# New Principles of Differential Equations V

Hong Lai Zhu *

*School of Physics and Electronic Information,Huaibei Normal University, Anhui 235000, China*

**Abstract**

In this paper, we use the previously proposed $Z$ Transformations to obtain the general solutions of many typical nonlinear partial differential equations, and use the general solutions to get the exact solutions of some definite solution problems.

**Keywords:** $Z$ Transformations; nonlinear partial differential equations; general solutions; problem of definite solution; exact solution.

## 1. Introduction

In the previous paper [1],

$$a_1u_{x_1}+a_2u_{x_2}+a_3u_{x_3}=0, \tag{1}$$

we got the general solution of Eq. (1) in $\mathbb{R}^3$ is

$$u=f\left(\frac{-a_2c_2-a_3c_3}{a_1}x_1+c_2x_2+c_3x_3,\frac{-a_2c_5-a_3c_6}{a_1}x_1+c_5x_2+c_6x_3\right). \tag{2}$$

According to (2), it is easy to find

$$u=f\left(\frac{-a_2-a_3}{a_1}x_1+x_2+x_3\right),u=f\left(\frac{-a_2+a_3}{a_1}x_1+x_2-x_3\right),\dots,$$

are general solutions of Eq. (1) too. For distinguish these general solutions, we propose Definition 1.

**Definition 1.** If the number of independent variables of an arbitrary function in the general solution of a PDE is at most $l$, $(l\geq 1)$, it is called **the $l$-th type general solution** of the PDE.

According to Definition 1, it could be known that $u=f\left(\frac{-a_2-a_3}{a_1}x_1+x_2+x_3\right)$ and $u=f\left(\frac{-a_2c_2-a_3c_3}{a_1}x_1+c_2x_2+c_3x_3\right)$ are the first type general solutions of Eq. (1) and Eq. (2) is the second type general solution of Eq. (1).

## 2. General solutions of nonlinear PDEs and exact solutions of definite solution problems

Next, we use Z Transformations to get general solutions of some typical nonlinear PDEs. Theorem 1 is presented first.

*E-mail address: honglaizhu@qq.com

**Theorem 1.** *In* $\mathbb{R}^3$*, if*

$$\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+\Lambda\left(a_7u_t+a_8u_x+a_9u_y\right)=0, \tag{3}$$

*where $a_i$ are any known constants $(1\le i\le 9)$, $\Theta=\Theta\left(t,x,y,u,u_t,\ldots u_{txy},\ldots\right)$, $\Lambda=\Lambda\left(t,x,y,u,u_t,\ldots u_{txy},\ldots\right)$, the second type general solution of Eq. (3) is*

$$u=f(v,w), \tag{4}$$

$$v=k_1t+k_2x+k_3y+k_4, \tag{5}$$

$$w=k_5t+k_6x+k_7y+k_8, \tag{6}$$

*where $f$ is an any first differentiable function, $v,w$ are independent of each other, and the constants $k_1,k_2,k_3,k_5,k_6,k_7$ must satisfy*

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0, \tag{7}$$

$$a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_5k_6+a_5k_6k_7+a_6k_5k_7=0, \tag{8}$$

$$2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+a_4\left(k_1k_6+k_2k_5\right)+a_5\left(k_2k_7+k_3k_6\right)+a_6\left(k_3k_5+k_1k_7\right)=0, \tag{9}$$

$$a_7k_1+a_8k_2+a_9k_3=0, \tag{10}$$

$$a_7k_5+a_8k_6+a_9k_7=0. \tag{11}$$

**Proof.** According to $Z_1$ transformation, set $u=f(v,w)$, $v=k_1t+k_2x+k_3y+k_4$, $w=k_5t+k_6x+k_7y+k_8$. $k_1,k_2,\ldots,k_8$ are undetermined constants, so

$$\begin{aligned}
&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+\Lambda\left(a_7u_t+a_8u_x+a_9u_y\right)\\
&=\Theta a_1(k_1f_v+k_5f_w)^2+\Theta a_2(k_2f_v+k_6f_w)^2+\Theta a_3(k_3f_v+k_7f_w)^2\\
&+\Theta a_4\left(k_1f_v+k_5f_w\right)\left(k_2f_v+k_6f_w\right)+\Theta a_5\left(k_2f_v+k_6f_w\right)\left(k_3f_v+k_7f_w\right)\\
&+\Theta a_6\left(k_3f_v+k_7f_w\right)\left(k_1f_v+k_5f_w\right)+\Lambda a_7\left(k_1f_v+k_5f_w\right)+\Lambda a_8\left(k_2f_v+k_6f_w\right)\\
&+\Lambda a_9\left(k_3f_v+k_7f_w\right)=0.
\end{aligned}$$

Namely

$$\begin{aligned}
&\Theta f_v^2\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3\right)\\
&+\Theta f_w^2\left(a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_5k_6+a_5k_6k_7+a_6k_5k_7\right)\\
&+\Theta f_vf_w\left(2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+a_4\left(k_1k_6+k_2k_5\right)+a_5\left(k_2k_7+k_3k_6\right)+a_6\left(k_3k_5+k_1k_7\right)\right)\\
&+\Lambda f_v\left(a_7k_1+a_8k_2+a_9k_3\right)+\Lambda f_w\left(a_7k_5+a_8k_6+a_9k_7\right)=0.
\end{aligned}$$

Set

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,$$

$$a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_5k_6+a_5k_6k_7+a_6k_5k_7=0,$$

$$2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+a_4\left(k_1k_6+k_2k_5\right)+a_5\left(k_2k_7+k_3k_6\right)+a_6\left(k_3k_5+k_1k_7\right)=0,$$

$$a_7k_1+a_8k_2+a_9k_3=0,$$

$$a_7k_5+a_8k_6+a_9k_7=0.$$

Therefore, the general solution of Eq. (3) is

$$u=f(v,w)$$

The theorem is proven. □

Next, we use Theorem 1 to study a definite solution problem.

**Example 1.** In $\mathbb{R}^3$, use Theorem 1 to get the exact solution of

$$\left(u_t^5+u_x^3\right)\left(3u_t^2+3u_x^2+3u_y^2-10u_tu_x+6u_xu_y-10u_yu_t\right)-u_y^2\left(9u_t+3u_x+3u_y\right)=0, \tag{12}$$

in the condition of $u(0,y,z)=g(x,y)$, $g$ is an random known first differentiable function.

**Solution.** According to Theorem 1, the general solution of (12) is

$$u\left(t,x,y\right)=f\left(t+k_2x+\left(3-k_2\right)y,t+k_6x+\left(3-k_6\right)y\right). \tag{13}$$

So

$$u\left(0,x,y\right)=f\left(k_2x+\left(3-k_2\right)y,k_6x+\left(3-k_6\right)y\right)=g\left(x,y\right).$$

Set

$$k_2x+\left(3-k_2\right)y=\beta,k_6x+\left(3-k_6\right)y=\gamma. \tag{14}$$

We obtain

$$x=\frac{\beta}{k_2}+\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}-\frac{k_6\beta-k_2\gamma}{k_2k_6-k_2^2},$$

$$y=\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}.$$

Namely

$$u\left(0,x,y\right)=f\left(k_2x+\left(3-k_2\right)yk_6x+\left(3-k_6\right)y\right)=g\left(\frac{\beta}{k_2}+\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}-\frac{k_6\beta-k_2\gamma}{k_2k_6-k_2^2},\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}\right). \tag{15}$$

Set

$$t+k_2x+\left(3-k_2\right)y=\beta,t+k_6x+\left(3-k_6\right)y=\gamma.$$

We get

$$\frac{\beta}{k_2}+\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}-\frac{k_6\beta-k_2\gamma}{k_2k_6-k_2^2}=\frac{t+3x}{3},$$

$$\frac{k_6\beta-k_2\gamma}{3k_6-3k_2}=\frac{t+3y}{3}.$$

Then

$$u\left(t,x,y\right)=f\left(t+k_2x+\left(3-k_2\right)y,t+k_6x+\left(3-k_6\right)y\right)=g\left(\frac{t+3x}{3},\frac{t+3y}{3}\right). \tag{16}$$

According to Example 1, we can directly get the exact solution of Eq. (12) in diversified initial value conditions. If the initial value condition is $u\left(0,x,y\right)=\sin\left(2x+y\right)+e^{4x-y}$, the exact solution is $u=\sin\left(t+2x+y\right)+e^{t+4x-y}$.

According to Theorem 1, set $k_5=k_6=k_7=k_8=0$, we can obtain Theorem 2.

**Theorem 2.** *In* $\mathbb{R}^3$, *if*

$$\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+\Lambda\left(a_7u_t+a_8u_x+a_9u_y\right)=0,$$

*where $a_i$ are any known constants $(1 \le i \le 9)$, $\Theta = \Theta\left(t, x, y, u, u_t, \ldots u_{txy}, \ldots\right)$, $\Lambda = \Lambda\left(t, x, y, u, u_t, \ldots u_{txy}, \ldots\right)$, the first type general solution of Eq. (3) is*

$$u = f(v), \tag{17}$$

*where $f$ is an arbitrary first differentiable function, $v = k_1 t + k_2 x + k_3 y + k_4$, and the constants $k_1, k_2, k_3$ must satisfy*

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_1 k_2 + a_5 k_2 k_3 + a_6 k_1 k_3 = 0,$$

$$a_7 k_1 + a_8 k_2 + a_9 k_3 = 0.$$

The reason why we propose Theorem 2 is that there is no the second type general solution of some forms of Eq. (3), such as examples 2 and 3.

**Example 2.** Prove that $a_1 u_t^2 + a_2 u_x^2 + a_3 u_y^2 + a_4 u_t u_x + a_5 u_x u_y + a_6 u_y u_t + a_7 u_t = 0$ does not have the second type general solution similar to Theorem 1, $a_i$ are arbitrary known constants $(1 \le i \le 7)$, $a_7 \ne 0$.

