ERRATUM TO "TABLES OF INTEGRAL TRANSFORMS" BY A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER & F. G. TRICOMI (1953), P. 61 (4)

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The table of integral transforms contains on page 61, equation (4), the Fourier cosine integral of an exponential multiplied by a sum of two Parabolic-Cylinder Functions [4]

$$\int_0^\infty e^{-x^2/2}\cos(bx)[D_{2\nu-1/2}(x)+D_{2\nu-1/2}(-x)]dx \stackrel{?}{=} \sin[(\nu+1/4)\pi]2^{1/4-2\nu}\sqrt{\pi}b^{2\nu-1/2}e^{-b^2/4}. \tag{1}$$

It also appears in that form in the Gradshteyn-Ryzhik tables [5, (7.741.5.)]. Standard numerical comparison with Riemann Approximations with a few hundred sampling points for finite ranges up to x a few hundred and b and ν in ranges around 0.5 demonstrate that this right hand side is not correct. A correct expression is derived here: equations (8) and (12).

The Parabolic-Cylinder Function is rephrased as a difference of two Confluent Hypergeometric Functions [5, (9.240)][1, 19.12.4][6, §8.1.1]

$$D_p(z) = 2^{p/2}e^{-z^2/4}\left[\frac{\sqrt{\pi}}{\Gamma(\frac{1-p}{2})}{}_1F_1(-p/2;1/2;z^2/2) - \frac{\sqrt{2\pi}z}{\Gamma(-p/2)}{}_1F_1(\frac{1-p}{2};3/2;z^2/2)\right] \tag{2}$$

in the standard notation [8]. In the even component, the sum of the two Parabolic-Cylinder Functions, one Confluent Hypergeometric Function remains:

$$D_{2\nu-1/2}(x)+D_{2\nu-1/2}(-x) = \frac{2^{\nu+3/4}\sqrt{\pi}}{\Gamma(3/4-\nu)}e^{-x^2/4}{}_1F_1(-\nu+1/4;1/2;x^2/2). \tag{3}$$

The series expansion of the Confluent Hypergeometric Function yields

$$\begin{aligned} I_{\alpha,\nu,b} &\equiv \int_0^\infty e^{-\alpha x^2}\cos(bx)[D_{2\nu-1/2}(x)+D_{2\nu-1/2}(-x)]dx = \\ &= \int_0^\infty e^{-\alpha x^2}\cos(bx)\frac{2^{\nu+3/4}\sqrt{\pi}}{\Gamma(3/4-\nu)}e^{-x^2/4}{}_1F_1(-\nu+1/4;1/2;x^2/2)dx = \\ &= \frac{2^{\nu+3/4}\sqrt{\pi}}{\Gamma(3/4-\nu)}\sum_{k=0}^\infty\int_0^\infty e^{-(\alpha+1/4)x^2}\cos(bx)\frac{(-\nu+1/4)_k}{(1/2)_k k!}(x^2/2)^k \\ &= \frac{2^{\nu+3/4}\sqrt{\pi}}{\Gamma(3/4-\nu)}\sum_{k=0}^\infty\frac{(-\nu+1/4)_k}{(1/2)_k k!2^k}\int_0^\infty e^{-(\alpha+1/4)x^2}\cos(bx)x^{2k}. \end{aligned} \tag{4}$$

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Interchange of summation and integration leads to integrals that are again Confluent Hypergeometric Functions [5, 3.952.8][3, p. 15 (14)]

(5)
$$I_{\alpha,\nu,b} = \frac{2^{\nu+3/4}\sqrt{\pi}}{\Gamma(3/4-\nu)}\sum_{k=0}^{\infty}\frac{(-\nu+1/4)_k}{(1/2)_k k! 2^k}\frac{\Gamma(\frac{2k+1}{2})}{2(\alpha+1/4)^{k+1/2}}{}_1F_1(k+1/2;1/2;-\frac{b^2}{4(\alpha+1/4)})$$
$$= \frac{2^{\nu-1/4}\pi}{\Gamma(3/4-\nu)\sqrt{\alpha+1/4}}\sum_{k=0}^{\infty}\frac{(-\nu+1/4)_k}{k!}\frac{1}{(2\alpha+1/2)^k}{}_1F_1(k+1/2;1/2;-\frac{b^2}{4\alpha+1}).$$

