A compact Solution of a Cubic Equation

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In this article, a simple solution of cubic equation is presented by the use of a new substitution $y = (\sqrt[3]{\alpha} + s/\sqrt[3]{\alpha})$, which can replace a complicated solution presented by G. Cardano, and François Viète's Vieta substitution. This paper also shows that one of the existing solution of the trigonometric function is to be changed to $-\cos(\phi - \frac{\pi}{3})$ instead of $\cos(\phi - \frac{4\pi}{3})$ due to the range limit of the inverse trigonometric function.

A. Derivation of a compact solution of a cubic equation

The solution of cubic equations is well known since an Italian mathematician Gerolamo Cardano had established. In order to replace G. Cardano's solution, I hereby settle a simple and compact substitution for an easier solution. A general form of cubic equations is written as,

$$ax^3 + bx^2 + cx + d = 0, \;\; a \neq 0. \tag{1}$$

Then, a simplified equation divided both sides by the coefficient a is given as,

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0. \tag{2}$$

Substituting $x = y - \frac{b}{3a}$ by using a Tschirnhaus transformation, we get [1]

$$y^3 + sy + t = 0. \tag{3}$$

Here the coefficients s and t represent respectively

$$\begin{aligned} s &= \frac{c}{a} - \frac{b^2}{3a^2}, \\ t &= \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}. \end{aligned} \tag{4}$$

A solution of the cubic equation [1] obtained by Cardano is quite complicated because it needs several steps to get solutions. For a simplified solution, I define a new substitution analogous to Vieta's substitution [2][4], which is basically similar to Vieta's, as is given in the form,

$$y = \sqrt[3]{\alpha} - \frac{s}{3\sqrt[3]{\alpha}}. \tag{5}$$

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[1] Cardano's basic form is

$$y^3 + 3py + q = 0.$$

Cardano's substitution is

$$u^3 + v^3 = -q, \; uv = -p,$$

where $y = (u + v)$ is a root of the given cubic equation.

[2] Vieta's solution of a cubic is read as follows

$$t^3 + pt + q = 0.$$

And the substitution is

$$t = w - \frac{p}{3w},$$

which provides $w^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$.

It is to be noted that this substitution provides only one solution of a cubic equation, while Cardano's and Vieta's provide three solutions. By using this substitution, we get from the equation (3),

$$\begin{aligned}&\left(\sqrt[3]{\alpha}-\frac{s}{3\sqrt[3]{\alpha}}\right)^3+s\left(\sqrt[3]{\alpha}-\frac{s}{3\sqrt[3]{\alpha}}\right)+t\\&=\alpha+t-\frac{s^3}{27\alpha}\\&=0.\end{aligned}\tag{6}$$

Multiplying both sides by α, we get a quadratic equation,

$$\alpha^2+t\alpha-\frac{s^3}{27}=0.\tag{7}$$

And, then we get a pair of solutions

$$\alpha=\frac{-t}{2}\pm\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}.\tag{8}$$

Though the resolvent quadratic equation (7) provides two roots as per the above, we can identify that they produce only one radical of the reduced cubic form (3). Substituting each α of (8) respectively, then we find the same result as follows

$$\begin{aligned}y&=\sqrt[3]{\frac{-t}{2}+\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}-\left(\frac{s}{3\sqrt[3]{\frac{-t}{2}+\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}}\right)\\&=\sqrt[3]{\frac{-t}{2}-\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}-\left(\frac{s}{3\sqrt[3]{\frac{-t}{2}-\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}}\right)\\&=\sqrt[3]{\frac{-t}{2}+\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}+\sqrt[3]{\frac{-t}{2}-\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}.\end{aligned}\tag{9}$$

With this result, we can get the two remaining solutions by letting the above solution as $y=y_1$ and solving the factorized quadratic equation of the right hand side

$$\begin{aligned}y^3+sy+t&=(y-y_1)(y^2+y_1y+{y_1}^2+s)\\&=0.\end{aligned}\tag{10}$$

From this, we get

$$\begin{aligned}y_2,y_3&=\frac{-y_1}{2}\pm\sqrt{\frac{-3{y_1}^2}{4}-s}\\&=-\frac{1}{2}\left(\sqrt[3]{\frac{-t}{2}+\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}+\sqrt[3]{\frac{-t}{2}-\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}\right)\\&\quad\pm\frac{i\sqrt{3}}{2}\left(\sqrt[3]{\frac{-t}{2}+\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}-\sqrt[3]{\frac{-t}{2}-\sqrt{\frac{t^2}{4}+\frac{s^3}{27}}}\right).\end{aligned}\tag{11}$$