**Proof.** If $a_1 u_t^2 + a_2 u_x^2 + a_3 u_y^2 + a_4 u_t u_x + a_5 u_x u_y + a_6 u_y u_t + a_7 u_t = 0$ has a general solution in the form of $u = f(v, w)$, and

$$v = k_1 t + k_2 x + k_3 y + k_4, w = k_5 t + k_6 x + k_7 y + k_8,$$

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_1 k_2 + a_5 k_2 k_3 + a_6 k_1 k_3 = 0,$$

$$a_1 k_5^2 + a_2 k_6^2 + a_3 k_7^2 + a_4 k_5 k_6 + a_5 k_6 k_7 + a_6 k_5 k_7 = 0,$$

$$2a_1 k_1 k_5 + 2a_2 k_2 k_6 + 2a_3 k_3 k_7 + a_4\left(k_1 k_6 + k_2 k_5\right) + a_5\left(k_2 k_7 + k_3 k_6\right) + a_6\left(k_3 k_5 + k_1 k_7\right) = 0,$$

$$a_7 k_1 = 0,$$

$$a_7 k_5 = 0.$$

For $a_7 \ne 0$, we get

$$k_1 = k_5 = 0.$$

So

$$a_2 k_2^2 + a_3 k_3^2 + a_5 k_2 k_3 = 0,$$

$$a_2 k_6^2 + a_3 k_7^2 + a_5 k_6 k_7 = 0,$$

$$2a_2 k_2 k_6 + 2a_3 k_3 k_7 + a_5\left(k_2 k_7 + k_3 k_6\right) = 0.$$

Since $f$ is a random first differentiable function, we may set

$$k_2 = k_6 = 1.$$

Then

$$a_2 + a_3 k_3^2 + a_5 k_3 = 0,$$

$$a_2 + a_3 k_7^2 + a_5 k_7 = 0,$$

$$2a_2 + 2a_3 k_3 k_7 + a_5\left(k_7 + k_3\right) = 0.$$

Set

$$k_3 = \frac{-a_5 + \sqrt{a_5^2 - 4a_2 a_3}}{2a_3},$$

$$k_7 = \frac{-a_5 - \sqrt{a_5^2 - 4a_2a_3}}{2a_3}.$$

So

$$2a_2 + 2a_3k_3k_7 + a_5(k_7 + k_3) = 2a_2 + 2a_2 - \frac{a_5^2}{a_3} = 0.$$

Namely

$$a_5^2 = 4a_2a_3.$$

We obtain $k_3 = k_7$, $v, w$ are not independent of each other. That is, $a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t + a_7u_t = 0$ does not have a solution similar to the $u = f(v, w)$ form of Theorem 1.

**Example 3.** Prove that $a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t + a_7u_t - a_7u_x = 0$ does not have the second type general solution similar to Theorem 1, $a_i$ are arbitrary known constants $(1 \le i \le 7), a_7 \ne 0$.

**Proof.** If $a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t + a_7u_t - a_7u_x = 0$ has a general solution in the form of $u = f(v, w)$, and

$$v = k_1t + k_2x + k_3y + k_4, w = k_5t + k_6x + k_7y + k_8,$$

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0,$$

$$a_1k_5^2 + a_2k_6^2 + a_3k_7^2 + a_4k_5k_6 + a_5k_6k_7 + a_6k_5k_7 = 0,$$

$$2a_1k_1k_5 + 2a_2k_2k_6 + 2a_3k_3k_7 + a_4(k_1k_6 + k_2k_5) + a_5(k_2k_7 + k_3k_6) + a_6(k_3k_5 + k_1k_7) = 0,$$

$$k_1 = k_2,$$

$$k_5 = k_6.$$

Since $f$ is an random first differentiable function, we may set

$$k_1 = k_2 = k_5 = k_6 = 1.$$

So

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = a_3k_3^2 + (a_5 + a_6)k_3 + a_1 + a_2 + a_4 = 0,$$

$$a_1k_5^2 + a_2k_6^2 + a_3k_7^2 + a_4k_5k_6 + a_5k_6k_7 + a_6k_5k_7 = a_3k_7^2 + (a_5 + a_6)k_7 + a_1 + a_2 + a_4 = 0.$$

Set

$$k_3 = \frac{-a_5 - a_6 + \sqrt{(a_5 + a_6)^2 - 4a_3(a_1 + a_2 + a_4)}}{2a_3},$$

$$k_7 = \frac{-a_5 - a_6 - \sqrt{(a_5 + a_6)^2 - 4a_3(a_1 + a_2 + a_4)}}{2a_3}.$$

Then

$$\begin{aligned} &2a_1k_1k_5 + 2a_2k_2k_6 + 2a_3k_3k_7 + a_4(k_1k_6 + k_2k_5) + a_5(k_2k_7 + k_3k_6) + a_6(k_3k_5 + k_1k_7) \\ &= 2a_1 + 2a_2 + 2a_3k_3k_7 + 2a_4 + a_5(k_7 + k_3) + a_6(k_3 + k_7) \\ &= 4(a_1 + a_2 + a_4) - \tfrac{(a_5+a_6)^2}{a_3} = 0. \end{aligned}$$

Namely

$$4a_3(a_1 + a_2 + a_4) = (a_5 + a_6)^2.$$

We obtain $k_3 = k_7, v, w$ are not independent of each other. That is, $a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t+a_7u_t-a_7u_x=0$ does not have a solution similar to the $u=f(v,w)$ form of Theorem 1.

Next, we use Theorem 2 to study a definite solution problem.

**Example 4.** In $\mathbb{R}^3$, use Theorem 2 to get the exact solution of

$$\left(u_t^2+u_xu_y\right)\left(u_t^2+u_x^2-u_y^2-u_tu_x+u_xu_y-u_yu_t\right)+\left(u_x^2-u_tu_x\right)\left(u_t-u_x\right)^2=0, \tag{18}$$

in the condition of $u\left(0,y,z\right)=\sum_i \varphi_i\left(\kappa_i x-\kappa_i y+\lambda_i\right)$, $\varphi_i$ are random known first differentiable functions, $\kappa_i$ and $\lambda_i$ are arbitrary known constants.

**Solution.** According to Theorem 2, the general solution of (18) is

$$u=f\left(k_1t+k_2x+k_3y+k_4\right),$$

and

$$k_1^2+k_2^2-k_3^2-k_1k_2+k_2k_3-k_1k_3=0,$$
$$k_1=k_2.$$

Then

$$k_1^2+k_2^2-k_3^2-k_1k_2+k_2k_3-k_1k_3=k_1^2-k_3^2=0,$$
$$k_3=\pm k_1.$$

That is, the general solution of (18) is

$$u=f\left(k_1t+k_1x+k_1y+k_4\right)=\sum_i f_i\left(k_{1_i}t+k_{1_i}x+k_{1_i}y+k_{4_i}\right). \tag{19}$$

Or

$$u=f\left(k_1t+k_1x-k_1y+k_4\right)=\sum_i f_i\left(k_{1_i}t+k_{1_i}x-k_{1_i}y+k_{4_i}\right). \tag{20}$$

Since the initial value condition is

$$u\left(0,x,y\right)=\sum_i \varphi_i\left(\kappa_i x-\kappa_i y+\lambda_i\right).$$

So the corresponding general solution of (18) is (20), set

$$f_i=\varphi_i, k_{1_i}=\kappa_i, k_{4_i}=\lambda_i.$$

Then the exact solution of the definite solution problem is

$$u\left(t,x,y\right)=\sum_i \varphi_i\left(\kappa_i t+\kappa_i x-\kappa_i y+\lambda_i\right). \tag{21}$$

According to Theorem 2, we can get Theorem 3.

**Theorem 3.** *In* $\mathbb{R}^2$*, if*

$$\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_tu_x\right)+\Lambda\left(a_4u_t+a_5u_x\right)=0, \tag{22}$$

*where $a_i$ are known constants $(1 \le i \le 5)$, $\Theta = \Theta(t, x, u, u_t, \ldots u_{tx}, \ldots)$, $\Lambda = \Lambda(t, x, u, u_t, \ldots u_{tx}, \ldots)$, the first type general solution of Eq. (22) is*

$$u = f\left(\frac{-a_5 k_2}{a_4} t + k_2 x + k_3\right), \tag{23}$$

*where $f$ is an random first differentiable function, $k_2$ and $k_3$ are arbitrary constants, and $a_i$ must satisfy*

$$a_1 a_5^2 + a_2 a_4^2 - a_3 a_4 a_5 = 0. \tag{24}$$

**Prove.** According to Theorem 2, the general solution of (22) is

$$u = f(k_1 t + k_2 x + k_3),$$

$k_1, k_2$ and $k_3$ satisfy

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 = 0,$$

$$a_4 k_1 + a_5 k_2 = 0.$$

So

$$k_1 = \frac{-a_5 k_2}{a_4},$$

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 = a_1 \frac{a_5^2}{a_4^2} k_2^2 + a_2 k_2^2 - \frac{a_3 a_5 k_2^2}{a_4} = 0 \Rightarrow a_1 a_5^2 + a_2 a_4^2 - a_3 a_4 a_5 = 0.$$

Therefore, the general solution of Eq. (22) is

$$u = f\left(\frac{-a_5 k_2}{a_4} t + k_2 x + k_3\right),$$

and $a_i$ need satisfy

$$a_1 a_5^2 + a_2 a_4^2 - a_3 a_4 a_5 = 0.$$

The theorem is proven. □

If the initial value condition of (22) is

$$u(0, x) = g(x).$$

Set $k_3 = 0$, so

$$u(t, x) = f\left(\frac{-a_5 k_2}{a_4} t + k_2 x\right) = g\left(\frac{-a_5}{a_4} t + x\right).$$

That is, the exact solution of the definite solution problem is $u(t, x) = g\left(\frac{-a_5}{a_4} t + x\right)$.

Next we propose Theorem 4.

