An Empirical Convergence Phenomenon related to Riemann Hypothesis

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Abstract

We stumble upon an empirical convergence phenomenon that is maybe related to Berry-Keating conjecture and the proof of Riemann hypothesis.

We assume that we have some sequence $0 < x_1 < x_2 < x_3 < \cdots$ fixed, and use it to carry out the following construction: We define a multiplication operator

$$M_x = \begin{pmatrix} x_1 & 0 & 0 & \cdots \\ 0 & x_2 & 0 & \cdots \\ 0 & 0 & x_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and a derivative operator

$$D_x = \begin{pmatrix} \frac{1}{x_1 - x_2} & \frac{1}{x_2 - x_1} & 0 & \cdots \\ 0 & \frac{1}{x_2 - x_3} & \frac{1}{x_3 - x_2} & \cdots \\ 0 & 0 & \frac{1}{x_3 - x_4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and then use these to define a Hermitian operator H by the formula

$$H = \frac{1}{2}\big(M_x(-iD_x) + (-iD_x)^\dagger M_x^\dagger\big).$$

This H turns out to be

$$H = -\frac{i}{2}\begin{pmatrix} 0 & \frac{x_1}{x_2 - x_1} & 0 & 0 & \cdots \\ \frac{x_1}{x_1 - x_2} & 0 & \frac{x_2}{x_3 - x_2} & 0 & \cdots \\ 0 & \frac{x_2}{x_2 - x_3} & 0 & \frac{x_3}{x_4 - x_3} & \cdots \\ 0 & 0 & \frac{x_3}{x_3 - x_4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We write down an eigenvalue equation

$$H\begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ \vdots \end{pmatrix} = z\begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ \vdots \end{pmatrix},$$

where $z \in \mathbb{C}$ is some complex variable. If the function sequence $f_1, f_2, f_3, \ldots$ satisfies the formulas

$$\begin{aligned} f_1(z) &= z \\ f_2(z) &= 2i\frac{x_2 - x_1}{x_1}z^2 \\ f_{n+1}(z) &= \frac{x_{n+1} - x_n}{x_n}\Big(2izf_n(z) + \frac{x_{n-1}}{x_n - x_{n-1}}f_{n-1}(z)\Big) \quad \text{for } n \in \{2, 3, 4, \ldots\}, \end{aligned}$$

the eigenvalue equation is satisfied too.

In a sense all complex numbers $z \in \mathbb{C}$ are eigenvalues of H, since the recursion formula obviously always generates some vector $(f_1(z), f_2(z), f_3(z), \ldots)$ for any z. Let's decide that we are only interested in vectors that have the property $\lim_{n\to\infty} f_n(z) = 0$. Then it is no longer obvious which complex numbers z qualify as the eigenvalues of H. We are interested in the question that how does the choice of sequence $x_1 < x_2 < x_3 < \cdots$ affect the possible eigenvalues of H.

Next step is that we write a computer program that works so that it takes some sequence $x_1 < x_2 < x_3 < \cdots$ as input, and as ouput the program shows the zeros of the functions $f_1, f_2, f_3, \ldots$.

Since the functions $f_1, f_2, f_3, \ldots$ are polynomials, they are also analytic, and it will make sense for our program to render the arguments $\arg(f_n(z))$. We render them so that red color means that the argument is close to 0, green means that argument is close to $\frac{2\pi}{3}$, and blue means that the argument is close to $-\frac{2\pi}{3}$. The zeros will be in locations where the three colors meet. If $f_n(z) = 0$ with some $n \in \{2, 3, 4, \ldots\}$, then z is an eigenvalue of a $(n-1) \times (n-1)$ Hermitian matrix, and is therefore real. This means that it makes sense to write our program so that it only shows some area close to the real axis.

Figure 1 shows what happens when we substitute some arbitrary choice to the sequence $x_1 < x_2 < x_3 < \cdots$. There are a lot of zeros, but they don't seem to converge to any values. Figure 2 shows what happens when we substitute prime numbers to the sequence $x_1 < x_2 < x_3 < \cdots$. This time the zeros appear to converge to some values, and it looks like that there exist numbers $z_1, z_2, z_3, \ldots$ that have the property $\lim_{n\to\infty} f_n(z_k) = 0$. It is not obvious why using prime numbers like this should make the zeros converge like this, so this is a very interesting empirical observation. Whether the numbers $z_1, z_2, z_3, \ldots$ really exist or not is now a conjecture. We know that Riemann zeta function is related to prime numbers, and according to Berry-Keating conjecture the operator $\frac{1}{2}(M_x(-iD_x) + (-iD_x)^\dagger M_x^\dagger)$ is related to Riemann hypothesis, so it looks like that we stumbled upon an empirical convergence phenomenon that is maybe related to the proof of Riemann hypothesis.

