Existence Conditions, Asymptotic Behavior And Properties Of A Class Of "*Rational-Equivalence*" Nonlinear Systems.

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Abstract.

Liptai, Németh, et. al. (2020) supposedly proved that in the diophantine equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$ in positive integers and where a≤b and c≤d, the only solution to the title equation is (a,b,c,d)=(1,2,1,1). This article analyzes the Complexity of, and introduces properties of the equations $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$ and $g^u=f^v$, new "*Existence Conditions*", new theories of "*Rational Equivalence*", and a new theorem pertaining to the equation $g^u=f^v$. The class of equations of the type $[(X^a-1)(X^b-1)=(Y^c-1)(Y^d-1)]$ (the "Rational-Equivalence Equation") includes the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$. This article also introduces simple Java codes for finding solutions to this class of equations for positive-integers up to$10^{2457600000}$ (and even greater positive-integers depending on available computing power).

Keywords: Nonlinearity; *Existence-Conditions*; *Rational-Equivalence Conditions*; Computational Complexity; Dynamical Systems; Mathematical Cryptography; Number Theory; Ill-posed Problems.

1. Introduction.

The equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$ is among a class of diophantine equations that have applications in many fields including Computer Science, Applied Math, Operations Research, Physics and Econophysics.

The nonlinear equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ represents a class of Nonlinear Systems of the type $[(X^a-1)(X^b-1)=(Y^c-1)(Y^d-1)]$ that have the following characteristics:

i) $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ is an ill-posed problem because it and both sides of the equation can vary and behave differently over the interval $0<a,b,c,d<+\infty$.

ii) The system is *covariant* since any change in any of *a*, *b*, *c* or *d* affects the other variables (regardless of whether or not the variables are integers or non-integer real numbers).

iii) The entire system is nonlinear, and each side of the equation is nonlinear.

iv) In the realm of integers, the system preserves the relationship a,b ≥ c,d.

The main problems/deficiencies in the Liptai, Németh, et. al. (2020) analysis are as follows:

i) Liptai, Németh, et. al. (2020) uses so many un-verified "assumptions" that its "proofs" are really just conjectures.

ii) Liptai, Németh, et. al. (2020) didn't prove that (a+b)>(c+d) or that a,b ≥ c,d, or that (a+b)/(c+d) ≥ 1.5; all of which are critical elements of the analysis.

iii) Liptai, Németh, et. al. (2020) didn't sufficiently discuss the effect(s) of the "structural" similarities of both sides of the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$.

iv) Liptai, Németh, et. al. (2020) didn't derive valid "*Existence*" conditions for the system (the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$ is an ill-posed problem).

v) On *Linear Recurrences*, see Kuhapatanakul & Laohakosol (2019), Schlickewei & Schmidt (1993) and Morgari, Steila & Elia (2000); but the formal definitions of *Linear Recurrences* and "*Recurrence Relations*" are somewhat different from the definitions used in Liptai, Németh, et. al. (2020).

2. Existing Literature.

Goedhart & Grundman (2015) analyzed the equation $(a^2cx^k-1)(b^2cy^k-1)=(abcz^k-1)^2$. Zhang (2014) studied the Diophantine equation $(ax^k-1)(by^k-1)=(abz^k-1)$. Bennett (2007) analyzed the Diophantine equation $(x^k-1)(y^k-1)=(z^k-1)^t$. Bugeaud (2004) analyzed the Diophantine equation $(x^k-1)(y^k-1)=(z^k-1)$. Stroeker (1981) studied the diophantine equation $(2y^2-3)^2=x^2(3x^2-2)$. On Homomorphisms, see: Wang & Chin (2012). Kreso & Tichy (2018) and Pakovich (2011) analyzed Diophantine Equations of the type f(x)=g(y).

