Linear Formulation of Square Peg Problem Test Function

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Abstract

In this paper, we developed a set of linear constraints to test whether 4 points form a square. Traditionally people use Euclidean distance to test whether the 4 points form a square. It forms a square if the four sides are of equal length and the diagonals are of equal length. My test function using a set of linear constraints is much simpler without the use of quadratic operations in Euclidean distance test function. This is needed in the future to prove Square Peg Problem for any arbitary closed curve.

Contents

1 Introduction

Square Peg Problem (Inscribed Square Problem) definition is

> Does every plane simple closed curve contain all four vertices of some square?

An example of a closed curve with 3 squares are shown below.

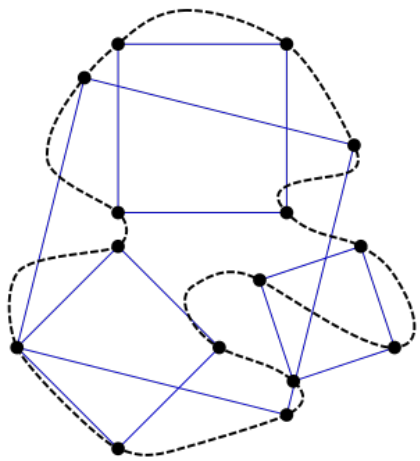

Figure 1: Example of a closed curve with 3 squares vertices touching the curve

2 Test function

A test function f() maps to a 1 if 4 points form a square [1].

Specifically

$$f(p_1, p_2, p_3, p_4) = \begin{cases} 1 & \text{if 4 points form a square} \\ 0 & \text{otherwise} \end{cases}$$

$f(p1, p2, p3, p4) = 1$ if $\|p_1 - p_2\| = \|p_2 - p_4\| = \|p_4 - p_3\| = \|p_3 - p_1\|$ and $\|p_1 - p_4\| = \|p_2 - p_3\|$

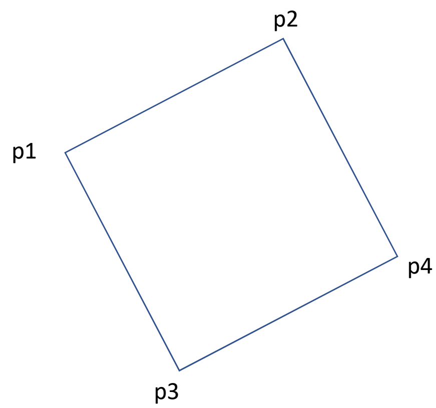

Figure 2: 4 points on a square

The lengths of the sides of the square are the same.

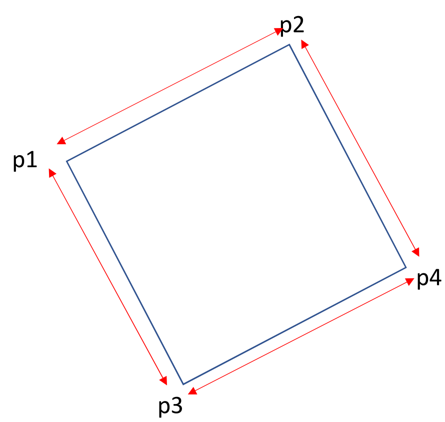

Figure 3: The lengths of the sides of the square are the same

The lengths of the diagonals of the square are the same.

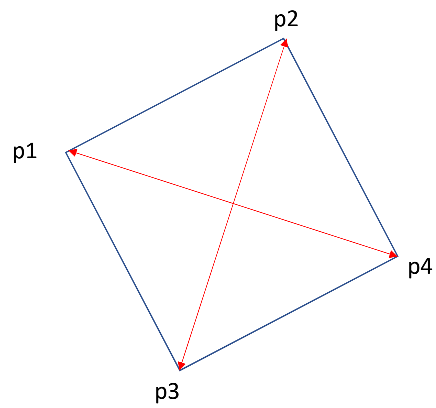

Figure 4: The lengths of the diagonals of the square are the same

This existing test function is complex with too many quadratic operations.