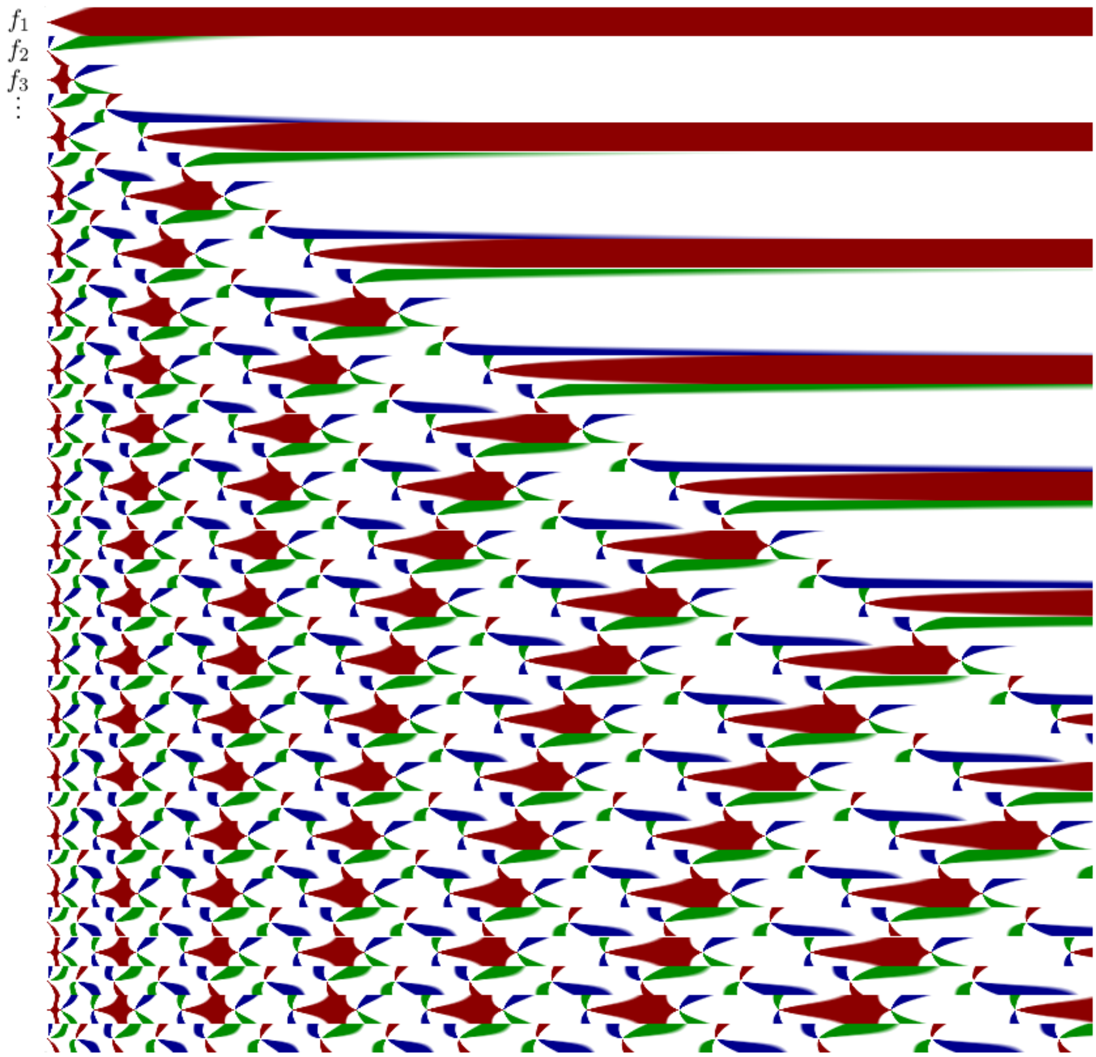

Figure 1: Arguments of the functions $f_1, f_2, f_3, \ldots$ near the real axis, when we use a sequence $(x_1, x_2, x_3, \ldots) = (1, 2, 3, 4, 5, \ldots)$. Zeros of f_{n+1} are usually in different positions than the zeros of f_n, so the zeros do not seem to converge anywhere. Other arbitrary choices for $(x_1, x_2, x_3, \ldots)$ usually produce other similar arbitrary looking patterns of zeros.