**Theorem 4.** *In $\mathbb{R}^3$, if*

$$\Theta\big(a_1 u_t^2 + a_2 u_x^2 + a_3 u_y^2 + a_4 u_t u_x + a_5 u_x u_y + a_6 u_y u_t\big)^n + \Lambda(a_7 u_t + a_8 u_x + a_9 u_y)^m = 0, \tag{25}$$

*where $a_i$ are any known constants $(1 \le i \le 9)$, $\Theta = \Theta(t, x, y, u, u_t, \ldots u_{txy}, \ldots)$, $\Lambda = \Lambda(t, x, y, u, u_t, \ldots u_{txy}, \ldots), n \ge 1, m \ge 1$, the second type general solution of Eq. (25) is*

$$u = f(v, w),$$

$$v = k_1 t + k_2 x + k_3 y + k_4, w = k_5 t + k_6 x + k_7 y + k_8,$$

*where $f$ is a random first differentiable function, $v, w$ are independent of each other, and the constants $k_1, k_2, k_3, k_5, k_6, k_7$ must satisfy*

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_1 k_2 + a_5 k_2 k_3 + a_6 k_1 k_3 = 0,$$

$$a_1 k_5^2 + a_2 k_6^2 + a_3 k_7^2 + a_4 k_5 k_6 + a_5 k_6 k_7 + a_6 k_5 k_7 = 0,$$

$$2a_1 k_1 k_5 + 2a_2 k_2 k_6 + 2a_3 k_3 k_7 + a_4 (k_1 k_6 + k_2 k_5) + a_5 (k_2 k_7 + k_3 k_6) + a_6 (k_3 k_5 + k_1 k_7) = 0,$$

$$a_7 k_1 + a_8 k_2 + a_9 k_3 = 0,$$

$$a_7 k_5 + a_8 k_6 + a_9 k_7 = 0.$$

**Proof.** Set

$$\Theta' = \Theta\big(a_1 u_t^2 + a_2 u_x^2 + a_3 u_y^2 + a_4 u_t u_x + a_5 u_x u_y + a_6 u_y u_t\big)^{n-1},$$

$$\Lambda' = \Lambda(a_7 u_t + a_8 u_x + a_9 u_y)^{m-1}.$$

Then

$$\begin{aligned}&\Theta\big(a_1 u_t^2 + a_2 u_x^2 + a_3 u_y^2 + a_4 u_t u_x + a_5 u_x u_y + a_6 u_y u_t\big)^n + \Lambda(a_7 u_t + a_8 u_x + a_9 u_y)^m \\ &= \Theta' \left(a_1 u_t^2 + a_2 u_x^2 + a_3 u_y^2 + a_4 u_t u_x + a_5 u_x u_y + a_6 u_y u_t\right) + \Lambda' \left(a_7 u_t + a_8 u_x + a_9 u_y\right) = 0.\end{aligned}$$

Obviously $\Theta' = \Theta'(t, x, y, u, u_t, \ldots u_{txy}, \ldots)$, $\Lambda' = \Lambda'(t, x, y, u, u_t, \ldots u_{txy}, \ldots)$, according to Theorem 1, the general solution of the above equation is (4), so the theorem is proved. □

Theorem 4 explains that the general solution of (25) is independent of $\Theta$, $\Lambda$, $n$ and $m$, namely, general solutions of these infinitely many nonlinear PDEs are the same. Theorems 2 and 3 have similar laws too, which we will not elaborate here.

Next we propose Theorem 5.

**Theorem 5.** *In $\mathbb{R}^3$, if*

$$\Theta\left(a_1 u_t^2 + a_2 u_x^2 + a_3 u_y^2 + a_4 u_t u_x + a_5 u_x u_y + a_6 u_y u_t\right) + A\left(a_7 u_t + a_8 u_x + a_9 u_y\right) = B, \tag{26}$$

*where $a_i$ are any known constants $(1 \le i \le 9)$, $\Theta = \Theta(t, x, y, u, u_t, \ldots u_{txy}, \ldots)$, $A = A(t, x, y)$, $B = B(t, x, y)$, the second type general solution of Eq. (26) is*

$$u = f(p, q) + \frac{\int \frac{B(p,q,r)}{A(p,q,r)} dr}{a_7 k_7 + a_8 k_8 + a_9 k_9}, \tag{27}$$

$$p = k_1 t + k_2 x + k_3 y, q = k_4 t + k_5 x + k_6 y, r = k_7 t + k_8 x + k_9 y, \tag{28}$$

*where $f$ is a random first differentiable function, and the constants $k_1, k_2, \ldots, k_9$ must satisfy*

$$-k_3 k_5 k_7 + k_2 k_6 k_7 + k_3 k_4 k_8 - k_1 k_6 k_8 - k_2 k_4 k_9 + k_1 k_5 k_9 \neq 0, \tag{29}$$

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_1 k_2 + a_5 k_2 k_3 + a_6 k_1 k_3 = 0, \tag{30}$$

$$a_1 k_4^2 + a_2 k_5^2 + a_3 k_6^2 + a_4 k_4 k_5 + a_5 k_5 k_6 + a_6 k_4 k_6 = 0, \tag{31}$$

$$a_1 k_7^2 + a_2 k_8^2 + a_3 k_9^2 + a_4 k_7 k_8 + a_5 k_8 k_9 + a_6 k_7 k_9 = 0, \tag{32}$$

$$2a_1 k_1 k_4 + 2a_2 k_2 k_5 + 2a_3 k_3 k_6 + a_4 (k_1 k_5 + k_2 k_4) + a_5 (k_2 k_6 + k_3 k_5) + a_6 (k_3 k_4 + k_1 k_6) = 0, \tag{33}$$

$$2a_1 k_4 k_7 + 2a_2 k_5 k_8 + 2a_3 k_6 k_9 + a_4 (k_4 k_8 + k_5 k_7) + a_5 (k_5 k_9 + k_6 k_8) + a_6 (k_6 k_7 + k_4 k_9) = 0, \tag{34}$$

$$2a_1k_1k_7 + 2a_2k_2k_8 + 2a_3k_3k_9 + a_4\left(k_1k_8 + k_2k_7\right) + a_5\left(k_2k_9 + k_3k_8\right) + a_6\left(k_3k_7 + k_1k_9\right) = 0, \quad (35)$$

$$a_7k_1 + a_8k_2 + a_9k_3 = 0, \quad (36)$$

$$a_7k_4 + a_8k_5 + a_9k_6 = 0. \quad (37)$$

**Proof.** By $Z_1$ transformation, set $u = u(p,q,r)$,$p = k_1t + k_2x + k_3y$, $q = k_4t + k_5x + k_6y, r = k_7t + k_8x + k_9y$. $k_1, k_2, \ldots, k_9$ are undetermined constants, $p, q$ and $r$ are independent of each other, so

$$-k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_1k_6k_8 - k_2k_4k_9 + k_1k_5k_9 \neq 0,$$

and

$$\begin{aligned}
&\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t\right) + A\left(a_7u_t + a_8u_x + a_9u_y\right)\\
&= a_1\Theta(k_1u_p + k_4u_q + k_7u_r)^2 + a_2\Theta(k_2u_p + k_5u_q + k_8u_r)^2\\
&+a_3\Theta(k_3u_p + k_6u_q + k_9u_r)^2 + a_4\Theta\left(k_1u_p + k_4u_q + k_7u_r\right)\left(k_2u_p + k_5u_q + k_8u_r\right)\\
&+a_5\Theta\left(k_2u_p + k_5u_q + k_8u_r\right)\left(k_3u_p + k_6u_q + k_9u_r\right)\\
&+a_6\Theta\left(k_3u_p + k_6u_q + k_9u_r\right)\left(k_1u_p + k_4u_q + k_7u_r\right)\\
&+A\left(a_7\left(k_1u_p + k_4u_q + k_7u_r\right) + a_8\left(k_2u_p + k_5u_q + k_8u_r\right) + a_9\left(k_3u_p + k_6u_q + k_9u_r\right)\right)\\
&= \Theta\left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3\right)u_p^2\\
&+\Theta\left(a_1k_4^2 + a_2k_5^2 + a_3k_6^2 + a_4k_4k_5 + a_5k_5k_6 + a_6k_4k_6\right)u_q^2\\
&+\Theta\left(a_1k_7^2 + a_2k_8^2 + a_3k_9^2 + a_4k_7k_8 + a_5k_8k_9 + a_6k_7k_9\right)u_r^2\\
&+\Theta\left(2a_1k_1k_4 + 2a_2k_2k_5 + 2a_3k_3k_6 + a_4\left(k_1k_5 + k_2k_4\right) + a_5\left(k_2k_6 + k_3k_5\right) + a_6\left(k_3k_4 + k_1k_6\right)\right)\\
&u_pu_q\\
&+\Theta\left(2a_1k_4k_7 + 2a_2k_5k_8 + 2a_3k_6k_9 + a_4\left(k_4k_8 + k_5k_7\right) + a_5\left(k_5k_9 + k_6k_8\right) + a_6\left(k_6k_7 + k_4k_9\right)\right)\\
&u_qu_r\\
&+\Theta\left(2a_1k_1k_7 + 2a_2k_2k_8 + 2a_3k_3k_9 + a_4\left(k_1k_8 + k_2k_7\right) + a_5\left(k_2k_9 + k_3k_8\right) + a_6\left(k_3k_7 + k_1k_9\right)\right)\\
&u_ru_p\\
&+A\left(\left(a_7k_1 + a_8k_2 + a_9k_3\right)u_p + \left(a_7k_4 + a_8k_5 + a_9k_6\right)u_q + \left(a_7k_7 + a_8k_8 + a_9k_9\right)u_r\right)\\
&= B\left(p,q,r\right).
\end{aligned}$$