Osgood (1975) analyzed bounds on the "diophantine approximation" of algebraic functions over fields, and applications to differential equations. Furthermore, the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$ can be viewed as a differential/difference equation where: $\{[(3^a-1)(3^b-1)]/[(5^c-1)(5^d-1)]\}$ -1= 0; and if $\partial y=[(3^a-1)(3^b-1)]$, and $\partial x=[(5^c-1)(5^d-1)]$, then $(\partial y/\partial x)$ -1 = 0.

In Mathematical Physics, Hesamiarshad (2021) developed *Equivalence Conditions* and *Invariants* for a class of equations; and Zeng, Deng & Wang (2021) developed *Global Existence conditions* for a class of equations. On solutions of diophantine equations in Mathematical Physics, Mathematical Chemistry and Computer Math, see: Matveev (2000), Ibarra & Dang (2006), Ren & Yang (2012), Bremner (1986), and Bitim & Keskin (2013). On solutions to Diophantine Equations in *Analysis*, see Zaidenberg (1988), Bitim & Keskin (2013), and Zadeh (2019).

Chu (2008) and Lu & Wu (2016) studied dynamical systems pertaining to Diophantine equations; and each of the *equations* $(X^a-1)(X^b-1)=a$, $(Y^c-1)(Y^d-1)=b$, and $(X^a-1)(X^b-1)=(Y^c-1)(Y^d-1)$ can approximate Dynamical Systems. Luca, Moree & Weger (2011) discussed *Group Theory* as it relates to Diophantine Equations. Jones, Sato, et. al. (1976) and Matijasevič (1981) noted that primes can also be represented as Diophantine equations or as polynomials (ie. each of the equations $[(X^a-1)(X^b-1)]+[(Y^c-1)(Y^d-1)]$, and $[(X^a-1)(X^b-1)]-[(Y^c-1)(Y^d-1)]$ can represent a prime). On uses of *Diophantine Equations* and *Mersenne Composite Numbers* in Cryptography, see: Ding, Kudo, et. al. (2018), Okumura (2015), Nemron (2008) and Ogura (2012) (ie. each of the equations $(X^a-1)(X^b-1)=a$; $(Y^c-1)(Y^d-1)=b$; and $[(X^a-1)(X^b-1)]-[(Y^c-1)(Y^d-1)]=c$; and $(X^a-1)(X^b-1)= (Y^c-1)(Y^d-1)$ can be used in cryptoanalysis and in the creation of public-keys).

3. The Theorems.

Theorem-1: For Any Two Exponentials $g^u = f^v$ In Real-Numbers, Regardless Of The Numerical Magnitude Of Their Exponents, The Larger The Numerical Difference Between Their "Bases" (eg. $-\infty<g, f<+\infty$), The Smaller The Probability That There Can Be More Than One Combination Of *u* And *v* That Makes $g^u = f^v$ Valid.

Proof:

This theorem is henceforth referred to as the *Exponential Equivalence Theory*.

If $g^u = f^v$, then:

i) As $|g-f| \to +\infty$, then $v \oplus u \to +\infty$;

ii) As $|g-f| \to +\infty$; then $|v-u| \to +\infty$;

iii) As $|g-f| \to +\infty$; then $[|+\infty-v| \to 0] \oplus [|+\infty-u| \to 0]$.

Thus as $|g-f|$ increases in magnitude, there are increasingly fewer "qualifying" or "feasible" combinations of integers *u* and *v* in the intervals $(v,+\infty)$ and or $(u,+\infty)$, and the probability that there can be more than one "feasible" combination of *u* and *v* decreases.

The equation $g^u = f^v$, is relevant because of recurrences of 3^x and 5^y, in the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$. ■

Theorem-2: For $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$ In Positive Integers, *Horizontal Equivalence* And *Vertical Equivalence* Can Exist Where Terms On Both Sides Of An Equation Have Similar Mathematical "Structures" And Bases.

Proof:

Assume that as a condition for $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$, it's possible that:

2.1) $(3^a-1)=(5^c-1)$, and $(3^b-1)=(5^d-1)$; or

2.2) $(3^a-1)=(5^d-1)$, and $(3^b-1)=(5^c-1)$.