The discriminant of the cubic is given as

$$D=\frac{t^2}{4}+\frac{s^3}{27}.\tag{12}$$

These three roots, (9) and (11), are the solutions of a cubic equation in terms of two unknown coefficients, s and t. In case the discriminant $D>0$ of a cubic equation, the roots of (9) and (11) can be used as they are, because the cubic equation has one real roots and two complex conjugate. However, in case of $D < 0$, the value in the square root of (9) is changed to an imaginary unit, in which case, it is convenient to use trigonometric functions,

$$D = \begin{cases} \frac{t^2}{4} + \frac{s^3}{27} > 0, & \text{a real root and a pair of complex conjugate} \\ \frac{t^2}{4} + \frac{s^3}{27} = 0, & \text{three real roots at least two of them equal} \\ \frac{t^2}{4} + \frac{s^3}{27} < 0, & \text{three real roots unequal to each other.} \end{cases}$$

In case $D < 0$, we may eliminate the imaginary unit by using the trigonometric functions, so we can define an intermediary coefficient θ as follows,

$$\begin{aligned} \cos 3\theta &= \frac{-3\sqrt{3}t}{2\sqrt{-s^3}}, \\ \sin 3\theta &= \frac{3\sqrt{3}}{\sqrt{-s^3}}\sqrt{-\frac{t^2}{4} - \frac{s^3}{27}}, \end{aligned} \tag{13}$$

where θ is given as

$$\theta = \frac{1}{3} \arccos\left(\frac{-3\sqrt{3}t}{2\sqrt{-s^3}}\right). \tag{14}$$

By substituting the above into the equation (9), we get the following result by using the de Moivre's formula [5] [3].

$$\begin{aligned} y_1 &= \sqrt[3]{\frac{-t}{2} + \sqrt{\frac{t^2}{4} + \frac{s^3}{27}}} + \sqrt[3]{\frac{-t}{2} - \sqrt{\frac{t^2}{4} + \frac{s^3}{27}}} \\ &= \frac{\sqrt{-s}}{\sqrt{3}}\left(\sqrt[3]{\frac{-3\sqrt{3}t}{2\sqrt{-s^3}} + i\frac{3\sqrt{3}}{\sqrt{-s^3}}\sqrt{-\frac{t^2}{4} - \frac{s^3}{27}}} + \sqrt[3]{\frac{-3\sqrt{3}t}{2\sqrt{-s^3}} - i\frac{3\sqrt{3}}{\sqrt{-s^3}}\sqrt{-\frac{t^2}{4} - \frac{s^3}{27}}}\right) \\ &= \frac{\sqrt{-s}}{\sqrt{3}}\left(\sqrt[3]{\cos 3\theta + i\sin 3\theta} + \sqrt[3]{\cos 3\theta - i\sin 3\theta}\right) \\ &= \frac{\sqrt{-s}}{\sqrt{3}}\left(\sqrt[3]{(\cos\theta + i\sin\theta)^3} + \sqrt[3]{(\cos\theta - i\sin\theta)^3}\right) \\ &= \frac{2\sqrt{-s}}{\sqrt{3}}\cos\theta, \end{aligned} \tag{15}$$

where i represents the imaginary unit.
And the remaining two roots are given from the equation (11)

$$\begin{aligned} y_2, y_3 &= -\frac{1}{2}\left(\sqrt[3]{\frac{-t}{2} + \sqrt{\frac{t^2}{4} + \frac{s^3}{27}}} + \sqrt[3]{\frac{-t}{2} - \sqrt{\frac{t^2}{4} + \frac{s^3}{27}}}\right) \pm \frac{i\sqrt{3}}{2}\left(\sqrt[3]{\frac{-t}{2} + \sqrt{\frac{t^2}{4} + \frac{s^3}{27}}} - \sqrt[3]{\frac{-t}{2} - \sqrt{\frac{t^2}{4} + \frac{s^3}{27}}}\right) \\ &= \frac{-\sqrt{-s}}{\sqrt{3}}\left(\frac{1}{2}\sqrt[3]{\frac{-3\sqrt{3}t}{2\sqrt{-s^3}} + i\frac{3\sqrt{3}}{\sqrt{-s^3}}\sqrt{-\frac{t^2}{4} - \frac{s^3}{27}}} + \frac{1}{2}\sqrt[3]{\frac{-3\sqrt{3}t}{2\sqrt{-s^3}} - i\frac{3\sqrt{3}}{\sqrt{-s^3}}\sqrt{-\frac{t^2}{4} - \frac{s^3}{27}}}\right) \\ &\pm \frac{\sqrt{-s}}{\sqrt{3}}\left(\frac{i\sqrt{3}}{2}\sqrt[3]{\frac{-3\sqrt{3}t}{2\sqrt{-s^3}} + i\frac{3\sqrt{3}}{\sqrt{-s^3}}\sqrt{-\frac{t^2}{4} - \frac{s^3}{27}}} - \frac{i\sqrt{3}}{2}\sqrt[3]{\frac{-3\sqrt{3}t}{2\sqrt{-s^3}} - i\frac{3\sqrt{3}}{\sqrt{-s^3}}\sqrt{-\frac{t^2}{4} - \frac{s^3}{27}}}\right) \end{aligned} \tag{16}$$

[3] De Moivre's formula proves that $\cos nx + i\sin nx = (\cos x + i\sin x)^n$.