Set

$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0,$$

$$a_1k_4^2 + a_2k_5^2 + a_3k_6^2 + a_4k_4k_5 + a_5k_5k_6 + a_6k_4k_6 = 0,$$

$$a_1k_7^2 + a_2k_8^2 + a_3k_9^2 + a_4k_7k_8 + a_5k_8k_9 + a_6k_7k_9 = 0,$$

$$2a_1k_1k_4 + 2a_2k_2k_5 + 2a_3k_3k_6 + a_4\left(k_1k_5 + k_2k_4\right) + a_5\left(k_2k_6 + k_3k_5\right) + a_6\left(k_3k_4 + k_1k_6\right) = 0,$$

$$2a_1k_4k_7 + 2a_2k_5k_8 + 2a_3k_6k_9 + a_4\left(k_4k_8 + k_5k_7\right) + a_5\left(k_5k_9 + k_6k_8\right) + a_6\left(k_6k_7 + k_4k_9\right) = 0,$$

$$2a_1k_1k_7 + 2a_2k_2k_8 + 2a_3k_3k_9 + a_4\left(k_1k_8 + k_2k_7\right) + a_5\left(k_2k_9 + k_3k_8\right) + a_6\left(k_3k_7 + k_1k_9\right) = 0,$$

$$a_7k_1 + a_8k_2 + a_9k_3 = 0,$$

$$a_7k_4 + a_8k_5 + a_9k_6 = 0.$$

We get

$$\left(a_7k_7 + a_8k_8 + a_9k_9\right)u_r = \frac{B\left(p,q,r\right)}{A\left(p,q,r\right)}. \quad (38)$$

The general solution of (38) is

$$u = f\left(p,q\right) + \frac{\int \frac{B(p,q,r)}{A(p,q,r)}dr}{a_7k_7 + a_8k_8 + a_9k_9},$$

where $f$ is a random first differentiable function, so the theorem is proved. □

Similar to the proof method of Theorem 5, we can obtain Theorem 6.

**Theorem 6.** *In* $\mathbb{R}^3$*, if*

$$A\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t\right)+\Theta\left(a_7u_t+a_8u_x+a_9u_y\right)=B, \tag{39}$$

*where $a_i$ are any known constants* $(1\le i\le 9)$, $\Theta=\Theta\left(t,x,y,u,u_t,\ldots u_{txy},\ldots\right)$, $A=A\left(t,x,y\right)$, $B=B(t,x,y)$*, the second type general solution of Eq. (39) is*

$$u=f\left(p,q\right)+\int\left(\frac{B\left(p,q,r\right)}{A\left(p,q,r\right)\left(a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9\right)}\right)^{\frac{1}{2}}dr, \tag{40}$$

$$p=k_1t+k_2x+k_3y, q=k_4t+k_5x+k_6y, r=k_7t+k_8x+k_9y,$$

*where $f$ is a random first differentiable function, and the constants $k_1,k_2,\ldots,k_9$ must satisfy*

$$-k_3k_5k_7+k_2k_6k_7+k_3k_4k_8-k_1k_6k_8-k_2k_4k_9+k_1k_5k_9\neq 0,$$

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,$$
$$a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6=0,$$
$$2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)=0,$$
$$2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)=0,$$
$$2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)+a_5\left(k_2k_9+k_3k_8\right)+a_6\left(k_3k_7+k_1k_9\right)=0,$$
$$a_7k_1+a_8k_2+a_9k_3=0,$$
$$a_7k_4+a_8k_5+a_9k_6=0,$$
$$a_7k_7+a_8k_8+a_9k_9=0.$$

Next, we use Theorem 5 to study a definite solution problem.

**Example 5.** In $\mathbb{R}^3$, use Theorem 5 to obtain the exact solution of

$$u^2\left(9u_t^2+4u_x^2+u_y^2-12u_tu_x-4u_xu_y+6u_yu_t\right)+3u_t-2u_x+u_y=4e^{t+x+y}+16e^{2t+2x+2y}, \tag{41}$$

in the condition of $u(0,y,z)=g(x,y)$, $g$ is an any known first differentiable function.

**Solution.** By Theorem 5, the general solution of (41) is

$$u=f\left(t+k_2x+\left(2k_2-3\right)y,t+k_5x+\left(2k_5-3\right)y\right)+2e^{t+x+y}.$$

So

$$u\left(0,x,y\right)=f\left(k_2x+\left(2k_2-3\right)y,k_5x+\left(2k_5-3\right)y\right)+2e^{x+y}=g\left(x,y\right).$$

Namely

$$f\left(k_2x+\left(2k_2-3\right)y,k_5x+\left(2k_5-3\right)y\right)=g\left(x,y\right)-2e^{x+y}.$$

Set

$$k_2x+\left(2k_2-3\right)y=\beta, k_5x+\left(2k_5-3\right)y=\gamma.$$

We obtain

$$x=\frac{(2k_2-3)\,\gamma+(3-2k_5)\,\beta}{3\,(k_2-k_5)},y=\frac{k_5\beta-k_2\gamma}{3\,(k_2-k_5)}.$$

That is

$$\begin{aligned}&f\left(k_2x+(2k_2-3)\,y,k_5x+(2k_5-3)\,y\right)\\&=g\left(\tfrac{(2k_2-3)\gamma+(3-2k_5)\beta}{3(k_2-k_5)},\tfrac{k_5\beta-k_2\gamma}{3(k_2-k_5)}\right)-2e^{\frac{(k_2-3)\gamma+(3-k_5)\beta}{3(k_2-k_5)}}.\end{aligned}$$

Set

$$t+k_2x+(2k_2-3)\,y=\beta,t+k_5x+(2k_5-3)\,y=\gamma.$$

Then

$$\frac{(2k_2-3)\,\gamma+(3-2k_5)\,\beta}{3\,(k_2-k_5)}=\frac{2t}{3}+x,$$
$$\frac{k_5\beta-k_2\gamma}{3\,(k_2-k_5)}=\frac{t}{3}+y,$$
$$\frac{(k_2-3)\,\gamma+(3-k_5)\,\beta}{3\,(k_2-k_5)}=t+x+y.$$

So

$$u\,(t,x,y)=f\,(t+k_2x+(2k_2-1)\,y,t+k_6x+(2k_6-1)\,y)+2e^{t+x+y}=g\left(\frac{2t}{3}+x,\frac{t}{3}+y\right).$$

According to Example 5, we can directly obtain the exact solution of Eq. (41) in diversified initial value conditions. If the initial value condition is $u\,(0,x,y)=(x+y)^2+\sin{(x-y)}+2e^{x+y}$, the exact solution is $u=(t+x+y)^2+\sin\left(\frac{t}{3}+x-y\right)+2e^{t+x+y}$.

For

$$\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_tu_x\right)+A\,(a_4u_t+a_5u_x)=B, \tag{42}$$
$$A\left(a_1u_t^2+a_2u_x^2+a_3u_tu_x\right)+\Theta\,(a_4u_t+a_5u_x)=B, \tag{43}$$

where $a_i$ are arbitrary known constants $(1\le i\le 5)$, $\Theta=\Theta\,(t,x,u,u_t,\dots u_{tx},\dots)$, $A=A\,(t,x)$, $B=B(t,x)$, and readers can try to get their general solutions by the method similar to Theorem 5.

Next we propose Theorem 7.

**Theorem 7.** *In* $\mathbb{R}^3$, *if*

$$\begin{aligned}&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t+a_7uu_t+a_8uu_x+a_9uu_y\right)\\&+A\,(a_{10}u_t+a_{11}u_x+a_{12}u_y+a_{13}u)=B,\end{aligned} \tag{44}$$

*where* $a_i$ *are arbitrary known constants* $(1\le i\le 13)$, $\Theta=\Theta\,(t,x,y,u,u_t,\dots u_{txy},\dots)$, $A=A\,(t,x,y)$, $B=B(t,x,y)$, *the second type general solution of Eq. (44) is*

$$u=e^{\frac{-a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}}\left(f\,(p,q)+\frac{\int e^{\frac{a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}}\frac{B(p,q,r)}{A(p,q,r)}dr}{a_{10}k_7+a_{11}k_8+a_{12}k_9}\right), \tag{45}$$

$$p=k_1t+k_2x+k_3y,q=k_4t+k_5x+k_6y,r=k_7t+k_8x+k_9y,$$

*where* $f$ *is a random first differentiable function, and the constants* $k_1,k_2,\dots,k_9$ *must satisfy*

$$-k_3k_5k_7+k_2k_6k_7+k_3k_4k_8-k_1k_6k_8-k_2k_4k_9+k_1k_5k_9\neq 0,$$

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,$$
$$a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6=0,$$
$$a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9=0,$$
$$2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)=0,$$
$$2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)=0,$$
$$2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)+a_5\left(k_2k_9+k_3k_8\right)+a_6\left(k_3k_7+k_1k_9\right)=0,$$
$$a_7k_1+a_8k_2+a_9k_3=0,$$
$$a_7k_4+a_8k_5+a_9k_6=0$$
$$a_7k_7+a_8k_8+a_9k_9=0, \tag{46}$$
$$a_{10}k_1+a_{11}k_2+a_{12}k_3=0, \tag{47}$$
$$a_{10}k_4+a_{11}k_5+a_{12}k_6=0. \tag{48}$$

