# An algebraic solution for the Rubik's Cube

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## 1 Introduction

The Rubik's Cube is a combination puzzle consisting on a cube typically tiled by 18 *cubelets* (corresponding to the 3x3x3 cube) each coloured according to 6 possible different colours. The cube is held in the inside by a device that allows for rotations of all faces, hence the problem is to rotate the faces in such a way to make any cube arrangement into the *identity* cube, as will be referred to in this paper, (the cube where each face is coloured in exactly one color).

This paper proposes an algebraic approach to solve the Rubik's cube, using Group Theory. The paper is concerned primarily with solving the 2x2x2 cube but the method is easily generalised to the 3x3x3 cube.

This paper assumes prior knowledge of elementary Abstract Algebra and Group Theory from the reader.

## 2 The Rubik's Cube as a group

Let us recall the definition of a group.

**Definition:** A group $G$ is a set of objects with an associated binary operation, $*$, that acts on the elements of the set. The following axioms must be satisfied for $(G, *)$ to be considered a group:

1. **Closure:** $\forall x, y \in G, x * y \in G$
2. **Identity:** $\exists e \in G$ such that $\forall x \in G, x * e = e * x = x$
3. **Inverse:** $\forall x \in G, \exists x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$
4. **Associativity:** $\forall x, y, z \in G, (x * y) * z = x * (y * z)$

We can now define the Rubik's Group, $(\mathbb{G}, *)$

**Definition:** let $\mathbb{G}$ be the set of all possible permutations applied to the Rubik's cube, in the original 3x3x3 Rubik's Cube there are around 43 Quintillion possible permutations of the original cube, hence we will not attempt to enumerate them extensively, but rather, we can see that every possible permutation is spanned by the rotations of each face, hence we can see that $\mathbb{G} = \langle\{e,\mathrm{U},\mathrm{D},\mathrm{L},\mathrm{R},\mathrm{F},\mathrm{B}\}\rangle$, where U corresponds to the rotation of the face pointing upwards by 90º anticlockwise and so on; $e$ is the identity element, i.e, no rotation. The Group operation $*$ is defined as the composition of two rotations

**Theorem 2.1:** $(\mathbb{G}, *)$ forms a group

**Proof:** Let us check every axiom individually

1. **Closure:** Follows from the definition of $\mathbb{G}$
2. **Identity:** The *identity cube*, which corresponds to not rotating any face
3. **Inverse:** The inverse of any rotation is the rotation of the same face but in opposite direction
4. **Associativity:** Left as an exercise

■

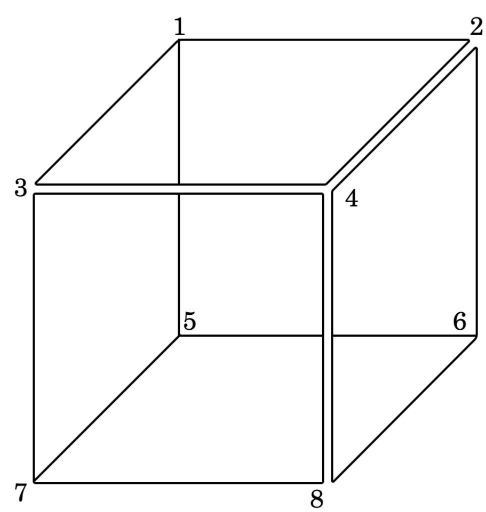
From now on we will restrict ourselves to the 2x2x2 cube for the sake of simplicity. We will now prove the most important result in this paper, which will be very beneficial when computing the solutions.

**Theorem 2.2:** The group $\mathbb{G}$ isomorphic to $H \subset S_8$, a subgroup of the permutation group $S_8$, where $H$ is the set of all legal permutations of the Rubik's Cube

**Proof:** Recall $\mathbb{G} \cong H \subset S_8 \iff$ there exists a bijective map $\phi : \mathbb{G} \to H$ between the two groups.

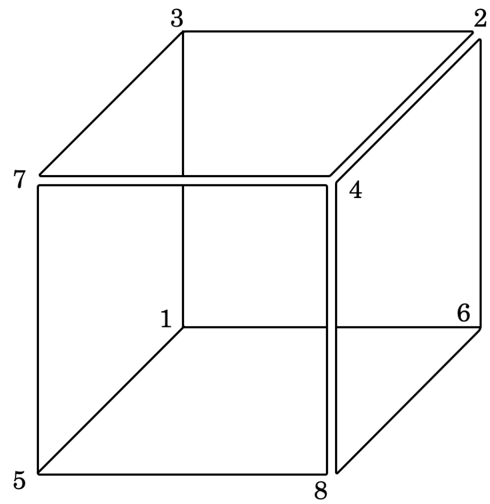
We now define such a map. Start with the identity cube, choose a fixed orientation for the cube and label each vertex with the numbers 1 through 8, notice each number corresponds to one *cubelet*. It is not relevant in which order the edges are labeled as long as it remains consistent.