$$= \frac{-\sqrt{-s}}{\sqrt{3}}\left(\frac{1}{2}\left(\sqrt[3]{\cos 3\theta + i\sin 3\theta} + \sqrt[3]{\cos 3\theta - i\sin 3\theta}\right) \pm \frac{i\sqrt{3}}{2}\left(\sqrt[3]{\cos 3\theta + i\sin 3\theta} - \sqrt[3]{\cos 3\theta - i\sin 3\theta}\right)\right)$$
$$= \frac{-2\sqrt{-s}}{\sqrt{3}}\cos\left(\theta \pm \frac{\pi}{3}\right).$$

with θ[2] [4] of (14).
As the results, in case that the discriminant of the cubic equation (3) is $D < 0$, the three real roots of (15) and (16) can be expressed as an inverse function of trigonometry as follows,

$$\begin{aligned} y_1 &= \frac{2\sqrt{-s}}{\sqrt{3}}\cos\theta \\ &= \frac{2\sqrt{-s}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{-3\sqrt{3}t}{2\sqrt{-s^3}}\right)\right), \\ y_{2,3} &= \frac{-2\sqrt{-s}}{\sqrt{3}}\cos\left(\theta \pm \frac{\pi}{3}\right) \\ &= \frac{-2\sqrt{-s}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{-3\sqrt{3}t}{2\sqrt{-s^3}}\right) \pm \frac{\pi}{3}\right). \end{aligned}$$

B. A generalized Solution of a Cubic Equation

A general form of a cubic equation

$$ax^3 + bx^2 + cx + d = 0, \;\; a \neq 0. \tag{17}$$

Dividing by the leading coefficient a, we get a monic cubic equation

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0. \tag{18}$$

A depressed form of the above by substituting with $x = y - \frac{b}{3a}$, we get

$$y^3 + sy + t = 0, \tag{19}$$

where

$$\begin{aligned} s &= \frac{c}{a} - \frac{b^2}{3a^2}, \\ t &= \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}. \end{aligned} \tag{20}$$

From the above (19), we get a solution of a cubic,

$$y_1 = \sqrt[3]{\frac{-t}{2} + \sqrt{D_3}} + \sqrt[3]{\frac{-t}{2} - \sqrt{D_3}}, \tag{21}$$

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[4] Refer to: The three real roots of a depressed form ($t^3 + pt + q = 0$) can be expressed as

$$t_k = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - k\frac{2\pi}{3}\right) \quad \text{for} \quad k = 0, 1, 2\,.$$

https://en.wikipedia.org/wiki/Cubic_equation where D_3 is the discriminant of a cubic equation given as

$$
\begin{aligned}
D_3 &= \frac{t^2}{4} + \frac{s^3}{27} \\
&= -\frac{b^2c^2}{108a^4} + \frac{b^3 d}{27a^4} - \frac{bcd}{6a^3} + \frac{c^3}{27a^3} + \frac{d^2}{4a^2} \\
&= -\frac{1}{108a^4}\left(18abcd - 4ac^3 - 27a^2d^2 + b^2c^2 - 4b^3 d\right),
\end{aligned}
\tag{22}
$$

and the remaining two roots are

$$
y_{2,3} = -\frac{1}{2}\left(\sqrt[3]{\frac{-t}{2} + \sqrt{D_3}} + \sqrt[3]{\frac{-t}{2} - \sqrt{D_3}}\right) \pm \frac{i\sqrt{3}}{2}\left(\sqrt[3]{\frac{-t}{2} + \sqrt{D_3}} - \sqrt[3]{\frac{-t}{2} - \sqrt{D_3}}\right). \tag{23}
$$

As the results, a solution of the cubic equation (18) is given as,

$$
x_1 = -\frac{b}{3a} + \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{D_3}} + \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} + \frac{d}{2a} - \sqrt{D_3}}, \tag{24}
$$

and the remaining two solutions,

$$
\begin{aligned}
x_{2,3} = & -\frac{b}{3a} - \frac{1}{2}\left(\sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{D_3}} + \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} + \frac{d}{2a} - \sqrt{D_3}}\right) \\
& \pm \frac{i\sqrt{3}}{2}\left(\sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{D_3}} - \sqrt[3]{-\frac{b^3}{27a^3} + \frac{bc}{6a^2} + \frac{d}{2a} - \sqrt{D_3}}\right).
\end{aligned}
\tag{25}
$$

with D_3 of (22).

In case $D_3 > 0$, the cubic equation has one real root of (24) and two complex conjugate of (25) .
In case $D_3 = 0$, it has three real roots, two of which are equal to each other.
If $D_3 < 0$, the cubic has three different real roots.

[1] J.H. Jeong *et al.*, "Concise Mathematics Dictionary", Changwonsa, Seoul, Korea, 1993
[2] https://en.wikipedia.org/wiki/Cubic_equation
[3] https://en.wikipedia.org/wiki/Tschirnhaus_transformation
[4] http://www.itu.dk/bibliotek/encyclopedia/math/c/c818.htm
[5] http://en.wikipedia.org/wiki/De_Moivre's_formula