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# WHAT ARE THE OPERATOR ERROR ESTIMATES? A MANUAL FOR REFEREES

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A very brief explanation what are the operator error estimates in periodic homogenization.

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**Motivation.** There are a lot of detail texts on operator error estimates in homogenization theory. Probably, this literature is too detail: sometimes the referees are not able to understand even the statement of the problem — what are the operator error estimates and what is the goal for the paper on operator error estimates. (See, e. g., Appendix containing the report on the author's preprint 1904.02781.) The aim of the present text is to fix this problem and to provide a citable explanation that can be ununderstood only by the malicious reader.

**Explanation.** Homogenization theory deals with equations in highly inhomogeneous media. To describe the asymptotic behaviour of such a medium different types of convergence were introduced and studied.

The subject of the present paper is *periodic homogenization*, there the object of studies is a PDE with periodic coefficients depending on $\mathbf{x}/\varepsilon$, $\mathbf{x} \in \mathbb{R}^d$. The parameter $\varepsilon > 0$ is assumed to be small. Problems can be stated in the whole space $\mathbb{R}^d$, in a bounded domain, in a layer,. . . The typical problem in $\mathbb{R}^d$ is

$$-\operatorname{div} g(\mathbf{x})\nabla u_\varepsilon + u_\varepsilon = F,$$

where $g$ is a bounded periodic positive definite matrix and $F \in L_2(\mathbb{R}^d)$ is a known function. The *classical result* for homogenization is the convergence $u_\varepsilon \to u_0$ as $\varepsilon \to 0$. Here $u_0$ is the solution for the problem of the same type but with the constant matrix $g^0$ called the effective matrix. Its definition is well-known and given in terms of a cell problem solution.

The *operator error estimate* for this problem have the form

$$\|u_\varepsilon - u_0\|_{L_2(\mathbb{R}^d)} \leqslant C\varepsilon \|F\|_{L_2(\mathbb{R}^d)},$$

where the constant $C$ depends only on the dimension $d$, the matrix $g$, and the lattice of periodicity. Or, equivalently,

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon.$$

Here $A_\varepsilon = -\operatorname{div} g(\mathbf{x})\nabla$ and $A^0 = -\operatorname{div} g^0\nabla$. According to the author's knowledge, the first estimate of such type can be found in [MoVo] for a Dirichlet problem with a smooth matrix $g$.

The systematic treatment of operator error estimates for periodic homogenization started from the paper [BSu], there strongly elliptic systems were studied via the spectral approach based on the Floquet–Bloch theory. Currently there is a rich literature on the subject, with different operators under consideration and estimates in various types of operator norms obtained by different methods, see [CDGr, Sh, ZhPas].

**Summary.** When one claims to prove operator error estimate in homogenization, the expected result is an estimate in the uniform operator topology ($L_2 \to L_2$, $L_2 \to H^1$[1],. . . ) with the traced dependence of constants from the problem data. The scientific novelty of each paper consists in the class operator $A_\varepsilon$ under consideration and the choose of type of the problem (elliptic, parabolic, or hyperbolic), i. e., the form of the operator function (the resolvent $(A_\varepsilon + I)^{-1}$, the semigroup $\exp(-A_\varepsilon t)$, or operator sine and cosine functions).

[1] To obtain approximation in the Sobolev class $H^1$, one needs to take into account the corrector corresponding to the second term of the asymptotic expansion for $u_\varepsilon$. Nevertheless, the full asymptotic expansion is not the goal for papers on operator error estimates.

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