The foregoing are some of the possible combinations of (3^a-1), (5^c-1), (3^b-1) and (5^d-1).

If $(3^a-1)=(5^c-1)$, and $(3^b-1)=(5^d-1)$; then by "*Horizontal Equivalence*":
2.3) $(3^a-1)=(5^c-1)$, and $3^a=5^c$
2.4) $(3^b-1)=(5^d-1)$, and $3^b=5^d$;

That is because (3^a-1) and (5^c-1) have similar or the same mathematical "structure" – namely, an exponential (whose base and exponent are both positive integers) from which one is subtracted. Similarly, (3^b-1) and (5^d-1), have similar or the same "structure" which is an exponential (whose base and exponent are both positive integers) from which one is subtracted. Also $(3^a-1)(3^b-1)$ and $(5^c-1)(5^d-1)$ have similar or the same mathematical "structure" – namely, the multiplicative product of exponentials (whose base and exponent are positive integers) from which one is subtracted. However, in the Equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$, $(3^a-1)(3^b-1)$ and $(5^c-1)(5^d-1)$ can behave differently over the interval $0<a,b,c,d<+\infty$ because of the differences in the magnitude of the bases and exponents.

Note that Eq.-2.3 and Eq.-2.4 apply only to a sub-set of solutions for the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ in positive integers $a\leq b$, and $c\leq d$.

To confirm "*Horizontal Equivalence*":

$(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$, and thus $[(3^a-1)/(5^c-1)] * [(3^b-1)/(5^d-1)] = 1$; and for a sub-set of solutions of the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$:

$[(3^a-1)/(5^c-1)] =1$; and $[(3^b-1)]/(5^d-1)] = 1$; and therefore:

$3^a= 5^c$; and $3^b=5^d$;

It follows that by "*Vertical Equivalence*" and in order for Equations 2.3 & 2.4 to be valid, then for a sub-set of solutions of the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$:

$$a=b;\ \text{and}\ c=d; \tag{2.5}$$

That is because (for a sub-set of solutions to the Equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$) the equations $(3^a-1)=(5^c-1)$, and $(3^b-1)=(5^d-1)$, have the same mathematical "structure", and are part of, or were derived from the same equation which is $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$.

To confirm "*Vertical Equivalence*":

As above, and for a sub-set of solutions of the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$, $[(3^a-1)/(5^c-1)] * [(3^b-1)/(5^d-1)] = 1$, and thus:

$[(3^a-1)/(5^c-1)] =1$; and $[(3^b-1)]/(5^d-1)] = 1$; and given that both foregoing equations have similar Bases (and also $3^a= 5^c$ and $3^b=5^d$), therefore:

$a=b$, and $c=d$

■

Theorem-3: Given The Differences In The Magnitudes Of The Bases Of Exponents On Both Sides Of The Equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ (ie. 3 versus 5), For The Equation To Be Valid, Then: $a,b \geq c,d$.

Proof:

$$\mathrm{Ln}[(3^a-1)(3^b-1)] = \mathrm{Ln}[(5^c-1)(5^d-1)]; \tag{3.1}$$

3.2) $$\mathrm{Ln}(3^a-1)+\mathrm{Ln}(3^b-1) = \mathrm{Ln}(5^c-1) + \mathrm{Ln}(5^d-1); \tag{3.2}$$

As $a,b,c,d \rightarrow +\infty$ (and for relatively medium and large values of a,b,c and d):

$(3^a-1) \rightarrow 3^a$
$(3^b-1) \rightarrow 3^b$
$(5^c-1) \rightarrow 5^c$
$(5^d-1) \rightarrow 5^d$

So that: $$[(3^a)(3^b)] = [(5^c)(5^d)]; \tag{3.3}$$

$Ln[(3^a)(3^b)] = Ln[(5^c)(5^d)]$; (3.4)
$Ln(3^a)+Ln(3^b) = Ln(5^c)+Ln(5^d)$
$aLn(3)+bLn(3) = cLn(5)+dLn(5)$
$(Ln3)(a+b) = (Ln5)(c+d)$