**Proof.** By $Z_1$ transformation, set $u=u(p,q,r)$, $p=k_1t+k_2x+k_3y, q=k_4t+k_5x+k_6y, r=k_7t+k_8x+k_9y$. $k_1,k_2,\ldots,k_9$ are undetermined constants, $p,q$ and $r$ are independent of each other, so

$$-k_3k_5k_7+k_2k_6k_7+k_3k_4k_8-k_1k_6k_8-k_2k_4k_9+k_1k_5k_9\neq 0,$$

and

$$\begin{aligned}
&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_tu_x+a_5u_xu_y+a_6u_yu_t+a_7uu_t+a_8uu_x+a_9uu_y\right)\\
&+A\left(a_{10}u_t+a_{11}u_x+a_{12}u_y+a_{13}u\right)\\
&=a_1\Theta(k_1u_p+k_4u_q+k_7u_r)^2+a_2\Theta(k_2u_p+k_5u_q+k_8u_r)^2\\
&+a_3\Theta(k_3u_p+k_6u_q+k_9u_r)^2+a_4\Theta\left(k_1u_p+k_4u_q+k_7u_r\right)\left(k_2u_p+k_5u_q+k_8u_r\right)\\
&+a_5\Theta\left(k_2u_p+k_5u_q+k_8u_r\right)\left(k_3u_p+k_6u_q+k_9u_r\right)\\
&+a_6\Theta\left(k_3u_p+k_6u_q+k_9u_r\right)\left(k_1u_p+k_4u_q+k_7u_r\right)\\
&+a_7\Theta u\left(k_1u_p+k_4u_q+k_7u_r\right)+a_8\Theta u\left(k_2u_p+k_5u_q+k_8u_r\right)+a_9\Theta u\left(k_3u_p+k_6u_q+k_9u_r\right)\\
&+A\left(a_{10}\left(k_1u_p+k_4u_q+k_7u_r\right)+a_{11}\left(k_2u_p+k_5u_q+k_8u_r\right)+a_{12}\left(k_3u_p+k_6u_q+k_9u_r\right)+a_{13}u\right)\\
&=\Theta\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3\right)u_p^2\\
&+\Theta\left(a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6\right)u_q^2\\
&+\Theta\left(a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9\right)u_r^2\\
&+\Theta\left(2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)\right)\\
&u_pu_q\\
&+\Theta\left(2a_1k_4k_7+2a_2k_5k_8+2a_3k_6k_9+a_4\left(k_4k_8+k_5k_7\right)+a_5\left(k_5k_9+k_6k_8\right)+a_6\left(k_6k_7+k_4k_9\right)\right)\\
&u_qu_r\\
&+\Theta\left(2a_1k_1k_7+2a_2k_2k_8+2a_3k_3k_9+a_4\left(k_1k_8+k_2k_7\right)+a_5\left(k_2k_9+k_3k_8\right)+a_6\left(k_3k_7+k_1k_9\right)\right)\\
&u_ru_p\\
&+\Theta\left(a_7k_1+a_8k_2+a_9k_3\right)uu_p+\Theta\left(a_7k_4+a_8k_5+a_9k_6\right)uu_q+\Theta\left(a_7k_7+a_8k_8+a_9k_9\right)uu_r\\
&+A\left(a_{10}k_1+a_{11}k_2+a_{12}k_3\right)u_p+A\left(a_{10}k_4+a_{11}k_5+a_{12}k_6\right)u_q+A\left(a_{10}k_7+a_{11}k_8+a_{12}k_9\right)u_r\\
&+a_{13}Au=B\left(p,q,r\right).
\end{aligned}$$

Set

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_2k_3+a_6k_1k_3=0,$$
$$a_1k_4^2+a_2k_5^2+a_3k_6^2+a_4k_4k_5+a_5k_5k_6+a_6k_4k_6=0,$$
$$a_1k_7^2+a_2k_8^2+a_3k_9^2+a_4k_7k_8+a_5k_8k_9+a_6k_7k_9=0,$$
$$2a_1k_1k_4+2a_2k_2k_5+2a_3k_3k_6+a_4\left(k_1k_5+k_2k_4\right)+a_5\left(k_2k_6+k_3k_5\right)+a_6\left(k_3k_4+k_1k_6\right)=0,$$

$$2a_1k_4k_7 + 2a_2k_5k_8 + 2a_3k_6k_9 + a_4(k_4k_8 + k_5k_7) + a_5(k_5k_9 + k_6k_8) + a_6(k_6k_7 + k_4k_9) = 0,$$
$$2a_1k_1k_7 + 2a_2k_2k_8 + 2a_3k_3k_9 + a_4(k_1k_8 + k_2k_7) + a_5(k_2k_9 + k_3k_8) + a_6(k_3k_7 + k_1k_9) = 0,$$
$$a_7k_1 + a_8k_2 + a_9k_3 = 0,$$
$$a_7k_4 + a_8k_5 + a_9k_6 = 0,$$
$$a_7k_7 + a_8k_8 + a_9k_9 = 0,$$
$$a_{10}k_1 + a_{11}k_2 + a_{12}k_3 = 0,$$
$$a_{10}k_4 + a_{11}k_5 + a_{12}k_6 = 0.$$

We get

$$(a_{10}k_7 + a_{11}k_8 + a_{12}k_9)\,u_r + a_{13}u = \frac{B(p,q,r)}{A(p,q,r)}. \tag{49}$$

The general solution of Eq. (49) is

$$u = e^{\frac{-a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}}\left(f(p,q) + \frac{\int e^{\frac{a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}}\frac{B(p,q,r)}{A(p,q,r)}dr}{a_{10}k_7 + a_{11}k_8 + a_{12}k_9}\right),$$

where $f$ is a random first differentiable function, so the theorem is proved. □

Similar to the proof method of Theorem 7, we can get Theorem 8.

**Theorem 8.** *In* $\mathbb{R}^3$, *if*

$$\begin{aligned}&\Theta\left(a_1u_t^2 + a_2u_x^2 + a_3u_y^2 + a_4u_tu_x + a_5u_xu_y + a_6u_yu_t + a_7uu_t + a_8uu_x + a_9uu_y\right)\\&+\Lambda(a_{10}u_t + a_{11}u_x + a_{12}u_y + a_{13}u) = 0,\end{aligned} \tag{50}$$

*where* $a_i$ *are arbitrary known constants* $(1 \le i \le 13)$, $\Theta = \Theta(t,x,y,u,u_t,\ldots u_{txy},\ldots)$, $\Lambda = \Lambda(t,x,y,u,u_t,\ldots u_{txy},\ldots)$, *the second type general solution of Eq. (50) is*

$$u = e^{\frac{-a_{13}r}{a_{10}k_7+a_{11}k_8+a_{12}k_9}} f(p,q), \tag{51}$$

$$p = k_1t + k_2x + k_3y,\ q = k_4t + k_5x + k_6y,\ r = k_7t + k_8x + k_9y,$$

*where* $f$ *is a random first differentiable function, and the constants* $k_1, k_2, \ldots, k_9$ *must satisfy*

$$-k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_1k_6k_8 - k_2k_4k_9 + k_1k_5k_9 \neq 0,$$
$$a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_2k_3 + a_6k_1k_3 = 0,$$
$$a_1k_4^2 + a_2k_5^2 + a_3k_6^2 + a_4k_4k_5 + a_5k_5k_6 + a_6k_4k_6 = 0,$$
$$a_1k_7^2 + a_2k_8^2 + a_3k_9^2 + a_4k_7k_8 + a_5k_8k_9 + a_6k_7k_9 = 0,$$
$$2a_1k_1k_4 + 2a_2k_2k_5 + 2a_3k_3k_6 + a_4(k_1k_5 + k_2k_4) + a_5(k_2k_6 + k_3k_5) + a_6(k_3k_4 + k_1k_6) = 0,$$
$$2a_1k_4k_7 + 2a_2k_5k_8 + 2a_3k_6k_9 + a_4(k_4k_8 + k_5k_7) + a_5(k_5k_9 + k_6k_8) + a_6(k_6k_7 + k_4k_9) = 0,$$
$$2a_1k_1k_7 + 2a_2k_2k_8 + 2a_3k_3k_9 + a_4(k_1k_8 + k_2k_7) + a_5(k_2k_9 + k_3k_8) + a_6(k_3k_7 + k_1k_9) = 0,$$
$$a_7k_1 + a_8k_2 + a_9k_3 = 0,$$
$$a_7k_4 + a_8k_5 + a_9k_6 = 0,$$
$$a_7k_7 + a_8k_8 + a_9k_9 = 0,$$

$$a_{10}k_1 + a_{11}k_2 + a_{12}k_3 = 0,$$
$$a_{10}k_4 + a_{11}k_5 + a_{12}k_6 = 0.$$

Next, we use Theorem 8 to study a definite solution problem.

**Example 6.** In $\mathbb{R}^4$, use Theorem 8 to get the exact solution of

$$2u_t^2 + u_x^2 + u_y^2 + 3u_tu_x + 3u_tu_y + 2u_xu_y + u\left(u_t + u_x + u_y\right) + 2u_t + u_x + u_y + u = 0, \qquad (52)$$

in the condition of $u(0, x, y) = g(x, y)$.

**Solution.** According to Theorem 8, the general solution of (52) is

$$u = e^{\frac{-t}{2}} f\left(\frac{\left(-c_1 - c_2\right)t}{2} + c_1x + c_2y, \frac{\left(-c_3 - c_4\right)t}{2} + c_3x + c_4y\right), \qquad (53)$$

or

$$u = e^{\frac{-x}{2}} f\left(\left(-c_1 - c_2\right)t + c_1x + c_2y, \left(-c_3 - c_4\right)t + c_3x + c_4y\right), \qquad (54)$$

where $c_1, c_2, c_3$ and $c_4$ are arbitrary constants. If the general solution is (53), so

$$u\left(0, x, y, z\right) = u\left(0, x, y\right) = f\left(c_1x + c_2y, c_3x + c_4y\right) = g\left(x, y\right).$$

Set

$$c_1x + c_2y = \beta, c_3x + c_4y = \gamma.$$

We obtain

$$x = \frac{c_4\beta - c_2\gamma}{c_1c_4 - c_2c_3}, y = \frac{c_1\gamma - c_3\beta}{c_1c_4 - c_2c_3}.$$

Namely

$$f\left(c_1x + c_2y, c_3x + c_4y\right) = g\left(\frac{c_4\beta - c_2\gamma}{c_1c_4 - c_2c_3}, \frac{c_1\gamma - c_3\beta}{c_1c_4 - c_2c_3}\right).$$

Set

$$\frac{\left(-c_1 - c_2\right)t}{2} + c_1x + c_2y = \beta, \frac{\left(-c_3 - c_4\right)t}{2} + c_3x + c_4y = \gamma.$$

Then

$$\frac{c_4\beta - c_2\gamma}{c_1c_4 - c_2c_3} = -\frac{t}{2} + x,$$
$$\frac{c_1\gamma - c_3\beta}{c_1c_4 - c_2c_3} = -\frac{t}{2} + y.$$

So the exact solution of the definite solution problem is

$$u\left(t, x, y\right) = g\left(-\frac{t}{2} + x, -\frac{t}{2} + y\right). \qquad (55)$$

If the general solution is (54), In a similar way, we can obtain

$$u\left(t, x, y\right) = g\left(-t + x, -t + y\right). \qquad (56)$$

Namely, if the general solution of a PDE is not unique, the exact solution of its definite solution problem might not be unique either.