3 My Test Function using Linear Operations

$f(p_1, p_2, p_3, p_4) = 1$ if

$$\begin{aligned} x_2 - x_1 &= x_4 - x_3 \\ y_2 - y_1 &= y_4 - y_3 \\ x_4 - x_1 &= y_2 - y_3 \\ y_4 - y_1 &= x_3 - x_2 \end{aligned} \tag{1}$$

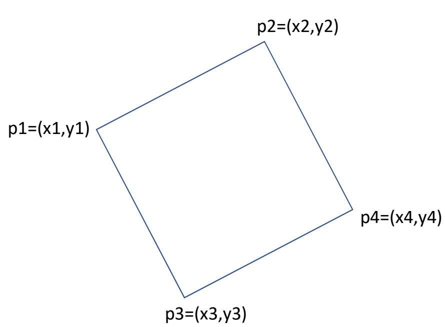

Figure 5: 4 points on a square with (x,y) variables

This test function is much simpler with only linear operations.
The gradients of the top and bottom sides are the same.

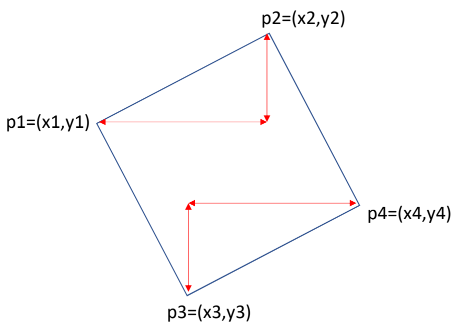

Figure 6: The gradients of the top and bottom sides are the same

The vectors of the diagonals are rotated by right angle.

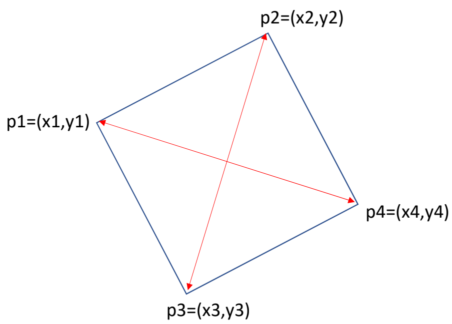

Figure 7: The vectors of the diagonals are rotated by right angle

4 Example

Parametric equation of a shape is

$$\begin{aligned} x &= cos(t) \\ y &= 2sin(t) \end{aligned} \tag{2}$$

The shape is an eclipse shown below.

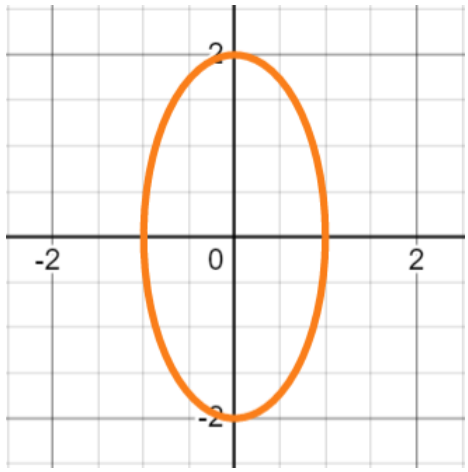

Figure 8: An eclipse

Find the inscribed square in the shape. Hint: let $\cos(t) = 2\sin(t)$.

$$\begin{aligned} \cos(t) &= 2\sin(t) \\ t &= 0.46364 \\ \text{Or } t = 0.46364 + \pi &= 3.60523 \end{aligned} \tag{3}$$

$$\begin{aligned} \cos(t) &= -2\sin(t) \\ t &= -0.46364 \\ \text{Or } t = -0.46364 + \pi &= 2.67795 \end{aligned} \tag{4}$$

$$\begin{aligned} t_2 &= 0.46364 \\ t_3 &= 3.60523 \\ t_1 &= 2.67795 \\ t_4 &= -0.46364 \end{aligned} \tag{5}$$

Put t_1, t_2, t_3, t_4 back into the linear equations,

$$\begin{aligned} x_2 - x_1 &= x_4 - x_3 \\ y_2 - y_1 &= y_4 - y_3 \\ x_4 - x_1 &= y_2 - y_3 \\ y_4 - y_1 &= x_3 - x_2 \end{aligned} \tag{6}$$

becomes

$$
\begin{aligned}
\cos(t_2) - \cos(t_1) &= \cos(t_4) - \cos(t_3) \\
2\sin(t_2) - 2\sin(t_1) &= 2\sin(t_4) - 2\sin(t_3) \\
\cos(t_4) - \cos(t_1) &= 2\sin(t_2) - 2\sin(t_3) \\
2\sin(t_4) - 2\sin(t_1) &= \cos(t_3) - \cos(t_2).
\end{aligned}
\tag{7}
$$

The equations

$$
\begin{aligned}
\cos(t_2) - \cos(t_1) &= \cos(t_4) - \cos(t_3) \\
2\sin(t_2) - 2\sin(t_1) &= 2\sin(t_4) - 2\sin(t_3) \\
\cos(t_4) - \cos(t_1) &= 2\sin(t_2) - 2\sin(t_3) \\
2\sin(t_4) - 2\sin(t_1) &= \cos(t_3) - \cos(t_2)
\end{aligned}
\tag{8}
$$

are all satisfied. Therefore these t_1, t_2, t_3, t_4 are points of inscribed square.

