An inequality on the rank of matrices of form of the matrices for generalised eigenvectors

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Abstract

This paper proves an inequality for the rank of matrices of a form resembling the one of matrices for generalised eigenvectors.

Introduction and motivation:

A non-zero vector $\mathbf{v} \in \mathbb{F}^n$ is said to be a generalised eigenvector of rank k of a matrix A with eigenvalue λ iff $k \in \mathbb{N}$ is the least such that:

$$(A - \lambda I)^k \mathbf{v} = \mathbf{0}$$

In this paper we will not look at the general case of such matrices but rather at the matrices of the form $(A - \lambda I)^k$, where A is an upper triangular matrix and prove an inequality regarding their rank.

Let $A \in M_n(\mathbb{F})$ be an upper triangular matrix whose characteristic polynomial is $P_A(x) = (-1)^n(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}...(x-\lambda_n)^{m_n}$
Consider the matrices $A_i := (A - \lambda_i)$ and $\tilde{A}_i := (A - \lambda_i)^{m_i}$

Theorem: $\text{rank}(\tilde{A}_i) \geq n - m_i$
Proof:
Let us define the following terminology:

Let $\mathcal{N}_{\mathcal{Z}}(A)$ denote number of non-zero entries in the diagonal of A.
Let $\mathcal{R}(A)$ be the rank of A.

The outline of the proof is as follows: we will show the claim is true for A_i by induction and then show it is also true for $\tilde{A}_i$.
To show the claim holds for A_i it is enough to show that for any triangular matrix A: $\mathcal{R}(A) \geq \mathcal{N}_{\mathcal{Z}}(A)$ since the diagonal entries of a triangular matrix are its eigenvalues, which appear with multiplicities m_i therefore, due to the fact that $n - m_i \leq \mathcal{N}_{\mathcal{Z}}(A_i)$ (since 0 may be an eigenvalue), if we show $\mathcal{R}(A) \geq \mathcal{N}_{\mathcal{Z}}(A)$ for all A, it follows that for A_i, $\mathcal{R}(A_i) \geq n - m_i$

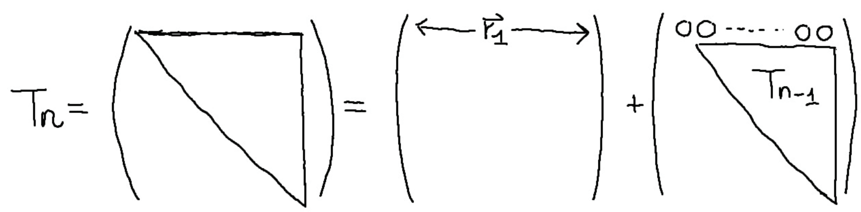
Base case: n=1: Trivial
Case: n=2: A is of the form: $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ therefore there are only three possible values for $\mathcal{N}_{\mathcal{Z}}(A)$, namely:

$\mathcal{N}_{\mathcal{Z}}(A)$=2 $\iff$ $a \neq 0$ and $c \neq 0$, in which case $\mathcal{R}(A)$=2 and the claim holds

$\mathcal{N}_{\mathcal{Z}}(A)$=1 $\iff$ $a \neq 0$ and $c = 0$ or $c \neq 0$ and $a = 0$. In either case $\mathcal{R}(A)$=1 and the claim still holds.

$\mathcal{N}_{\mathcal{Z}}(A)$=0 $\iff$ $a = c = 0$. In which case $\mathcal{R}(A)$=1 and the claim still holds.

Inductive hypothesis: Assume the theorem holds for all triangular matrices of up to size $(n-1)$. Let us consider a triangular matrix of size n, T_n, and split it into two matrices as follows:
$T_n = R + \tilde{T}_{n-1}$, where R is a matrix which only has one row vector in the first row, and $\tilde{T}_{n-1}$ Is an $n \times n$ matrix with zeroes in the first row and column and an upper triangular matrix of size $n-1$ by $n-1$ occupying the rest of the matrix:



(illustration of T_n)

We now recall the following lemma:

Lemma: Let A and B be matrices of the same size, $\mathcal{R}(A+B) \leq \mathcal{R}(A)+\mathcal{R}(B)$ with equality if and only if $Image(A) \cap Image(B) = \{0\}$
The proof equality, which is what we are interested in, is not included in this text but it follows intuitively by inspecting the proof of the inequality in the first place.