$(a+b)/(c+d) = Ln5/Ln3 = 1.47 \approx 1.5$ (3.5)
$(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ can be expressed as $(X^a-1)(X^b-1) = (Y^c-1)(Y^d-1)$

By *Horizontal Equivalence* above (and for a sub-set of solutions to the Equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$), and since X<Y in positive integers, then a≤b, and c≤d, and:

$X^a=Y^c$, and thus a≥c. (3.6)
$X^b=Y^d$; and thus b≥d. (3.7)

Note that Eq.-3.6 and Eq.-3.7 apply to a sub-set of solutions for the equation $(X^a-1)(X^b-1) = (Y^c-1)(Y^d-1)$ in positive integers a≤b, and c≤d.

By "*Vertical Equivalence*" (and for a sub-set of solutions to the Equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$), then:

a=b; and c=d; for most solutions to the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ (3.8)

Note that Eq.-3.8 applies only to a sub-set of solutions for the equations $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ and $(X^a-1)(X^b-1) = (Y^c-1)(Y^d-1)$ in positive integers a≤b, and c≤d.

Given that X<Y, and that in the equation $(X^a-1)(X^b-1) = (Y^c-1)(Y^d-1)$ in positive integers a≤b, and c≤d, both sides of the equation have the same or similar mathematical "structure", for the equation to be valid, it follows that:

a,b ≥ c,d. (3.9)

Some of the foregoing results, "conditions" and inequalities differ substantially from the Liptai, Németh, et. al. (2020) conjectures and result [(a,b,c,d) =(1,2,1,1)] which implies that there can be more than one solution for the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$. Also, a,b ≥ c,d may imply that there is more than one solution for the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$. ■

$(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$
$[(3^a-1)/(5^c-1)] * [(3^b-1)]/(5^d-1)] = 1$; and thus for a sub-set of solutions of equation$(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$:

$[(3^a-1)/(5^c-1)] =1$; and $[(3^b-1)]/(5^d-1)] = 1$; and:
$3^a= 5^c$; and $3^b=5^d$;

■

Theorem-4: The Liptai, Németh, et. al. (2020) Solution Doesn't Satisfy All The *Existence-1* Conditions.

It's given that: a ≤ b, and c ≤ d. As explained herein and above, and by "*Horizontal Equivalence*":
4.1) $(3^a-1)=(5^c-1)$, and $3^a=5^c$
4.2) And simultaneously: $(3^b-1)=(5^d-1)$, and $3^b=5^d$

As explained above, and by "*Vertical Equivalence*", then:
4.3) a=b; and c=d;
As noted above, and given Theorem-3 above, and the differences between the bases (3 and 5 respectively) of the exponentials, for $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ to be valid:
4.4) a,b ≥ c,d

From *Equation-4.1*: $3^a=5^c$, and given that: a ≤ b, and c ≤ d, then if a,c = 0, then a,b,c,d = 0; and then: $(3^a-1)(3^a-1) = (5^c-1)(5^c-1)$; and thus, the Liptai, Németh, et. al. (2019) conjecture and result [ie. (a,b,c,d)=(1,2,1,1,)] don't apply to (a,b,c,d)=(0,0,0,0).

From Equation-4.1, $3^a=5^c$, and then:
4.5) $Log_3(5^c) = a = Ln(5^c)/Ln(3) = c[Ln(5)/Ln(3)] = a = c1.465 \approx c1.5$
4.6) $Log_5(3^a) = c = Ln(3^a)/Ln(5) = a[Ln(3)/Ln(5)] = c = a0.683 \approx a0.7$