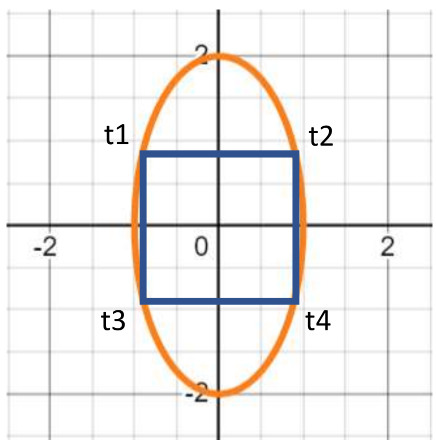

Figure 9: Inscribed square on an eclipse

5 Why I study test function for square peg problem?

I want to study pure mathematics to invent new algorithms. Most physical objects are moving along curves that are sum of sinusoidal waves. It can be represented by a curve and constraints in Square Peg Problem.

6 I'll be back

I will come back and explain how to find the inscribed squares of curves of more complex parametric equations and complex Fourier series. Examples of curves are shown below.

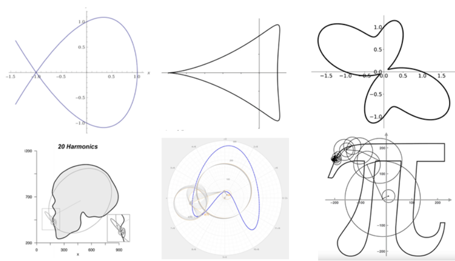

Figure 10: Examples of curves

7 Problem Formulation

There are many ways to formulate Square Peg Problem into algebraic form. Naive formulation will result in an algebra that will takes infinite steps to evaluate. Since naive formulation will result in infinite steps to prove Square Peg Problem, it is not able to show that Square Peg Problem conjecture is true for any closed 2D shape.

An example of naive formulation is to represent a closed 2D shape by infinite number of points. The 2D shape is formed by $a_0, a_1, ..., a_n$ points where each pair of consecutive points are linked by a straight line, a_i and a_{i+1} are linked by a straight line. Note that a_0 and a_n are the same point so that the 2D shape formed a closed curve. We assume that n is very large so the points can represent any arbitrary curve. The Square Peg Problem conjecture is true if the following equation is greater than zero,

$$\begin{aligned} g(a_0, a_1, ..., a_n) = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n-1} \int_0^1 \int_0^1 \int_0^1 \int_0^1 & \\ f(\alpha_0 a_i + (1-\alpha_0) a_{i+1}, & \\ \alpha_1 a_j + (1-\alpha_1) a_{j+1}, & \\ \alpha_2 a_k + (1-\alpha_2) a_{k+1}, & \\ \alpha_3 a_l + (1-\alpha_3) a_{l+1}) & \\ d\alpha_0 d\alpha_1 d\alpha_2 d\alpha_3. & \end{aligned} \tag{9}$$

If $g(a_0, a_1, ..., a_n) > 0$ then Square Peg Problem conjecture is true. Else it is false. Since this function $g()$ takes infinite number of steps to evaluate as we assume n is infinity, we are unable to show that the Square Peg Problem conjecture is true.

Closed curve can also be formulated using complex Fourier series. If for all possible 2D closed curves by the Fourier series each contains at least one inscribed square, then the Square Peg Problem conjecture is true. In this paper, I proved for an eclipse 2D curve which is a subset of all possible Fourier series, has an inscribed square.

In my later upcoming papers, I will try to prove Square Peg Problem by formulating the problem that can be solved in finite number of steps. I will formulate the closed curve using complex Fourier series.

Read my papers on NP vs P [2] and Discrete Markov Random Field Relaxation [3].

References

[1] R. Rius Casado, "The square peg problem," *http://diposit.ub.edu/dspace/bitstream/2445/151918/2/151918.pdf*, 2019.

[2] S. K. Tan, "Prove np not equal p using markov random field and boolean algebra simplification," *https://vixra.org/pdf/2105.0181v1.pdf*, 2021.

[3] ——, "Discrete markov random field relaxation," *https://vixra.org/pdf/2112.0151v1.pdf*, 2021.