For example: the arrangement that will be used throughout this paper is the following




Now we observe that a rotation on any of the faces will produce a permutation on the labeled edges, we now give an example:

Consider the rotation $L \in \mathbb{G}$. The cube will look now as follows after rotating the left face 90º anticlockwise:




We observe that we now have a permutation of the numbered edges, in which edge 3 has gone where edge 1 was, edge 7 has gone where edge three was and so on. We may write this with the usual matrix notation of the permutation group $S_k$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 1 & 4 & 7 & 6 & 3 & 8 \end{pmatrix}$$

Which may be read for convenience as: *"Edge 1 has gone to slot 5, edge 3 has gone to slot 1 and so on..."*. We may now simplify the matrix notation by using cycle notation, which represents permutations as disjoint cycles, and hence:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 1 & 4 & 7 & 6 & 3 & 8 \end{pmatrix} = (1573)$$

Moreover, for even greater compactness of notation, we may *factor* the disjoint cycle into a product of two-cycles:

$$(1573) = (13)(17)(15)$$

By the construction of this mapping, one observes that it is bijective. ■

Here are some tools that we will find useful in our computations later on:
**Lemma 2.3:** Every cycle $\sigma \in S_k$ can be factored as a product of two-cycles.
**Lemma 2.4:** $(ab)(bc) = (ac)(ab)$.
**Corollary 2.5:** $(ab)(bc)(ab) = (ac)$.

Now we are ready to tabulate every rotation in $\mathbb{G}$ as a product of two-cycles $\prod_{i=1}^{n}(a_ib_i) \in S_8$

| | |
|---|---|
| $\phi(\mathbf{U}) : (12)(14)(13)$ | $\phi(\mathbf{U^{-1}}) : (13)(14)(12)$ |
| $\phi(\mathbf{D}) : (56)(58)(57)$ | $\phi(\mathbf{D^{-1}}) : (57)(58)(56)$ |
| $\phi(\mathbf{L}) : (13)(17)(15)$ | $\phi(\mathbf{L^{-1}}) : (15)(17)(13)$ |
| $\phi(\mathbf{R}) : (42)(46)(48)$ | $\phi(\mathbf{R^{-1}}) : (48)(46)(42)$ |
| $\phi(\mathbf{F}) : (34)(38)(37)$ | $\phi(\mathbf{F^{-1}}) : (37)(38)(34)$ |
| $\phi(\mathbf{B}) : (12)(16)(15)$ | $\phi(\mathbf{B^{-1}}) : (15)(16)(12)$ |

Finally, we are ready to show and prove the algorithm for solving the 2x2x2 Rubik's Cube:

**Algorithm:**

1. We start with any given arrangement of the Cube, $\pi \in \mathbb{G}$

2. $\mathbb{G} \cong H \subset S_8$ (Theorem 2.2), hence, we obtain $\kappa \in S_8$ by $\kappa = \phi(\pi)$ with isomorphism of groups $\phi$ described above by using the method of labelling the edges of the identity cube, and observing where each edge has landed after the permutation, then we write this result in cycle notation and factor it out, much like in the example.

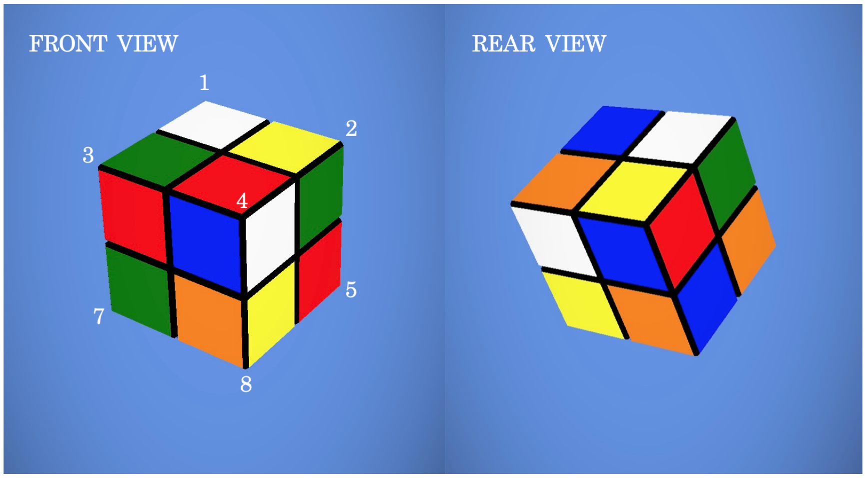
   Note, $\pi = \prod_{i=1}^{n} A_i$, where $A_i \in \mathbb{G}$ ($\mathbb{G}$ is a group)

   Therefore, $\kappa = \phi(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} \phi(A_i)$ ($\phi$ is an isomorphism of groups), note we know every $\phi(A_i)$ by using the tabulated table above

3. Since we wish to find a set of moves to get back to the identity cube, we must find $\pi^{-1} \in \mathbb{G}$, which is equivalent to finding $\kappa^{-1} \in S_8$.
   Hence, $\kappa^{-1} = (\prod_{i=1}^{n} \phi(A_i))^{-1} = \prod_{i=n}^{1} \phi(A_i)^{-1} = \prod_{i=n}^{1} \phi((A_i)^{-1})$
   However, we know every single $\phi((A_i)^{-1})$, since we have also tabulated them. We are done.