Next we propose Theorem 9.

**Theorem 9.** *In* $\mathbb{R}^4$, *if*

$$\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_z^2+a_5u_tu_x+a_6u_tu_y+a_7u_tu_z+a_8u_xu_y+a_9u_xu_z+a_{10}u_yu_z\right)$$
$$+\Lambda\left(a_{11}u_t+a_{12}u_x+a_{13}u_y+a_{14}u_z\right)=0, \tag{57}$$

*where* $a_i$ *are arbitrary known constants* $(1\le i\le 14)$, $\Theta=\Theta\left(t,x,y,z,u,u_t,\ldots u_{txyz},\ldots\right)$, $\Lambda=\Lambda\left(t,x,y,z,u,u_t,\ldots u_{txyz},\ldots\right)$, *the third type general solution of Eq. (57) is*

$$u=f(p,q,r), \tag{58}$$

$$p=k_1t+k_2x+k_3y+k_4z, \tag{59}$$

$$q=k_5t+k_6x+k_7y+k_8z, \tag{60}$$

$$r=k_9t+k_{10}x+k_{11}y+k_{12}z, \tag{61}$$

*where* $f$ *is a random first differentiable function;* $p,q$ *and* $r$ *are independent of each other, and the constants* $k_1,k_2,\ldots,k_{12}$ *must satisfy*

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2+a_5k_1k_2+a_6k_1k_3+a_7k_1k_4+a_8k_2k_3+a_9k_2k_4+a_{10}k_3k_4=0, \tag{62}$$

$$a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_8^2+a_5k_5k_6+a_6k_5k_7+a_7k_5k_8+a_8k_6k_7+a_9k_6k_8+a_{10}k_7k_8=0, \tag{63}$$

$$a_1k_9^2+a_2k_{10}^2+a_3k_{11}^2+a_4k_{12}^2+a_5k_9k_{10}+a_6k_9k_{11}+a_7k_9k_{12}+a_8k_{10}k_{11}+a_9k_{10}k_{12}+a_{10}k_{11}k_{12}=0, \tag{64}$$

$$2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+2a_4k_4k_8+a_5\left(k_1k_6+k_2k_5\right)+a_6\left(k_1k_7+k_3k_5\right)+a_7\left(k_1k_8+k_4k_5\right)$$
$$+a_8\left(k_2k_7+k_3k_6\right)+a_9\left(k_2k_8+k_4k_6\right)+a_{10}\left(k_3k_8+k_4k_7\right)=0, \tag{65}$$

$$2a_1k_1k_9+2a_2k_2k_{10}+2a_3k_3k_{11}+2a_4k_4k_{12}+a_5\left(k_1k_{10}+k_2k_9\right)+a_6\left(k_1k_{11}+k_3k_9\right)$$
$$+a_7\left(k_1k_{12}+k_4k_9\right)+a_8\left(k_2k_{11}+k_3k_{10}\right)+a_9\left(k_2k_{12}+k_4k_{10}\right)+a_{10}\left(k_3k_{12}+k_4k_{11}\right)=0, \tag{66}$$

$$2a_1k_5k_9+2a_2k_6k_{10}+2a_3k_7k_{11}+2a_4k_8k_{12}+a_5\left(k_5k_{10}+k_6k_9\right)+a_6\left(k_5k_{11}+k_7k_9\right)$$
$$+a_7\left(k_5k_{12}+k_8k_9\right)+a_8\left(k_6k_{11}+k_7k_{10}\right)+a_9\left(k_6k_{12}+k_8k_{10}\right)+a_{10}\left(k_7k_{12}+k_8k_{11}\right)=0, \tag{67}$$

$$a_{11}k_1+a_{12}k_2+a_{13}k_3+a_{14}k_4=0, \tag{68}$$

$$a_{11}k_5+a_{12}k_6+a_{13}k_7+a_{14}k_8=0, \tag{69}$$

$$a_{11}k_9+a_{12}k_{10}+a_{13}k_{11}+a_{14}k_{12}=0. \tag{70}$$

**Proof.** By $Z_1$ transformation, set $u=f(p,q,r)$, $p=k_1t+k_2x+k_3y+k_4z, q=k_5t+k_6x+k_7y+k_8z, k_9t+k_{10}x+k_{11}y+k_{12}z$; $k_1,k_2,\ldots,k_{12}$ are undetermined constants, $p,q$ and $r$ are independent of each other, so

$$
\begin{aligned}
&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_z^2+a_5u_tu_x+a_6u_tu_y+a_7u_tu_z+a_8u_xu_y+a_9u_xu_z+a_{10}u_yu_z\right)\\
&+\Lambda\left(a_{11}u_t+a_{12}u_x+a_{13}u_y+a_{14}u_z\right)\\
&=a_1\Theta(k_1f_p+k_5f_q+k_9f_r)^2+a_2\Theta(k_2f_p+k_6f_q+k_{10}f_r)^2+a_3\Theta(k_3f_p+k_7f_q+k_{11}f_r)^2\\
&+a_4\Theta(k_4f_p+k_8f_q+k_{12}f_r)^2+a_5\Theta\left(k_1f_p+k_5f_q+k_9f_r\right)\left(k_2f_p+k_6f_q+k_{10}f_r\right)\\
&+a_6\Theta\left(k_1f_p+k_5f_q+k_9f_r\right)\left(k_3f_p+k_7f_q+k_{11}f_r\right)\\
&+a_7\Theta\left(k_1f_p+k_5f_q+k_9f_r\right)\left(k_4f_p+k_8f_q+k_{12}f_r\right)\\
&+a_8\Theta\left(k_2f_p+k_6f_q+k_{10}f_r\right)\left(k_3f_p+k_7f_q+k_{11}f_r\right)\\
&+a_9\Theta\left(k_2f_p+k_6f_q+k_{10}f_r\right)\left(k_4f_p+k_8f_q+k_{12}f_r\right)\\
&+a_{10}\Theta\left(k_3f_p+k_7f_q+k_{11}f_r\right)\left(k_4f_p+k_8f_q+k_{12}f_r\right)+a_{11}\Lambda\left(k_1f_p+k_5f_q+k_9f_r\right)\\
&+a_{12}\Lambda\left(k_2f_p+k_6f_q+k_{10}f_r\right)+a_{13}\Lambda\left(k_3f_p+k_7f_q+k_{11}f_r\right)+a_{14}\Lambda\left(k_4f_p+k_8f_q+k_{12}f_r\right)\\
&=\Theta f_p^2\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2+a_5k_1k_2+a_6k_1k_3+a_7k_1k_4+a_8k_2k_3+a_9k_2k_4+a_{10}k_3k_4\right)\\
&+\Theta f_q^2\left(a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_8^2+a_5k_5k_6+a_6k_5k_7+a_7k_5k_8+a_8k_6k_7+a_9k_6k_8+a_{10}k_7k_8\right)\\
&+\Theta f_r^2a_1k_9^2+\Theta f_r^2a_2k_{10}^2+\Theta f_r^2a_3k_{11}^2+\Theta f_r^2a_4k_{12}^2+\Theta f_r^2a_5k_9k_{10}+\Theta f_r^2a_6k_9k_{11}+\Theta f_r^2a_7k_9k_{12}\\
&+\Theta f_r^2a_8k_{10}k_{11}+\Theta f_r^2a_9k_{10}k_{12}+\Theta f_r^2a_{10}k_{11}k_{12}+2\Theta f_pf_qa_1k_1k_5+2\Theta f_pf_qa_2k_2k_6\\
&+2\Theta f_pf_qa_3k_3k_7+2\Theta f_pf_qa_4k_4k_8+\Theta f_pf_qa_5\left(k_1k_6+k_2k_5\right)+\Theta f_pf_qa_6\left(k_1k_7+k_3k_5\right)\\
&+\Theta f_pf_qa_7\left(k_1k_8+k_4k_5\right)+\Theta f_pf_qa_8\left(k_2k_7+k_3k_6\right)+\Theta f_pf_qa_9\left(k_2k_8+k_4k_6\right)\\
&+\Theta f_pf_qa_{10}\left(k_3k_8+k_4k_7\right)+2\Theta f_pf_ra_1k_1k_9+2\Theta f_pf_ra_2k_2k_{10}+2\Theta f_pf_ra_3k_3k_{11}+2\Theta f_pf_ra_4k_4k_{12}\\
&+\Theta f_pf_ra_5\left(k_1k_{10}+k_2k_9\right)+\Theta f_pf_ra_6\left(k_1k_{11}+k_3k_9\right)+\Theta f_pf_ra_7\left(k_1k_{12}+k_4k_9\right)\\
&+\Theta f_pf_ra_8\left(k_2k_{11}+k_3k_{10}\right)+\Theta f_pf_ra_9\left(k_2k_{12}+k_4k_{10}\right)+\Theta f_pf_ra_{10}\left(k_3k_{12}+k_4k_{11}\right)\\
&+2\Theta f_qf_ra_1k_5k_9+2\Theta f_qf_ra_2k_6k_{10}+2\Theta f_qf_ra_3k_7k_{11}+2\Theta f_qf_ra_4k_8k_{12}+\Theta f_qf_ra_5\left(k_5k_{10}+k_6k_9\right)\\
&+\Theta f_qf_ra_6\left(k_5k_{11}+k_7k_9\right)+\Theta f_qf_ra_7\left(k_5k_{12}+k_8k_9\right)+\Theta f_qf_ra_8\left(k_6k_{11}+k_7k_{10}\right)\\
&+\Theta f_qf_ra_9\left(k_6k_{12}+k_8k_{10}\right)+\Theta f_qf_ra_{10}\left(k_7k_{12}+k_8k_{11}\right)+\Lambda f_p\left(a_{11}k_1+a_{12}k_2+a_{13}k_3+a_{14}k_4\right)\\
&+\Lambda f_q\left(a_{11}k_5+a_{12}k_6+a_{13}k_7+a_{14}k_8\right)+\Lambda f_r\left(a_{11}k_9+a_{12}k_{10}+a_{13}k_{11}+a_{14}k_{12}\right)=0.
\end{aligned}
$$