From *Equation-4.1*, $3^a=5^c$, and it follows that the absolute number of possible (both "matching" and "incorrect") combinations of the positive integers 0< *a,c* <+∞ (0< a, b, c, d < +∞) exceeds ten billion and may be as much as infinity. Because the numerical difference between 3 and 5 is not large (on a scale of zero to +∞) then it follows that there is a high probability that there can be more than one combination of positive integers *a* and *c* (0< a, b, c, d < +∞) that satisfy all the following conditions (the "*Existence-1 Conditions*") that make the equation and inequalities $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$, a ≤ b and c ≤ d valid:

4.7) *a* =b= c1.465 (≈c1.5)
4.8) *c* =d ≈ a0.683 ≈ a0.7
4.9) 0< a,c < +∞; and 0< a,b,c,d < +∞ are integers
4.10) a=b; and c=d;
4.11) a,b ≥ c,d;
4.12) a ≤ b, and c ≤ d;
4.13) $(3^a-1)(3^a-1) = (5^c-1)(5^c-1)$; and thus $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$.
4.14) $3^a=5^c$
4.15) $3^b=5^d$

Given the foregoing conditions, *a*, *b*, *c* and *d* can be calculated by iteration and or optimization.

It also follows that there are no upper bounds on *a*, *b*, *c* and *d*. Thus, there is a high probability that there is more than one solution for the Diophantine equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$, where a ≤ b, and c ≤ d are integers, because:

i) the Liptai, Németh, et. al. (2020) result (a,b,c,d)=(1,2,1,1) doesn't satisfy all the "*Existence-1 Conditions*"; and
ii) the absolute number of possible (both "matching" and "incorrect") combinations of the two positive integers 0< *a,c* <+∞ (0< a,b,c,d < +∞) exceeds ten billion and may be as much as infinity. ■

Theorem-5: The Liptai, Németh, et. al. (2020) Conjecture And Result Don't Satisfy All The *Existence-2 Conditions*; And For The Equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ To be Valid In Positive Integers a≤b, and c≤d As Construed, Then:

i) (a+b)/(c+d) ≥1; And
ii) a,b ≥ c,d;
iii) b-a ≥ d-c;

Proof:
a ≤ b, and c ≤ d
As explained herein and above, and by "*Horizontal Equivalence*":
$(3^a-1)(3^a-1) = (5^c-1)(5^c-1)$
If $(3^a-1) = (5^c-1)$, and $(3^b-1)=(5^d-1)$; then
5.1) $(3^a-1) = (5^c-1)$, and $3^a=5^c$
5.2) $(3^b-1) = (5^d-1)$, and $3^b=5^d$;

and by "*Vertical Equivalence*":
5.3) a=b; and c=d;
From **Theorem-3**:
5.4) a,b ≥ c,d

Given "*Vertical Equivalence*" and the differences in the magnitudes of the integers on both sides of the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ (ie. 3 versus 5):

5.6) $(3^a-1)(3^a-1) = (5^c-1)(5^c-1)$

Furthermore and as explained herein and above:
5.7) $(a+b)/(c+d) = Ln5/Ln3 = 1.47\approx 1.5$

If $(3^a-1)=(5^c-1)$, and $(3^b-1)=(5^d-1)$; then
$3^a=5^c$, and $3^b=5^d$; and thus by "vertical equivalence":
5.8) **$b-a \geq d-c$;**
5.9) if $b\geq a$, and $d\geq c$; and a,b,c, and d are positive integers then $b\geq 1$ and $d\geq 1$.

Thus, the conditions for the validity of the equation and inequalities $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$, $a \leq b$, and $c \leq d$ in positive integers (the "*Existence-2 Conditions*") are as follows:

5.10) $(a+b)/(c+d) \approx 1.5$
5.11) $3^a=5^c$; and $3^b=5^d$;
5.12) $(3^a-1)(3^a-1) = (5^c-1)(5^c-1)$
5.13) $0< a,c < +\infty$; and $0< a,b,c,d < +\infty$ are positive integers
5.14) a=b; and c=d are positive integers
5.15) $a,b \geq c,d$;
5.16) $a \leq b$, and $c \leq d$; are positive integers.
5.17) $b-a \geq d-c$;
5.18) if $b\geq a$, and $d\geq c$; and a,b,c, and d are positive integers then $b\geq 1$ and $d\geq 1$.
5.19) That means that for each of *c* and *d* to be positive integers, they must be even numbers (and not odd numbers) which when multiplied by 1.5, produces another positive integer (there is no odd number which when multiplied by 1.5, produces a positive integer). The smallest such even number integer is 2.