**Summary of the algorithm:** we start with a scrambled cube, we observe in which "slot" each of the original edges has ended, we write that as a matrix form and then as a factored cycle form, which we call $\kappa$. To find the set of moves to get back to the identity, we want to find elements from the table above such that when multiplying them to $\kappa$ from the left, we slowly get to simpler forms by cancellation of elements and finally back to the identity, note $(a_ib_i)^{-1} = (a_ib_i), \forall (a_ib_i) \in S_k$, hence the choice of which element of the table to multiply by becomes obvious most of the time, we just choose the element of the table that will cancel the rightmost elements of the permutation two-cycle product.
**Note:** sometimes, the choice will not be obvious at first, so some manipulation will be required to put the cycles into a neater form that will cancel some terms.

**Example:**




Given this state of the Cube, let us find the product of moves that will return it to the identity cube: First, we compare each vertex with the ones from the identity cube and we get that the product of rotations $\pi \in \mathbb{G} \cong \kappa \in S_8$.

Where $\kappa = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 1 & 4 & 2 & 3 & 7 & 6 \end{pmatrix} = (13)(16)(18)(25)$

It is not immediately obvious which element of the table we should multiply $\kappa$ by, but if we use **Lemma 2.4** we obtain that $\kappa = (36)(38)(13)(25)$. This may not seem helpful but this manipulation allows us to use **Lemma 2.5** in conjunction with $\phi(\mathbf{F^{-1}})$, because:

$\phi(\mathbf{F^{-1}})\kappa = (37)(34)(38)(36)(38)(13)(25)$ and we may now cancel $(38)(36)(38)$ and replace it with $(68)$, carrying out the full multiplication of the cycle to clear it up we get $\phi(\mathbf{F^{-1}})\kappa = (13)(17)(25)(68)(64)$ now it is clear that we use $\phi(\mathbf{L^{-1}})$ and now $\phi(\mathbf{L^{-1}})\phi(\mathbf{F^{-1}})\kappa = (15)(25)(68)(64)$. Now, we observe that if we manipulate $\phi(\mathbf{R^{-1}})$ into $\phi(\mathbf{R^{-1}}) = (64)(68)(42)$ the last digits match the last digits on $\phi(\mathbf{L^{-1}})\phi(\mathbf{F^{-1}})\kappa$, so we try: $\phi(\mathbf{R^{-1}})\phi(\mathbf{L^{-1}})\phi(\mathbf{F^{-1}})\kappa$ and we get:
$\phi(\mathbf{R^{-1}})\phi(\mathbf{L^{-1}})\phi(\mathbf{F^{-1}})\kappa = (26)(15)(25)$
Observe that if we multiply on the left now by $\phi(\mathbf{B^{-1}})$ we will get:
$\phi(\mathbf{B^{-1}})\phi(\mathbf{R^{-1}})\phi(\mathbf{L^{-1}})\phi(\mathbf{F^{-1}})\kappa = (15)(16)(12)(26)(15)(25)$ but $(16)(12)(26)$ cancels neatly into $(12)$, hence$\phi(\mathbf{B^{-1}})\phi(\mathbf{R^{-1}})\phi(\mathbf{L^{-1}})\phi(\mathbf{F^{-1}})\kappa = (15)(12)(15)(25) = (25)(25) = e$

Thus we have found a set of moves that takes us back to the identity, namely: $\mathbf{B^{-1}R^{-1}L^{-1}F^{-1}}$ when read from right to left by using the algorithm proven in the paper.

**Final remarks:** This paper has proven a solution for the 2x2x2 Rubik's Cube by using an algebraic method that transforms the problem of the Cube into a problem of performing a multiplication between a permutation and several combinations of two-cycles until the identity element of $S_8$ is achieved. This method offers an advantage in certain cases, since the move to perform becomes obvious as there will be one move that will cancel the most amount of elements, however in some scenarios it will be necessary to manipulate the expressions in order to achieve one that is easier to work with, just like other forms of algebra, like algebraic fractions in calculus. However, this approach becomes very cumbersome when dealing with a permutation of the Rubik's Cube where a large number of moves have been performed.

# 3 Generalising to larger Rubik's Cubes

This method may be generalised to a Rubik's Cube with more *cubelets* per side e.g: the 3x3x3 cube. The only adjustment needed would be to consider more *edges*, in such case there would be 27 *edges*, nonetheless, the same algorithm would apply.