Set

$$a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2+a_5k_1k_2+a_6k_1k_3+a_7k_1k_4+a_8k_2k_3+a_9k_2k_4+a_{10}k_3k_4=0,$$

$$a_1k_5^2+a_2k_6^2+a_3k_7^2+a_4k_8^2+a_5k_5k_6+a_6k_5k_7+a_7k_5k_8+a_8k_6k_7+a_9k_6k_8+a_{10}k_7k_8=0,$$

$$
\begin{aligned}
&a_1k_9^2+a_2k_{10}^2+a_3k_{11}^2+a_4k_{12}^2+a_5k_9k_{10}+a_6k_9k_{11}+a_7k_9k_{12}+a_8k_{10}k_{11}+a_9k_{10}k_{12}\\
&+a_{10}k_{11}k_{12}=0,
\end{aligned}
$$

$$
\begin{aligned}
&2a_1k_1k_5+2a_2k_2k_6+2a_3k_3k_7+2a_4k_4k_8+a_5\left(k_1k_6+k_2k_5\right)+a_6\left(k_1k_7+k_3k_5\right)\\
&+a_7\left(k_1k_8+k_4k_5\right)+a_8\left(k_2k_7+k_3k_6\right)+a_9\left(k_2k_8+k_4k_6\right)+a_{10}\left(k_3k_8+k_4k_7\right)=0,
\end{aligned}
$$

$$
\begin{aligned}
&2a_1k_1k_9+2a_2k_2k_{10}+2a_3k_3k_{11}+2a_4k_4k_{12}+a_5\left(k_1k_{10}+k_2k_9\right)+a_6\left(k_1k_{11}+k_3k_9\right)\\
&+a_7\left(k_1k_{12}+k_4k_9\right)+a_8\left(k_2k_{11}+k_3k_{10}\right)+a_9\left(k_2k_{12}+k_4k_{10}\right)+a_{10}\left(k_3k_{12}+k_4k_{11}\right)=0,
\end{aligned}
$$

$$
\begin{aligned}
&2a_1k_5k_9+2a_2k_6k_{10}+2a_3k_7k_{11}+2a_4k_8k_{12}+a_5\left(k_5k_{10}+k_6k_9\right)+a_6\left(k_5k_{11}+k_7k_9\right)\\
&+a_7\left(k_5k_{12}+k_8k_9\right)+a_8\left(k_6k_{11}+k_7k_{10}\right)+a_9\left(k_6k_{12}+k_8k_{10}\right)+a_{10}\left(k_7k_{12}+k_8k_{11}\right)=0,
\end{aligned}
$$

$$a_{11}k_1+a_{12}k_2+a_{13}k_3+a_{14}k_4=0,$$

$$a_{11}k_5+a_{12}k_6+a_{13}k_7+a_{14}k_8=0,$$

$$a_{11}k_9+a_{12}k_{10}+a_{13}k_{11}+a_{14}k_{12}=0.$$

Therefore, the general solution of Eq. (57) is

$$u = f(p, q, r)$$

The theorem is proven. □

In $\mathbb{R}^4$, if

$$\begin{aligned}&\Theta\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_z^2+a_5u_tu_x+a_6u_tu_y+a_7u_tu_z+a_8u_xu_y+a_9u_xu_z+a_{10}u_yu_z\right)\\&+A\left(a_{11}u_t+a_{12}u_x+a_{13}u_y+a_{14}u_z+u\right)=B,\end{aligned} \tag{71}$$

$$\begin{aligned}&A\left(a_1u_t^2+a_2u_x^2+a_3u_y^2+a_4u_z^2+a_5u_tu_x+a_6u_tu_y+a_7u_tu_z+a_8u_xu_y+a_9u_xu_z+a_{10}u_yu_z\right)\\&+\Theta\left(a_{11}u_t+a_{12}u_x+a_{13}u_y+a_{14}u_z+u\right)=B,\end{aligned} \tag{72}$$

where $a_i$ are any known constants $(1 \le i \le 14)$, $\Theta = \Theta(t, x, y, z, u, u_t, \ldots u_{txyz}, \ldots)$, $A = A(t, x, y, z)$, $B = B(t, x, y, z)$, set

$$T = k_1t + k_2x + k_3y + k_4z, \tag{73}$$

$$X = k_5t + k_6x + k_7y + k_8z, \tag{74}$$

$$Y = k_9t + k_{10}x + k_{11}y + k_{12}z, \tag{75}$$

$$Z = k_{13}t + k_{14}x + k_{15}y + k_{16}z, \tag{76}$$

$$\frac{\partial(X, Y, Z, T)}{\partial(x, y, z, t)} \neq 0. \tag{77}$$

Similar to the calculation of Theorem 5, the general solutions of (71,72) can be get, and readers could try it by themselves.

Next, we use Theorem 9 to study a definite solution problem.

**Example 7.** In $\mathbb{R}^4$, use Theorem 9 to get the exact solution of

$$\left(u_t^2 + u_x^2 - 2u_tu_x + u_tu_y + u_tu_z - u_xu_y - u_xu_z\right)^2 + (u_t - u_x)^3 = 0, \tag{78}$$

in the condition of $u(0, x, y, z) = g(x, y, z)$.

**Solution.** According to Theorem 9, the general solution of (78) is

$$u(x, y, z) = f(t + x + k_3y + k_4z, t + x + k_7y + k_8z, t + x + k_{11}y + k_{12}z), \tag{79}$$

where $k_3, k_4, k_7, k_8, k_{11}$ and $k_{12}$ are random constants, so

$$u(0, x, y, z) = f(x + k_3y + k_4z, x + k_7y + k_8z, x + k_{11}y + k_{12}z) = g(x, y, z).$$

Set

$$x + k_3y + k_4z = \alpha, x + k_7y + k_8z = \beta, x + k_{11}y + k_{12}z = \gamma.$$

We obtain

$$x = -\frac{-\gamma k_4k_7 + \gamma k_3k_8 + \beta k_4k_{11} - \alpha k_8k_{11} - \beta k_3k_{12} + \alpha k_7k_{12}}{k_4k_7 - k_3k_8 - k_4k_{11} + k_8k_{11} + k_3k_{12} - k_7k_{12}}, \tag{80}$$

$$y = -\frac{-\beta k_4 + \gamma k_4 + \alpha k_8 - \gamma k_8 - \alpha k_{12} + \beta k_{12}}{k_4k_7 - k_3k_8 - k_4k_{11} + k_8k_{11} + k_3k_{12} - k_7k_{12}}, \tag{81}$$

$$z=-\frac{-\beta k_3+\gamma k_3+\alpha k_7-\gamma k_7-\alpha k_{11}+\beta k_{11}}{-k_4k_7+k_3k_8+k_4k_{11}-k_8k_{11}-k_3k_{12}+k_7k_{12}}. \tag{82}$$

Namely

$$\begin{aligned}&u\left(0,x,y,z\right)=f\left(x+k_3y+k_4z,x+k_7y+k_8z,x+k_{11}y+k_{12}z\right)\\&=g\left(-\tfrac{-\gamma k_4k_7+\ldots+\alpha k_7k_{12}}{k_4k_7-\ldots-k_7k_{12}},-\tfrac{-\beta k_4+\ldots+\beta k_{12}}{k_4k_7-\ldots-k_7k_{12}},-\tfrac{-\beta k_3+\ldots+\beta k_{11}}{-k_4k_7+\ldots+k_7k_{12}}\right).\end{aligned}$$

Set

$$t+x+k_3y+k_4z=\alpha,t+x+k_7y+k_8z=\beta,t+x+k_{11}y+k_{12}z=\gamma.$$

Then

$$\begin{aligned}-\frac{-\gamma k_4k_7+\gamma k_3k_8+\beta k_4k_{11}-\alpha k_8k_{11}-\beta k_3k_{12}+\alpha k_7k_{12}}{k_4k_7-k_3k_8-k_4k_{11}+k_8k_{11}+k_3k_{12}-k_7k_{12}}&=t+x,\\-\frac{-\beta k_4+\gamma k_4+\alpha k_8-\gamma k_8-\alpha k_{12}+\beta k_{12}}{k_4k_7-k_3k_8-k_4k_{11}+k_8k_{11}+k_3k_{12}-k_7k_{12}}&=y,\\-\frac{-\beta k_3+\gamma k_3+\alpha k_7-\gamma k_7-\alpha k_{11}+\beta k_{11}}{-k_4k_7+k_3k_8+k_4k_{11}-k_8k_{11}-k_3k_{12}+k_7k_{12}}&=z.\end{aligned}$$

So the exact solution of the definite solution problem is

$$u(t,x,y,z)=g(t+x,y,z)$$

According to Example 7, if the initial value condition is $u\left(0,x,y,z\right)=\left(x+2y+z\right)^3+\cos\left(x+y+2z\right)+\tan\left(2x-y-z\right)$, the exact solution is $u=\left(t+x+2y+z\right)^3+\cos\left(t+x+y+2z\right)+\tan\left(2t+2x-y-z\right)$.

Theorem 10 is presented below.