It follows that the number of possible (both "matching" and "incorrect") combinations of positive integers $0<a,c<+\infty$ ($0<a, b,c,d<+\infty$) that satisfy the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ can be many. It also follows that there are no upper bounds on *a*, *b*, *c* and *d*. Because the Liptai, Németh, et. al. (2020) result (*a*,*b*,*c*,*d*)=(1,2,1,1) doesn't satisfy all the "*Existence-2 Conditions*", there may be more than one solution for the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ where $a \leq b$, and $c \leq d$ are positive integers. ■

Theorem-6: The Liptai, Németh, et. al. (2020) Conjecture And Result (For The Equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ To be Valid In Positive Integers $a\leq b$, and $c\leq d$) Satisfy All The Following *Existence-3 Conditions*:

i) **$(a+b)/(c+d) \geq 1$; And $(a+b)/(c+d) =1.5$**
ii) **c=d;**
iii) **$a,b \geq c,d$;**
iv) **$b-a \geq d-c$;**
v) **$(3^a3^b)-3^a-3^b = (5^c5^d)-5^c-5^d$;**
vi) **The Lower-Bound is (a,b,c,d)=(1,2,1,1).**

Proof:
6.1) $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$
6.2) $(3^a-1)(3^b-1) = (3^a3^b)-3^a-3^b+1$
6.3) $(5^c-1)(5^d-1) = (5^c5^d)-5^c-5^d+1$
6.4) Thus: $(3^a3^b)-3^a-3^b+1 = (5^c5^d)-5^c-5^d+1$; and $(3^a3^b)-3^a-3^b = (5^c5^d)-5^c-5^d$
6.5) $3^{(a+b)}-3^a-3^b = 5^{(c+d)}-5^c-5^d$, and $3^{(a+b)}- 5^{(c+d)}= 3^a+3^b- 5^c-5^d$

Thus, $(a+b)/(c+d)>1$; and
$a,b \geq c,d$

As noted above, c=d.

Since a,b,c and d are positive integers, in order for the inequalities $b \geq a$ and $c \geq d$ to be valid, then $b \geq 1$ and $d \geq 1$, $a \geq 1$ and $c \geq 1$ and since c=d, then: $c,d \geq 1$.
If c,d = 1, then (a+b) = 3.

By trying a = 1 or 2 and b = 1 or 2, and substituting in the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$, **the lower bound is (a,b,c,d) = (1,2,1,1)**.
Alternatively (and without proving that c=d), if d=1 and c=1, and since (a+b) = (1.5c+1.5d), then (a+b) = 3. By trying a = 1 or 2 and b = 1 or 2, and substituting in the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$, **the lower bound is (a,b,c,d) = (1,2,1,1)**.

The *Existence-3 Conditions* are as follows:

i) $(a+b)/(c+d) \geq 1$; And $(a+b)/(c+d) = 1.5$
ii) c=d;
iii) $a,b \geq c,d$;
iv) $b-a \geq d-c$;
v) $3^{(a+b)}-3^a-3^b = 5^{(c+d)}-5^c-5^d$

But there are potentially many combinations of "qualifying" *a* and *c* (or *a*, *b*, *c* and *d*) such that the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ most probably has more than one solution.

■

Theorem-7: The Equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ In Positive Integers Is Subject To "*Matching Reduction*", But The Liptai, Németh, et. al. (2020) Conjecture And Result [(a,b,c,d)=(1,2,1,1)] Don't Conform To "*Existence-4 Conditions*" Which Are As Follows:

i) $3^a=5^c$
ii) $3^b=5^d$
iii) $3^{(a+b)}=5^{(c+d)}$
iv) $[X^{(a+b)}-Y^{(c+d)}-X^b-X^a+Y^c+Y^d]=0$.