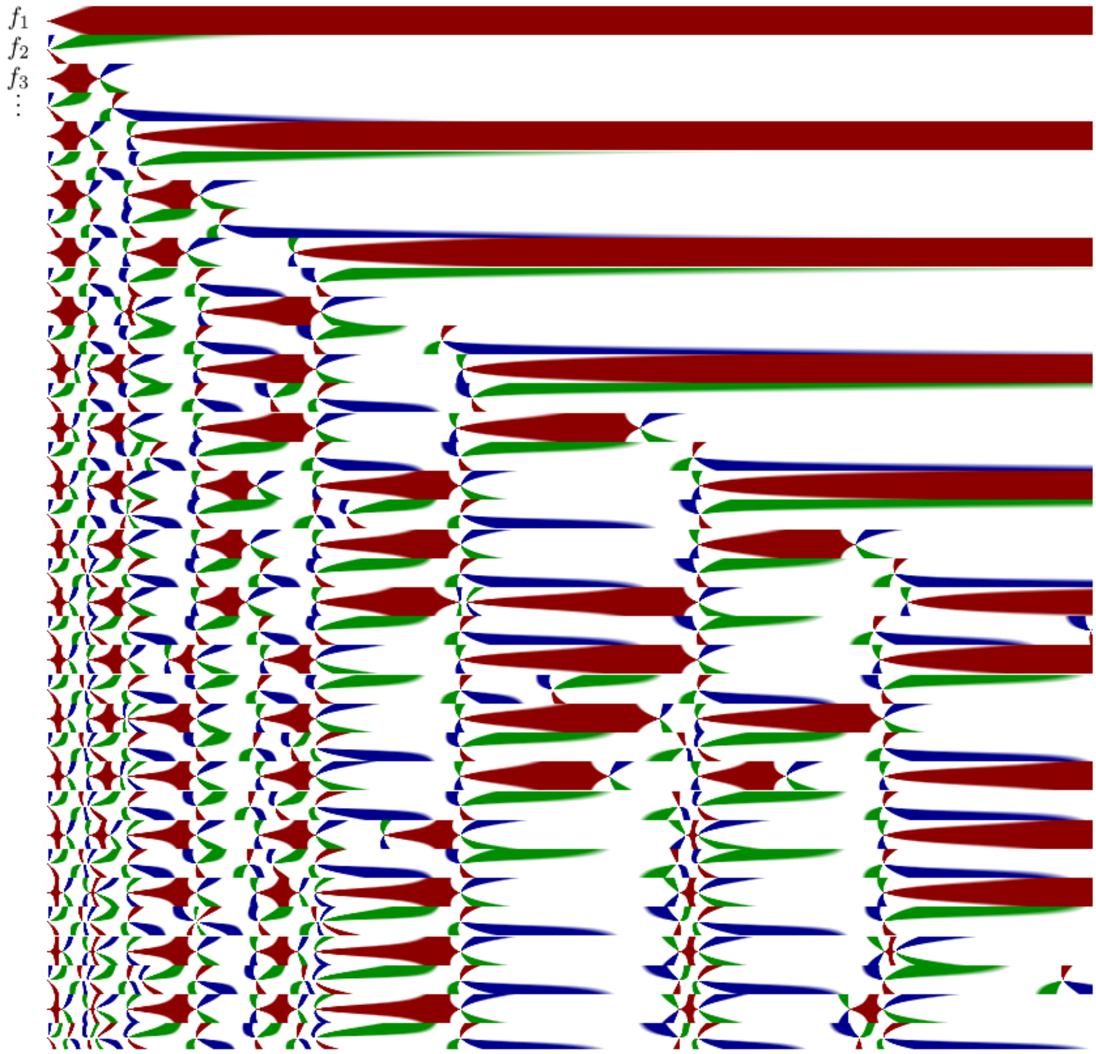

Figure 2: Arguments of the functions $f_1, f_2, f_3, \ldots$ near the real axis, when we use the sequence $(x_1, x_2, x_3, \ldots) = (2, 3, 5, 7, 11, 13, 17, \ldots)$. In many places the zeros of f_{n+1} are in almost the same positions as the zeros of f_n, so the zeros appear to converge to some values.