**Theorem 10.** *In $\mathbb{R}^3$, if*

$$a_1u_t^3+a_2u_x^3+a_3u_y^3+a_4u_t^2u_x+a_5u_t^2u_y+a_6u_x^2u_t+a_7u_x^2u_y+a_8u_y^2u_t+a_9u_y^2u_x=0, \tag{83}$$

*where $a_i$ are arbitrary known constants $(1\le i\le 9)$, the second type general solution of Eq. (83) is*

$$u=f(v,w)$$

$$v=k_1t+k_2x+k_3y+k_4,w=k_5t+k_6x+k_7y+k_8,$$

*where $f$ is a random first differentiable function, $v,w$ are independent of each other, and the constants $k_1,k_2,k_3,k_5,k_6,k_7$ must satisfy*

$$a_1k_1^3+a_2k_2^3+a_3k_3^3+a_4k_1^2k_2+a_5k_1^2k_3+a_6k_1k_2^2+a_7k_2^2k_3+a_8k_1k_3^2+a_9k_2k_3^2=0, \tag{84}$$

$$a_1k_5^3+a_2k_6^3+a_3k_7^3+a_4k_5^2k_6+a_5k_5^2k_7+a_6k_5k_6^2+a_7k_6^2k_7+a_8k_5k_7^2+a_9k_6k_7^2=0, \tag{85}$$

$$\begin{aligned}&3a_1k_1^2k_5+3a_2k_2^2k_6+3a_3k_3^2k_7+a_4\left(k_1^2k_6+2k_1k_2k_5\right)+a_5\left(k_1^2k_7+2k_1k_3k_5\right)\\&+a_6\left(k_2^2k_5+2k_1k_2k_6\right)+a_7\left(k_2^2k_7+2k_2k_3k_6\right)+a_8\left(k_3^2k_5+2k_1k_3k_7\right)+a_9\left(k_2k_7^2+2k_2k_3k_7\right)=0,\end{aligned} \tag{86}$$

$$\begin{aligned}&3a_1k_1k_5^2+3a_2k_2k_6^2+3a_3k_3k_7^2+a_4\left(k_2k_5^2+2k_1k_5k_6\right)+a_5\left(k_3k_5^2+2k_1k_5k_7\right)\\&+a_6\left(k_1k_6^2+2k_2k_5k_6\right)+a_7\left(k_3k_6^2+2k_2k_6k_7\right)+a_8\left(k_1k_7^2+2k_3k_5k_7+a_9\left(k_3^2k_6+2k_3k_6k_7\right)\right)=0.\end{aligned} \tag{87}$$

**Proof.** By $Z_1$ transformation, set $u = f(v, w)$, $v = k_1t+k_2x+k_3y+k_4$, $w = k_5t+k_6x+k_7y+k_8$. $k_1, k_2, \ldots, k_8$ are undetermined constants, $v, w$ are independent of each other, so

$$\begin{aligned}
&a_1u_t^3 + a_2u_x^3 + a_3u_y^3 + a_4u_t^2u_x + a_5u_t^2u_y + a_6u_x^2u_t + a_7u_x^2u_y + a_8u_y^2u_t + a_9u_y^2u_x \\
&= a_1(k_1f_v + k_5f_w)^3 + a_2(k_2f_v + k_6f_w)^3 + a_3(k_3f_v + k_7f_w)^3 \\
&+a_4(k_1f_v + k_5f_w)^2 (k_2f_v + k_6f_w) + a_5(k_1f_v + k_5f_w)^2 (k_3f_v + k_7f_w) \\
&+a_6(k_2f_v + k_6f_w)^2 (k_1f_v + k_5f_w) + a_7(k_2f_v + k_6f_w)^2 (k_3f_v + k_7f_w) \\
&+a_8(k_3f_v + k_7f_w)^2 (k_1f_v + k_5f_w) + a_9(k_3f_v + k_7f_w)^2 (k_2f_v + k_6f_w) \\
&= \left(a_1k_1^3 + a_2k_2^3 + a_3k_3^3 + a_4k_1^2k_2 + a_5k_1^2k_3 + a_6k_1k_2^2 + a_7k_2^2k_3 + a_8k_1k_3^2 + a_9k_2k_3^2\right) f_v^3 \\
&+ \left(a_1k_5^3 + a_2k_6^3 + a_3k_7^3 + a_4k_5^2k_6 + a_5k_5^2k_7 + a_6k_5k_6^2 + a_7k_6^2k_7 + a_8k_5k_7^2 + a_9k_6k_7^2\right) f_w^3 \\
&+3f_v^2f_wa_1k_1^2k_5 + 3f_v^2f_wa_2k_2^2k_6 + 3f_v^2f_wa_3k_3^2k_7 + f_v^2f_wa_4 \left(k_1^2k_6 + 2k_1k_2k_5\right) \\
&+f_v^2f_wa_5 \left(k_1^2k_7 + 2k_1k_3k_5\right) + f_v^2f_wa_6 \left(k_2^2k_5 + 2k_1k_2k_6\right) + f_v^2f_wa_7 \left(k_2^2k_7 + 2k_2k_3k_6\right) \\
&+f_v^2f_wa_8 \left(k_3^2k_5 + 2k_1k_3k_7\right) + f_v^2f_wa_9 \left(k_2k_7^2 + 2k_2k_3k_7\right) + 3f_vf_w^2a_1k_1k_5^2 + 3f_vf_w^2a_2k_2k_6^2 \\
&+3f_vf_w^2a_3k_3k_7^2 + f_vf_w^2a_4 \left(k_2k_5^2 + 2k_1k_5k_6\right) + f_vf_w^2a_5 \left(k_3k_5^2 + 2k_1k_5k_7\right) \\
&+f_vf_w^2a_6 \left(k_1k_6^2 + 2k_2k_5k_6\right) + f_vf_w^2a_7 \left(k_3k_6^2 + 2k_2k_6k_7\right) \\
&+f_vf_w^2a_8 \left(k_1k_7^2 + 2k_3k_5k_7 + f_vf_w^2a_9 \left(k_3^2k_6f_v^2f_w + 2k_3k_6k_7\right)\right) = 0.
\end{aligned}$$

Set

$$a_1k_1^3 + a_2k_2^3 + a_3k_3^3 + a_4k_1^2k_2 + a_5k_1^2k_3 + a_6k_1k_2^2 + a_7k_2^2k_3 + a_8k_1k_3^2 + a_9k_2k_3^2 = 0,$$

$$a_1k_5^3 + a_2k_6^3 + a_3k_7^3 + a_4k_5^2k_6 + a_5k_5^2k_7 + a_6k_5k_6^2 + a_7k_6^2k_7 + a_8k_5k_7^2 + a_9k_6k_7^2 = 0,$$

$$\begin{aligned}
&3a_1k_1^2k_5 + 3a_2k_2^2k_6 + 3a_3k_3^2k_7 + a_4 \left(k_1^2k_6 + 2k_1k_2k_5\right) + a_5 \left(k_1^2k_7 + 2k_1k_3k_5\right) \\
&+a_6 \left(k_2^2k_5 + 2k_1k_2k_6\right) + a_7 \left(k_2^2k_7 + 2k_2k_3k_6\right) + a_8 \left(k_3^2k_5 + 2k_1k_3k_7\right) + a_9 \left(k_2k_7^2 + 2k_2k_3k_7\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&3a_1k_1k_5^2 + 3a_2k_2k_6^2 + 3a_3k_3k_7^2 + a_4 \left(k_2k_5^2 + 2k_1k_5k_6\right) + a_5 \left(k_3k_5^2 + 2k_1k_5k_7\right) \\
&+a_6 \left(k_1k_6^2 + 2k_2k_5k_6\right) + a_7 \left(k_3k_6^2 + 2k_2k_6k_7\right) + a_8 \left(k_1k_7^2 + 2k_3k_5k_7 + a_9 \left(k_3^2k_6 + 2k_3k_6k_7\right)\right) = 0.
\end{aligned}$$

Therefore, the general solution of Eq. (83) is

$$u = f(v, w).$$

The theorem is proven. □

Next, we use Theorem 10 to study a definite solution problem.

**Example 8.** In $\mathbb{R}^3$, use Theorem 10 to get the exact solution of

$$u_x^3 + u_y^3 + u_t^2u_x + u_t^2u_y - u_x^2u_t + u_x^2u_y + u_y^2u_t + u_y^2u_x = 0, \tag{88}$$

in the condition of $u(t, 0, y) = g(t, y)$.

**Solution.** According to Theorem 10, the general solution of (88) is

$$u = f\left(t + x - y, t - x + y\right). \tag{89}$$

So

$$u(t, 0, y) = f(t - y, t + y) = g(t, y).$$

Set

$$t - y = \beta, t + y = \gamma.$$

We obtain

$$t = \frac{\beta + \gamma}{2}, y = \frac{-\beta + \gamma}{2}.$$

Namely

$$u(t, 0, y) = f(t - y, t + y) = g\left(\frac{\beta + \gamma}{2}, \frac{-\beta + \gamma}{2}\right).$$

Set

$$t + x - y = \beta, t - x + y = \gamma.$$

We get

$$\frac{\beta + \gamma}{2} = \frac{t + x - y + t - x + y}{2} = t,$$

$$\frac{-\beta + \gamma}{2} = \frac{-t - x + y + t - x + y}{2} = -x + y.$$

Then

$$u(t, x, y) = f(t + x - y, t - x + y) = g(t, -x + y). \tag{90}$$

### 3. Discussion and summary

In this paper, we have proved that $Z$ Transformations are important methods to get general solutions of nonlinear PDEs through specific cases, and it is also a efficient method to get exact solutions of definite solution problems by using general solutions. In practical applications, we find that the general solutions of nonlinear PDEs are sometimes not unique, and the exact solutions of the corresponding definite solution problems may not be unique. In subsequent papers, we will continue to research more kinds of non-linear PDEs by $Z$ Transformations.

## References

[1] Zhu H.L.: General solutions of the Laplace equation. Partial Diff. Equ. Appl. Math. **5**, 100302 (2022)