Proof:
7.1) $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$ can be expressed as $(X^a-1)(X^b-1)=(Y^c-1)(Y^d-1)$, which is:
7.2) $[X^aX^b-X^b-X^a+1]=[Y^cY^d-Y^c-Y^d+1]$; and $[X^{(a+b)}-X^b-X^a]=[Y^{(c+d)}-Y^c-Y^d]$
7.3) $[X^{(a+b)}-Y^{(c+d)}-X^b-X^a+Y^c+Y^d]=0$. If "similar" terms are matched in this Eq-7.3 (ie. matched with regards to opposite-signs, LHS/RHS of Eq.-7.1, and the structures of the variables), this equation supports the position that for a sub-set of solutions to the equation $[(3^a-1)(3^b-1) = (5^c-1)(5^d-1)]$, $X^a=Y^c$ and $X^b=Y^d$ and $X^{(a+b)}=Y^{(c+d)}$; which is a *necessary condition* for validity of Eq.-7.3. This matching process and equivalency is henceforth referred to as the "*Matching Reduction*" of an equation.

Thus, for a sub-set of solutions to the equation $[(3^a-1)(3^b-1) = (5^c-1)(5^d-1)]$, $3^a=5^c$ and $3^b=5^d$, and $3^{(a+b)}=5^{(c+d)}$; and taking the natural log of both sides of $3^{(a+b)}=5^{(c+d)}$, the result is:

7.4) $(a+b)\text{Ln}3=(c+d)\text{Ln}5$
7.5) $(a+b)/(c+d) = \text{Ln}5/\text{Ln}3=1.465 \approx 1.5$
7.6) $\text{Log}_3 5^{(c+d)}=(a+b)$

There are potentially many combinations of $X^{(a+b)}$ and $Y^{(c+d)}$ in positive integers that make the equation $X^{(a+b)}-Y^{(c+d)}=0$ valid. There are potentially and infinitely many combinations of X, Y, *a*, *b*, *c* and *d* that make the equations $X^a=Y^c$ and $X^b=Y^d$ valid. Thus its highly probable that there is more than one solution for the equation $(3^a-1)(3^b-1) = (5^c-1)(5^d-1)$. ■

3. Computer Codes For Verifying The Rational-Equivalence Equation.

The following simple Java code (and its variants) can be used for finding feasible solutions for (a,b,c,d) in the *Rational-Equivalence Equation* $[(3^a-1)(3^b-1)]=[(5^c-1)(5^d-1)]$ and similar classes of equations and for Positive Integers up to $10^{2457600000}$ (and even greater positive-integers depending on available computing power):

```
Int a, b, c, d i, j, k, m;
Int p = 3;
Int q =5;
Int e = (p**a)-1;
Int f = (p**b)-1;
Int g = (q**c)-1;
Int h = (q**d)-1;
Int x = (((((1000000000000000000000000000000000000000000000000000000000000000000000000000000000000 **
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)**
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)**
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)**
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)**
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000);

For (a=i; i=1; i <= x):
    For (b=j; j=1; j <= x):
        For (c=k; k=1; k <= x):
            For (d=m; m=1; m <= x):
                if((e*f) == (g*h)):
                    System.out.println("A feasible solution is: "+a+", "+b+ ", "+c+ "," +d+ ".");
                d += 1;
            c += 1;
        b += 1;
    a += 1;
```

4. Conclusion.

The equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$ is an ill-posed problem because its properties change over different intervals. Contrary to Liptai, Németh, et. al. (2020), it's likely that the equation $(3^a-1)(3^b-1)=(5^c-1)(5^d-1)$ in positive integers $a \leq b$ and $c \leq d$, has more than one solution.

List of Abbreviations Used:

No abbreviations were used.

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