + cx + d$ is to find r_2 and r_3 given that $c = -3r_2r_3$ and $d = r_2^3 + r_3^3$. If we are able to to do this we can use (7) and read off the roots: $R_1 = -r_2 - r_3$, $R_2 = -\omega r_2 - \omega^2 r_3$, and $R_3 = -\omega^2 r_2 - \omega r_3$.

Here is the "aha" moment: we can find two numbers knowing the product and sum of the numbers using a quadratic equation; and, as a bonus, quadratic solutions are complex conjugates. Solving the quadratic $x^2 + dx - (c/3)^3$ does it. If the roots are q_1 and q_2, then $\sqrt[3]{q_1} = r_2$ and $\sqrt[3]{q_2} = r_3$. For our example $x^3 - 18x + 35$, $-3r_2r_3 = -18$, so $r_2r_3 = 6$, that cubed is the quadratic's constant term and $d = r_2^3 + r_3^3$. The quadratic associated with $x^3 - 18x + 35$ is $x^2 - 35x + 216$. The quadratic formula yields

$$q_1 = \frac{35 - \sqrt{35^2 - 4(1)(216)}}{2} = 8 = r_2^3$$

and

$$q_2 = \frac{35 + \sqrt{35^2 - 4(1)(216)}}{2} = 27 = r_3^2.$$

Some algebraic manipulations of these ideas yields Blitzer's formula for the depressed cubic equation: $x^3 + mx - n = 0$. Here goes: $m = -3r_2r_3$ and $-n = r_2^3 + r_3^3$. So the quadratic associated with this cubic is $x^2 + nx - (m/3)^3$. The quadratic formula gives

$$\frac{-n + \sqrt{n^2 - 4(1)(-m^3/27)}}{2} \text{ and } \frac{-n - \sqrt{n^2 + 4(1)(-m^3/27)}}{2}$$

or

$$\frac{\sqrt{(4/4)n^2 - 4(1)(-m^3/27)} - n}{2} \text{ and } \frac{-\sqrt{(4/4)n^2 + 4(1)(-m^3/27)} - n}{2}$$

or

$$\sqrt[3]{\frac{\sqrt{(1/4)n^2 - (1)(-m^3/27)} - n}{2}} \text{ and } \sqrt[3]{\frac{-\sqrt{(1/4)n^2 + (1)(-m^3/27)} - n}{2}}$$

or

$$\sqrt{(n/2)^2 + (m/3)^3} - n/2 \text{ and } -\sqrt{(n/2)^2 + (m/3)^3} - n/2.$$

Taking the cube root of each and the negative of the sum gives (1).

General cubic formula

Given $p_3(x) = x^3 + bx^2 + cx + d$ can we eliminate the x^2 term by a transformation [1, Section 2.5], solve the resulting *depressed* cubic $d_3(x)$, and then solve the original $p_3(x)$? We will then be in a position to formulate a general solution and implement it as a calculator program. As a test case, let's take

$$p_3(x) = (x - 1)(x - (2 - 2i))(x + (2 - 2i)) = x^3 - 5x^2 + 12x - 8. \quad (8)$$

Left and right horizontal shifts just move a graph left and right. So in theory, if as a result of a horizontal shift we eliminate the x^2 term, we should be able to shift roots found back. Consider

$$p_3(x + h) = (x + h)^3 + b(x + h)^2 + c(x + h) + d$$

$$= x^3 + 3x^2h + 3xh^2 + h^3 + b(x^2 + 2hx + h^2) + cx + ch + d$$

$$3hx^2 + bx^2 + \text{ other terms} = (3h + b)x^2 + \text{ other terms}.$$

Setting $3h + b = 0$, we get $h = -b/3$ eliminates the x^2 term. We can crunch the new $d_3(x)$ polynomial using Maple, Figure 6.

$f := x \rightarrow (x - 1) \cdot (x - (2 - 2 \cdot I)) \cdot (x - (2 + 2 \cdot I));$
$\qquad f := x \mapsto (x - 1) \cdot (x - 2 + 2 \cdot I) \cdot (x - 2 - 2 \cdot I) \qquad (1)$

$expand(f(x));$
$\qquad\qquad x^3 - 5x^2 + 12x - 8 \qquad\qquad (2)$

$expand\left(f\left(x + \frac{5}{3}\right)\right);$
$\qquad\qquad x^3 + \frac{74}{27} + \frac{11}{3}x \qquad\qquad (3)$

Figure 6: Maple can perform the transformation that eliminates the x^2 term.

```
NORMAL FLOAT AUTO α+bί RADIAN CL    ◈▯
N=?-74/27
                     -0.6666666667
A=?1
B=?0
C=?11/3
D=?74/27
                     -0.6666666667
              0.3333333333-2ί
              0.3333333333+2ί
```

Figure 7: Option 3 gives the single root $-2/3$, and Option 4 gives all three roots.