Kummer's transformation [1, 13.1.27][5, 9.212][7, 13.2.39][6, §6.1.2] demonstrates that these are *terminating* Hypergeometric Functions: [3, §6.3]

(6)
$$I_{\alpha,\nu,b} = \frac{2^{\nu-1/4}\pi}{\Gamma(3/4-\nu)\sqrt{\alpha+1/4}}e^{-b^2/(4\alpha+1)}\sum_{k=0}^{\infty}\frac{(-\nu+1/4)_k}{k!}\frac{1}{(2\alpha+1/2)^k}{}_1F_1(-k;1/2;\frac{b^2}{4\alpha+1}).$$

This sum over Confluent Hypergeometric Functions is (again) a Confluent Hypergeometric Function [2][6, §6.4.3]

(7)
$$I_{\alpha,\nu,b} = \frac{2^{\nu-1/4}\pi}{\Gamma(3/4-\nu)\sqrt{\alpha+1/4}}e^{-b^2/(4\alpha+1)}(\frac{\alpha-1/4}{\alpha+1/4})^{\nu-1/4}{}_1F_1(-\nu+1/4;1/2;-\frac{b^2}{4\alpha+1}\cdot\frac{2}{4\alpha-1}),$$

and our main result is

(8)
$$I_{\alpha,\nu,b} = \frac{2^{\nu-1/4}\pi}{\Gamma(3/4-\nu)}\frac{(\alpha-1/4)^{\nu-1/4}}{(\alpha+1/4)^{\nu+1/4}}e^{-b^2/(4\alpha+1)}{}_1F_1(-\nu+1/4;1/2;-\frac{2b^2}{(4\alpha+1)(4\alpha-1)}),$$
$$\alpha > 1/4, \Re\nu > 1/4, b > 0.$$

In particular at $\alpha = 1/2$ a correct expression of the right hand side of (1) is

(9)
$$I_{1/2,\nu,b} = \frac{2^{\nu-1/4}\pi}{\Gamma(3/4-\nu)}\frac{(1/4)^{\nu-1/4}}{(3/4)^{\nu+1/4}}e^{-b^2/3}{}_1F_1(-\nu+1/4;1/2;-\frac{2b^2}{3})$$
$$= \frac{\sqrt{2}\pi}{\Gamma(3/4-\nu)}(\frac{2}{3})^{\nu+1/4}e^{-b^2/3}{}_1F_1(-\nu+1/4;1/2;-\frac{2b^2}{3}).$$

In the limit $\alpha \to 1/4^+$ the Confluent Hypergeometric Series in (8) has an asymptotic value of [1, 13.1.5][6, §6.8.2]

(10)
$$I_{1/4,\nu,b} \to \frac{2^{2\nu-1/2}\pi}{\Gamma(3/4-\nu)\sqrt{1/2}}e^{-b^2/2}(\alpha-1/4)^{\nu-1/4}\frac{\Gamma(1/2)}{\Gamma(1/4+\nu)}(\frac{2b^2}{(4\alpha+1)(4\alpha-1)})^{\nu-1/4}$$
$$= \frac{\sqrt{2}\pi^{3/2}}{\Gamma(3/4-\nu)}e^{-b^2/2}\frac{1}{\Gamma(1/4+\nu)}b^{2\nu-1/2}.$$

The reflection formula of the Γ-function yields [1, 6.1.17][7, 5.5.3]

(11)
$$I_{1/4,\nu,b} = \frac{\sqrt{2}\pi^{3/2}}{\pi}\sin[\pi(\nu+1/4)]e^{-b^2/2}b^{2\nu-1/2},$$

so a suitable substitution for both sides (!) of (1) is

(12)
$$I_{1/4,\nu,b} = \sqrt{2\pi}\sin[\pi(\nu+1/4)]e^{-b^2/2}b^{2\nu-1/2}, \quad \Re\nu > 1/4, b > 0.$$

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