Note: in the decomposition of T_n, $Image(R) \cap Image(\tilde{T}_{n-1}) = \{0\}$, hence by our lemma, $\mathcal{R}(T_n) = \mathcal{R}(R) + \mathcal{R}(\tilde{T}_{n-1})$. Also, $0 \leq \mathcal{R}(R) \leq 1$ and $\mathcal{R}(\tilde{T}_{n-1}) \geq \mathcal{N}_{\mathcal{Z}}(T_{n-1})$ by the induction hypothesis. Note $0 \leq \mathcal{N}_{\mathcal{Z}}(R) \leq 1$ and $\mathcal{N}_{\mathcal{Z}}(T_n) = \mathcal{N}_{\mathcal{Z}}(T_{n-1}) + \mathcal{N}_{\mathcal{Z}}(R)$.

Substituting into $\mathcal{R}(T_n)$ gives: $\mathcal{R}(T_n) \geq \mathcal{N}_{\mathcal{Z}}(R) + \mathcal{N}_{\mathcal{Z}}(T_{n-1}) = \mathcal{N}_{\mathcal{Z}}(T_n)$.
Thus proving the first part of the theorem.

Recap: we have shown that for $A_i = (A - \lambda_i)$, $\mathcal{R}(A_i) \geq n - m_i$ because $\mathcal{N}_{\mathcal{Z}}(A_i) \geq n - m_i$ and $\mathcal{R}(A) \geq \mathcal{N}_{\mathcal{Z}}(A)$ for **any** matrix A. In particular, this means $\mathcal{R}(A_i^m) \geq \mathcal{N}_{\mathcal{Z}}(A_i^m)$
Hence it is enough to show that $\mathcal{N}_{\mathcal{Z}}(A_i^{m_i}) = \mathcal{N}_{\mathcal{Z}}(A_i)$
Because that would imply our final result. Namely:
$\mathcal{R}(A_i^m) \geq \mathcal{N}_{\mathcal{Z}}(A_i^m) = \mathcal{N}_{\mathcal{Z}}(A_i) \geq n - m_i$

Claim: $\mathcal{N}_{\mathcal{Z}}(A_i^{m_i}) = \mathcal{N}_{\mathcal{Z}}(A_i)$

Proof:
Consider upper triangular matrices A and B of equal size, from the usual matrix multiplication rules it follows that. $(AB)_{ii} = \sum_{j=1}^{n}(A)_{ij}(B)_{ji}$ which we can split as follows:
$(AB)_{ii} = \sum_{j=1}^{n}(A)_{ij}(B)_{ji} = \sum_{j=1}^{i-1}(A)_{ij}(B)_{ji} + (A)_{ii}(B)_{ii} + \sum_{j=i+1}^{n}(A)_{ij}(B)_{ji}$
It follows from the definition of upper triangular matrices that:
$\sum_{j=1}^{i-1}(A)_{ij}(B)_{ji} = 0$ and $\sum_{j=i+1}^{n}(A)_{ij}(B)_{ji} = 0$. Thus: $(AB)_{ii} = (A)_{ii}(B)_{ii}$.
In particular:
$(AB)_{ii} = 0 \iff (A)_{ii} = 0$ or $(B)_{ii} = 0$. Therefore: $(A^2)_{ii} = 0 \iff (A)_{ii} = 0$, it follows inductively that: $(A^m)_{ii} = 0 \iff (A)_{ii} = 0$. Therefore we can conclude that $\mathcal{N}_{\mathcal{Z}}(A_i^{m_i}) = \mathcal{N}_{\mathcal{Z}}(A_i)$
Conlcluding this claim and proving the theorem. ■