We determine one root of $x^3 + 11/3x + 74/27$ is $x = -2/3$ using Option 3 with $M = 11/3$ and $N = -74/27$, Figure 7 at the top. Using synthetic division (or Maple) a division yields the quadratic $x^2 - 2/3x + 37/9$ and this has roots $1/3 \pm 2i$ via the quadratic formula (Maple, again). But we can get all three (in theory) using Option 1 and entering the coefficients $A = 1$, $B = 0$, $C = 11/3$, $D = 74/27$ and then Option 4. Viola: Figure 7 shows all three roots. When $5/3$ is added to these roots we have a confirmation of (8); we get the correct roots.

Negative discriminants

What could possibly go wrong? There is some fine print to these cubic equations. Chrystal, page 549 touches on negative discriminants of the quadratic we derive from a given cubic renders the omega math we've been using *useless for purposes of numerical calculation.*

Consider the cubic $x^3 - 12x + 10$. Its graph is given in Figure 8. It shows three real roots, but our program shows three complex roots, Figure 9.
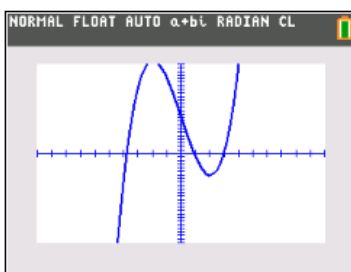

Figure 8: Graph for the cubic $x^3 - 12x + 10$ indicating it has three real roots.
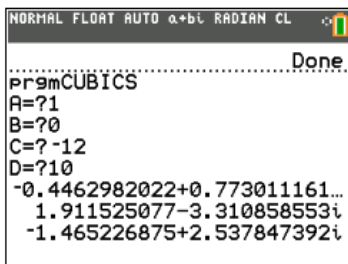

Figure 9: Option 4 gives three complex roots – not even close!

There are three cube roots of a complex number, so the problem resides in the calculator's not returning all three of these roots or figuring out which one of the three it does return, given it returns one. We need to determine the principal root and then use this with the omega math. The TI-83 family of calculators does support polar coordinates and exponential forms. Let's see if we can use these to correctly get the three real roots of $x^3 - 12x + 10$.

We can use the 2nd Calc feature to determine using the graph of $x^3 - 12x + 10$ that the roots are $r_1 = -3.82305$, $r_2 = 0.8925964$, and $r_3 = 2.9304537$.
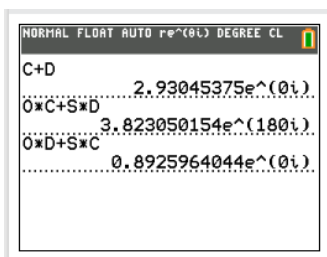
Figure 10: In exponential and degree mode, the TI-84 CE correctly crunches the roots of $x^3 - 12x + 10$.

For $x^3 - 12x + 10$ we have the quadratic $x^2 + 10x + 64$ and its roots are

$$\frac{-10 \pm \sqrt{100 - 4(1)(64)}}{2} = \frac{-10 \pm i\sqrt{156}}{2} = -5 + i\sqrt{39} \text{ and } -5 - \sqrt{39}.$$

The plan: put the calculator in exponential, degree mode, set variables A and B to the above roots. Set C and D to the cube roots of these. Then see what the calculator returns for $C + D$, $O * C + S * D$, and $O * D + S * C$. Figure 10 shows the result of these omega crunches; correctamento!

Conclusion

Figure 11: The variables for the quadratic formula are assigned, cube roots of these are crunched and assigned, and the omega arithmetic is displayed.

The insert needed for our program is given in Figure 11. Does it work for the three cubics we've investigated. Yes! And you can by changing modes see the rectangular coordinates, the $a + ib$ versions.

References

[1] Blitzer, R. (2010). *Algebra and Trigonometry*, 3rd ed., Pearson: Boston.

[2] Chrystal, G. (1999). *Textbook of Algebra*, Vol. 1, 7th ed., AMS: Providence, RI.

[3] Herstein, I.N. (1976). *Topics in Algebra*, 2nd ed., Wiley: New York.

[4] Tignol, J.P. (2001). *Galois' Theory of Algebraic Equations*, World Scientific: London.